INVERSION FORMULAE FOR A GENERALIZED LAPLACE INTEGRAL

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Dr. R. S. Varma (1952) has recently given a generalization of the Laplace Integral

$$f(s) = \int_0^\infty e^{-st}\, d\alpha(t) \quad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

in the form

$$f(s) = \int_0^\infty (st)^{m-\frac{1}{2}} e^{-\frac{1}{2}st}\, W_{k,m}(st)\, d\,\alpha(t) \quad \ldots \quad \ldots \quad (1.2)$$

where $\alpha(t)$ is a function of bounded variation in $0 \leqslant t \leqslant R$ for every R and $R(m) > 0$.

If we put $k+m = \frac{1}{2}$, the integral (1.2) reduces to (1.1) as

$$(st)^{m-\frac{1}{2}} e^{-\frac{1}{2}st}\, W_{\frac{1}{2}-m,\,m}(st) \equiv e^{-st}.$$

If $\alpha(t)$ be absolutely continuous and $d\alpha(t) = \phi(t)\,dt$, then

$$f(s) = \int_0^\infty (st)^{m-\frac{1}{2}} e^{-\frac{1}{2}st}\, W_{k,m}(st)\, \phi(t)\, dt. \quad \ldots \quad \ldots \quad (1.3)$$

In this paper I have worked out a few inversion formulae for the generalized Laplace transform (1.3). The variable s has been assumed to be real.

2. *First Inversion Formula.*

Multiplying both the sides of

$$f(s) = \int_0^\infty (st)^{m-\frac{1}{2}} e^{-\frac{1}{2}st}\, W_{k,m}(st)\, \phi(t)\, dt$$

by s^{-l} and integrating from 0 to ∞ we have

$$\int_0^\infty s^{-l} f(s)\, ds = \int_0^\infty \int_0^\infty s^{m-l-\frac{1}{2}}\, t^{m-\frac{1}{2}}\, e^{-\frac{1}{2}st}\, W_{k,m}(st)\, \phi(t)\, dt\, ds.$$

Let

$$\left.\begin{aligned} \phi(t) &= 0(t^{\rho}) \text{ for small } t, \\ &= 0(e^{-t^{\rho}}) \text{ for large } t. \end{aligned}\right\}, \; R(\nu) > 0. \quad \ldots \quad \ldots \quad (2.1)$$

Then if

$$A(s) = s^{m-l-\frac{1}{2}} \int_0^{\epsilon} t^{m-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k,m}(st)\,\phi(t)\,dt \quad \text{where } \epsilon \text{ is small,}$$

and

$$B(t) = t^{m-\frac{1}{2}} \phi(t) \int_0^{\infty} s^{m-l-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k,m}(st)\,ds$$

we see that $A(s)$ is uniformly convergent in $s \geqslant 0$ provided that

$$R(m-l\pm m) \geqslant 0, \quad R(m\pm m+\rho+1) > 0,$$

and $B(t)$ is uniformly convergent in $t \geqslant 0$ provided that

$$R(m\pm m+\rho) \geqslant 0, \quad R(m-l\pm m+1) > 0,$$

as (Whittaker and Watson, 1946, p. 346)

$$W_{k,m}(x) = 0\left(x^{\pm m+\frac{1}{2}}\right) \text{ for small } x.$$

Also if we consider the integral

$$\int_T^{\infty} t^{m-\frac{1}{2}} \phi(t)\,dt \int_{T'}^{\infty} s^{m-l-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k,m}(st)\,ds$$

where T and T' are large we find on account of the values (2.1) of $\phi(t)$ and the estimate (Whittaker and Watson, 1946, p. 343)

$$W_{k,m}(x) = 0\left(x^{k} e^{-\frac{1}{2}x}\right) \text{ where } x \text{ is large,} \qquad \ldots \qquad (2.2)$$

of $W_{k,m}(x)$, that the integral does not exceed a constant multiple of

$$\int_T^{\infty} |t^{k+m-\frac{1}{2}} e^{-t^{\nu}}|\,dt \int_{T'}^{\infty} |s^{k+m-l-\frac{1}{2}} e^{-st}|\,ds$$

which tends to zero provided that $R(\nu) > 0$.

