NON-CONCYCLIC SETS OF POINTS

by HANSRAJ GUPTA, *Hoshiarpur, Punjab.*

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1. Sylvester conjectured and Grünwald proved that—

If a finite number of distinct points in a plane are such that a line through any two of them passes through a third then all the points lie on a line.

An analogue of this is—

THEOREM A. *If a finite number of distinct points in a plane are such that a circle through any three of them passes through a fourth then all the points lie on a circle.*

A generalization would be

THEOREM B. *If a finite number of distinct points in space are such that a sphere through any four of them passes through a fifth then all the points lie on a sphere.*

The generalization can be extended to the k-dimensional space also.

The object of this note is to give a proof of Theorem A.

The argument would apply to Sylvester's theorem with slight modifications. Throughout this note we shall confine ourselves to real points in the finite part of an extended real Euclidean plane. The term circle shall include a 'straight circle'.

2. Proof of Theorem A.

Theorem A can be stated in the form:

If a finite number of distinct points in a plane are non-concyclic and a circle is drawn through every three of them then at least one of the circles so obtained contains exactly three of the points.

We shall call such a circle a 'g-circle' with respect to the system of points.

The theorem can be easily verified for sets of 4, 5 and 6 points.

Let us assume that the theorem holds for every set of $(n-1)$ or fewer coplanar points (not less than four, of course). Then we show that it holds when another point distinct from the $(n-1)$ points of the set is added to the set. We label the points $p_1, p_2, p_3, \ldots, p_n$ and take $n \geqslant 7$. By a k-set we shall mean a set of k distinct coplanar points $p_1, p_2, p_3, \ldots, p_k$.

Firstly, let the $(n-1)$ points $p_1, p_2, \ldots, p_{n-1}$ be concyclic.

Then the point p_n may or may not lie on the circle through the $(n-1)$ points $p_1, p_2, \ldots, p_{n-1}$.

In the former case, the n points are concyclic. In the latter case, every circle through p_n and two of the points $p_1, p_2, \ldots, p_{n-1}$ is a g-circle.

Secondly, when the $(n-1)$ points $p_1, p_2, \ldots, p_{n-1}$ are not concyclic.

If p_n does not lie on every g-circle of the set of $(n-1)$ points $p_1, p_2, \ldots, p_{n-1}$, then there is nothing to prove; if it does then take one of the g-circles of the $(n-1)$ set containing the points p_1, p_2, p_3, say.

This passes through p_n by supposition.

If the points $p_4, p_5, p_6, \ldots, p_n$ are concyclic and this circle does not pass through any of the points p_1, p_2 or p_3, then the circles $p_n\, p_1\, p_m$, $p_n\, p_2\, p_m$, $p_n\, p_3\, p_m$, $4 \leqslant m \leqslant n-1$; are g-circles for the n-set. If the circle through $p_4, p_5, p_6, \ldots, p_n$ passes also through one of the points p_1, p_2 or p_3, say p_1, then the circles $p_n\, p_2\, p_m$, $p_n\, p_3\, p_m$, $4 \leqslant m \leqslant n-1$; are g-circles for the n-set.

Finally, suppose that the points $p_4, p_5, p_6, \ldots, p_n$ are not concyclic. Consider the sets of non-concyclic points :

(1) $p_4, p_5, p_6, p_7, \ldots, p_n$;
(2) $p_1, p_4, p_5, p_6, \ldots, p_n$;
(3) $p_2, p_4, p_5, p_6, \ldots, p_n$;
(4) $p_3, p_4, p_5, p_6, \ldots, p_n$;
(5) $p_1, p_2, p_4, p_5, \ldots, p_{n-1}$;
(6) $p_2, p_3, p_4, p_5, \ldots, p_{n-1}$;
(7) $p_3, p_1, p_4, p_5, \ldots, p_{n-1}$;

Each of these sets has at least one g-circle pertaining to it. If the g-circle pertaining to set (1) is also a g-circle for each of the sets (2), (3), (4) then there is nothing left to prove. If a g-circle pertaining to set (1) is not a g-circle for one or more of the other sets then