Hence the order of integration can be changed if the conditions

$$R(m-l\pm m) \geqslant 0, \quad R(m\pm m+\rho) \geqslant 0, \quad R(\nu) > 0,$$

which, by the principle of analytic continuation, can be waived to

$$R(m-l\pm m+1) > 0, \quad R(m\pm m+\rho+1) > 0, \quad R(\nu) > 0, \qquad \ldots \qquad (2.3)$$

are satisfied.

Therefore changing the order of integration we have

$$\int_0^{\infty} s^{-l} f(s)\,ds = \int_0^{\infty} t^{m-\frac{1}{2}} \phi(t)\,dt \int_0^{\infty} s^{m-l-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k,m}(st)\,ds$$

provided conditions (2.3) are satisfied.

If we now apply Goldstein's formula (1932)

$$\int_0^\infty x^{l_1-1} e^{-(\alpha^2+\frac{1}{2})x} W_{k_1, m_1}(x)\, dx = \frac{\Gamma(l_1+m_1+\frac{1}{2})\, \Gamma(l_1-m_1+\frac{1}{2})}{\Gamma(l_1-k_1+1)}$$

$$\times {}_2F_1[l_1+m_1+\tfrac{1}{2}, l_1-m_1+\tfrac{1}{2};\ l_1-k_1+1;\ -\alpha^2] \quad .. \quad (2.4)$$

where

$$R(l_1 \pm m_1+\tfrac{1}{2}) > 0, \quad R(\alpha^2+1) > 0,$$

we get

$$\int_0^\infty s^{-l} f(s)\, ds = \frac{\Gamma(2m-l+1)\, \Gamma(1-l)}{\Gamma(m-k-l+\frac{3}{2})} \int_0^\infty t^{l-1} \phi(t)\, dt$$

where

$$R(2m-l+1) > 0, \quad R(1-l) > 0.$$

Applying Mellin's inversion formula (Titchmarsh, 1937, p. 46) to the integral on the right we have

$$\tfrac{1}{2}\{\phi(t+0)+\phi(t-0)\} = \frac{1}{2\pi i} \underset{T\to\infty}{\mathrm{Lt}} \int_{c-iT}^{c+iT} \frac{\Gamma(m-k-l+\frac{3}{2})}{\Gamma(2m-l+1)\, \Gamma(1-l)} t^{-l} \psi(l)\, dl$$

where
$$\psi(l) = \int_0^\infty s^{-l} f(s)\, ds$$

provided that

(i) the integral $\int_0^\infty x^{c-1} \phi(x)\, dx$ converges absolutely,

(ii) the integral $\int_0^\infty x^{-l} f(x)\, dx$ also converges absolutely $(l = c+i\tau,\ -\infty < \tau < \infty)$

and

(iii) $\phi(x)$ is of bounded variation in the neighbourhood of the point $x = t$ $(t>0)$.

If we put $k = \frac{1}{2}-m$, we have

$$\tfrac{1}{2}\{\phi(t+0)+\phi(t-0)\} = \frac{1}{2\pi i} \underset{T\to\infty}{\mathrm{Lt}} \int_{c-iT}^{c+iT} \frac{t^{-l}}{\Gamma(1-l)} \int_0^\infty s^{-l} f(s)\, ds\, dl$$

which is the corresponding result in the theory of ordinary Laplace transform (Titchmarsh, 1937, p. 316).

3. *Second Inversion Formula.*

Let $F(s)$ be a continuous function of s in $(0, \infty)$ such that its derivatives up to the n-th order are all continuous in $(0, \infty)$ and

$$\frac{d^n}{ds^n}[F(s)] = f(s) = \int_0^\infty (st)^{m-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k, m}(st)\, \phi(t)\, dt.$$

Then if we integrate (1.3) with respect to s, n times we have

$$F(s)=(-)^n\int_0^\infty t^{-n}(st)^{m+\frac{1}{2}(n-1)}e^{-\frac{1}{2}st}W_{k-\frac{1}{2}n,\,m+\frac{1}{2}n}(st)\,\phi(t)\,dt$$

as

$$\int (st)^{m-\frac{1}{2}}e^{-\frac{1}{2}st}W_{k,m}(st)\,ds=-t^{-1}(st)^m e^{-\frac{1}{2}st}W_{k-\frac{1}{2},\,m+\frac{1}{2}}(st). \quad .. \quad (3.1)$$

Multiplying both sides by

$${}_{n+1}F_{2n+1}\left[\begin{matrix}\alpha_1,\alpha_2,\ldots\ldots\alpha_{n+1},\\ \beta_1,\beta_2,\ldots\ldots\beta_{n+1},\gamma_1,\gamma_2,\ldots\ldots\gamma_n\end{matrix};-p\left(\frac{s}{n+1}\right)^{n+1}\right]$$

and integrating with respect to s from 0 to ∞, we have

$$\int_0^\infty F(s)\,{}_{n+1}F_{2n+1}\left[\begin{matrix}\alpha_1,\alpha_2,\ldots\ldots\alpha_{n+1},\\ \beta_1,\beta_2,\ldots\ldots\beta_{n+1},\gamma_1,\gamma_2,\ldots\gamma_n\end{matrix};-p\left(\frac{s}{n+1}\right)^{n+1}\right]ds$$

$$=(-)^n\int_0^\infty\int_0^\infty {}_{n+1}F_{2n+1}\left[\begin{matrix}\alpha_1,\alpha_2,\ldots\alpha_{n+1},\\ \beta_1,\beta_2,\ldots\beta_{n+1},\gamma_1,\gamma_2,\ldots\gamma_n\end{matrix};-p\left(\frac{s}{n+1}\right)^{n+1}\right]$$

$$\times\, t^{-n}(st)^{m+\frac{1}{2}(n-1)}e^{-\frac{1}{2}st}W_{k-\frac{1}{2}n,\,m+\frac{1}{2}n}(st)\,\phi(t)\,dt\,ds$$

$$=(-)^n\int_0^\infty t^{-n-1}\phi(t)\,dt\int_0^\infty v^{\{m+\frac{1}{2}(n+1)\}-1}e^{-\frac{1}{2}v}W_{k-\frac{n}{2},\,m+\frac{n}{2}}(v)$$

$$\times\,{}_{n+1}F_{2n+1}\left[\begin{matrix}\alpha_1,\alpha_2,\ldots\ldots\alpha_{n+1},\\ \beta_1,\beta_2,\ldots\ldots\beta_{n+1},\gamma_1,\gamma_2\ldots\gamma_n\end{matrix};-\frac{p}{\{(n+1)t\}^{n+1}}v^{n+1}\right]dv \qquad (3.2)$$

on changing the order of integration in the right-hand integral and making a slight change of variable, provided that

$$R\left(m\pm m+\frac{n}{2}\pm\frac{n}{2}+1\right)>0,\quad R\left(m\pm m-\frac{n}{2}\pm\frac{n}{2}+\rho+1\right)>0,\quad R(\nu)>0.$$

Evaluating the v-integral with the help of the following result due to Pasricha (1943)

$$\int_0^\infty x^{\mu-1}e^{-(\alpha^2+\frac{1}{2})x}W_{k,\nu}(x)\,{}_rF_s\left(\begin{matrix}l_1,l_2,\ldots\ldots l_r\\ m_1,m_2,\ldots\ldots m_s\end{matrix};yx^p\right)dx$$

$$=\sum_{m=0}^{\infty}\frac{(l_1)_m(l_2)_m\ldots\ldots(l_r)_m\,\Gamma(a+pm)\,\Gamma(b+pm)}{(m_1)_m(m_2)_m\ldots\ldots(m_s)_m\,\Gamma(c+pm)}$$

$$\times\,{}_2F_1(a+pm,b+pm;c+pm;-\alpha^2) \quad .. \quad (3.3)$$

where $a=\mu+\nu+\frac{1}{2},\ b=\mu-\nu+\frac{1}{2},\ c=\mu-k+1$

and $R(a)>0,\ R(b)>0,\ p<s+1-r,\ R(\alpha^2+1)>0$

we get

$$I = (-)^n \int_0^\infty t^{-n-1} J\, \phi(t)\, dt \qquad \ldots \qquad \ldots \quad (3.4)$$

where

$$I \equiv \int_0^\infty F(s)\ {}_{n+1}F_{2n+1}\left[\begin{matrix} \alpha_1, \alpha_2 \ldots \alpha_{n+1}, \\ \beta_1, \beta_2, \ldots \beta_{n+1}, \gamma_1, \gamma_2, \ldots \gamma_n \end{matrix} : -p\left(\frac{s}{n+1}\right)^{n+1}\right] ds$$

and

$$J \equiv \sum_{r=0}^{\infty} \frac{(\alpha_1)_r \ldots (\alpha_{n+1})_r\, \Gamma(a+\overline{n+1}\,r)\, \Gamma(b+\overline{n+1}\,r)}{(\beta_1)_r \ldots (\beta_{n+1})_r\, (\gamma_1)_r \ldots (\gamma_n)_r\, \Gamma(c+\overline{n+1}\,r)\, \lfloor r} \left\{-\frac{p}{\{(n+1)\,t\}^{n+1}}\right\}^r$$

a being equal to $2m+n+1$, $b = 1$, $c = m-k+n+\frac{3}{2}$.

Putting

$$\left.\begin{aligned} \alpha_i &= \frac{c+i-1}{n+1}, \ \beta_i = \frac{a+i-1}{n+1}\ (i = 1, 2, \ldots n+1) \\ \gamma_j &= \frac{b+j-1}{n+1}\ (j = 1, 2, \ldots . n) \end{aligned}\right\} \qquad \ldots \quad (3.5)$$

and using the multiplication formula (Gupta, 1948).

$$\Gamma(a+nr) = n^{nr}\, \Gamma(a) \left(\frac{a}{n}\right)_r \left(\frac{a+1}{n}\right)_r \ldots\ldots \left(\frac{a+n-1}{n}\right)_r \qquad \ldots \quad (3.6)$$

the above result gives

$$H = (-)^n \frac{\Gamma(2m+n+1)}{\Gamma(m-k+n+\frac{3}{2})} \int_0^\infty t^{-n-1}\ {}_1F_0\left(1\,;\ -\frac{p}{t^{n+1}}\right) \phi(t)\, dt$$

where H is the value of I with the parameters α's, β's and γ's having values given by (3.5).

We therefore have

$$\int_0^\infty \frac{\beta(u)\, du}{p+u} = \psi(p) \equiv (-)^n \frac{\Gamma(m-k+n+\frac{3}{2})}{\Gamma(2m+n+1)} H \qquad \ldots \quad (3.7)$$

where

$$\beta(t) = \frac{\phi\left(t^{1/(n+1)}\right)}{(n+1)\, t^{n/(n+1)}}$$

and $\phi(t)$ is such that the integral on the left converges.

The left-hand side is the ordinary Stieltjes transform.

If we now define an operator $L_{l,u}[\psi(x)]$ for any real positive number u by the equations

$$\left.\begin{aligned} L_{0,u}[\psi(x)] &= \psi(u) \\ L_{1,u}[\psi(x)] &= \frac{d}{du}[u\, \psi(u)] \\ L_{l,u}[\psi(x)] &= \frac{(-u)^{l-1}}{\lfloor l \lfloor l-2} \cdot \frac{d^{2l-1}}{du^{2l-1}}[u^l\, \psi(u)] \\ (l &= 2, 3, \ldots\ldots) \end{aligned}\right\} \qquad \ldots \qquad \ldots \quad (3.8)$$

and $\psi(u)$ possesses derivatives of all orders less than $2l$, then, by a result of Widder (1941, p. 345), we get

$$\beta(u) = \operatorname*{Lt}_{l\to\infty} \{L_{l,u}[\psi(p)]\}$$

for almost all positive u.

Hence

$$\phi(t) = (n+1)t^n \left\{ \operatorname*{Lt}_{l\to\infty} [L_{l,u}[\psi(p)]] \right\}_{u=t^{n+1}}$$

for almost all positive t.

It remains to justify the change in the order of integration in (3.2). To do so let us put

$$\delta(t) = t^{m-\frac{n}{2}-\frac{1}{2}} \phi(t) \int_0^\epsilon s^{m+\frac{n}{2}-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k-\frac{1}{2}n,\, m+\frac{1}{2}n}(st)$$
$$\times {}_{n+1}F_{2n+1}\left[\begin{matrix}\alpha_1, \alpha_2, \ldots \alpha_{n+1}, \\ \beta_1, \beta_2, \ldots \beta_{n+1}, \gamma_1, \gamma_2, \ldots \gamma_n\end{matrix}; -p\left(\frac{s}{n+1}\right)^{n+1}\right] ds$$

where ϵ is small,

$$\chi(s) = s^{m+\frac{n}{2}-\frac{1}{2}} {}_{n+1}F_{2n+1}\left[\begin{matrix}\alpha_1, \alpha_2, \ldots \ldots \alpha_{n+1} \\ \beta_1, \beta_2, \ldots \beta_{n+1}, \gamma_1, \gamma_2, \ldots \gamma_n\end{matrix}; -p\left(\frac{s}{n+1}\right)^{n+1}\right]$$
$$\times \int_0^\infty t^{m-\frac{n}{2}-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k-\frac{1}{2}n,\, m+\frac{1}{2}n}(st)\, \phi(t)\, dt$$

and let the behaviour of $\phi(t)$ be given by (2.1).

Then $\delta(t)$ is uniformly convergent in $t \geqslant 0$ provided that

$$R\left(m \pm m - \frac{n}{2} \pm \frac{n}{2} + \rho\right) \geqslant 0, \quad R\left(m \pm m + \frac{n}{2} \pm \frac{n}{2} + 1\right) > 0$$

and $\chi(s)$ is uniformly convergent in $s \geqslant 0$ provided that

$$R\left(m \pm m + \frac{n}{2} \pm \frac{n}{2}\right) \geqslant 0, \quad R\left(m \pm m - \frac{n}{2} \pm \frac{n}{2} + \rho + 1\right) > 0.$$

Again if we consider the integral

$$I_1 = \int_T^\infty t^{m-\frac{n}{2}-\frac{1}{2}} \phi(t)\, dt \int_{T'}^\infty s^{m+\frac{n}{2}-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k-\frac{1}{2}n,\, m+\frac{1}{2}n}(st)$$
$$\times {}_{n+1}F_{2n+1}\left[\begin{matrix}\alpha_1, \alpha_2, \ldots \alpha_{n+1}, \\ \beta_1, \beta_2, \ldots \beta_{n+1}, \gamma_1, \gamma_2, \ldots \gamma_n\end{matrix}; -p\left(\frac{s}{n+1}\right)^{n+1}\right] ds$$

where T and T' are large, we find, on account of the estimates (2.1) and on account of the following behaviour of ${}_{n+1}F_{2n+1}$ (Gupta, 1948).

$${}_{n+1}F_{2n+1}\left[\begin{matrix}\alpha_1, \alpha_2, \ldots \ldots \alpha_{n+1}, \\ \beta_1, \beta_2, \ldots \beta_{n+1}, \gamma_1, \gamma_2, \ldots \gamma_n\end{matrix}; -p\left(\frac{s}{n+1}\right)^{n+1}\right]$$
$$\sim \left(p^{\frac{1}{n+1}} s\right)^{\theta'} \exp\left\{\left(p^{\frac{1}{n+1}} s\right) e^{\frac{\pi i}{n+1}}\right\} + \sum_{r=1}^{n+1} a_n \left(s\, p^{\frac{1}{n+1}}\right)^{-(n+1)\alpha_r} \quad (s \to \infty)$$

where $$\theta' = \Sigma\alpha_n - \Sigma\beta_n - \Sigma\gamma_n + \frac{n}{2},$$

that I_1 does not exceed a constant multiple of

$$\int_T^\infty |t^{m-n+k-\frac{1}{2}} e^{-t^\nu}| dt \int_{T'}^\infty \left| s^{m+k-\frac{1}{2}} \left\{ s^{\theta'} e^{sp^{\frac{1}{n+1}} e^{\frac{\pi i}{n+1}}} + \sum_{r=1}^{n+1} s^{-(n+1)\alpha_r} \right\} e^{-st} \right| ds$$

which tends to zero for a given p provided that $R(\nu)>0$.

Hence the change in the order of integration is justified under the conditions

$$R\left(m \pm m + \frac{n}{2} \pm \frac{n}{2}\right) > 0, \quad R\left(m \pm m - \frac{n}{2} \pm \frac{n}{2} + \rho\right) > 0, \quad R(\nu) > 0,$$

which can be relaxed to

$$R\left(m \pm m + \frac{n}{2} \pm \frac{n}{2} + 1\right) > 0, \quad R\left(m \pm m - \frac{n}{2} \pm \frac{n}{2} + \rho + 1\right) > 0, \quad R(\nu) > 0,$$

by the principle of analytic continuation.

4. *Third Inversion Formula :*

Here we shall assume k and m to be both real.

Let us define differential operators U, V_q by

$$\left.\begin{aligned} U_0[f(s)] &= f(s) \\ U_1[f(s)] &= (-)s^{m-k-\frac{3}{2}} D[s^{k-m+\frac{1}{2}} f(s)] \\ U_q[f(s)] &= (-)^q s^{m-k-q-\frac{1}{2}} D^q[s^{k-m+\frac{1}{2}} f(s)] \\ &(q = 2, 3, \ldots\ldots) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (4.1)$$

where $$D \equiv s^2 \frac{d}{ds}$$

and

$$V_q[f(u)] = \frac{u^{-1}}{\Gamma(q+k+m-\frac{1}{2})} \Big[U_q\{ f(s) \Big]_{s = \frac{q+k+m-\frac{1}{2}}{u}}$$

Since (Sharma, 1939)

$$\frac{d}{dx}\left[x^k e^{-\frac{1}{2}x} W_{k,m}(x) \right] = -x^{k-1} e^{-\frac{1}{2}x} W_{k+1,m}(x)$$

applying the operator U to $g(s) \equiv (st)^{m-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k,m}(st)$

we find that $$U_q[g(s)] = (st)^{m-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k+q,m}(st).$$

If (1.3) converges, we can evaluate the derivatives of $f(s)$ by differentiation under the integral sign.

Therefore

$$U_q[f(s)] = \int_0^\infty (st)^{m-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k+q,m}(st)\, \phi(t)\, dt.$$

If the variable t be changed to v by the relation $t = uv$, we have

$$U_q[f(s)] = u\int_0^\infty (suv)^{m-\frac{1}{2}} e^{-\frac{1}{2}suv} W_{k+q,m}(suv)\,\phi(uv)\,dv.$$

Putting $su = l$, the right-hand side becomes

$$\frac{l}{s}\int_0^\infty (lv)^{m-\frac{1}{2}} e^{-\frac{1}{2}lv} W_{k+q,m}(lv)\,\phi(uv)\,dv.$$

Therefore substituting the asymptotic estimate (2.2) of $W_{k+q,m}(lv)$ in the integral we have

$$\operatorname*{Lt}_{l\to\infty}\left[\frac{sU_q\{f(s)\}}{\lfloor l}\right] = \operatorname*{Lt}_{l\to\infty}\frac{l}{\lfloor l}\int_0^\infty (lv)^{k+q+m-\frac{1}{2}} e^{-lv}\phi(uv)\,dv$$

If we take $l = k+q+m-\frac{1}{2}$, the right-hand side becomes

$$\operatorname*{Lt}_{l\to\infty}\int_0^\infty \frac{l^{l+1}}{\lfloor l} v^l e^{-lv}\phi(uv)\,dv = \phi(u)$$

as it is known (Widder, 1934) that for almost all positive u,

$$\operatorname*{Lt}_{n\to\infty}\frac{n^{n+1}}{\lfloor n}\int_0^\infty y^n e^{-ny}\{\phi(uy)-\phi(u)\}\,dy = 0.$$

Hence for almost all positive u

$$\phi(u) = \operatorname*{Lt}_{l\to\infty}\left[\frac{\frac{1}{u}U_q\{f(s)\}}{\lfloor l-1}\right]_{s=\frac{l}{u}}$$

$$= \operatorname*{Lt}_{q\to\infty} V_q[f(u)].$$

I am indebted to Dr. R. S. Varma for his kind help and guidance in the preparation of this paper.

SUMMARY.

Dr. R. S. Varma has recently given a generalization of the Laplace transform

$$f(s) = \int_0^\infty e^{-st}\,d\alpha(t)$$

in the form

$$f(s) = \int_0^\infty (st)^{m-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k,m}(st)\,d\alpha(t)$$

which assumes the form

$$f(s) = \int_0^\infty (st)^{m-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k,m}(st)\,\phi(t)\,dt$$

when $\alpha(t)$ is absolutely continuous. Three inversion formulae for this generalized Laplace transform have been obtained in this paper—one by making use of the Mellin's Inversion Formula, second by using a result in the theory of Stieltjes transform due to Widder, and third by employing a set of differential operators.

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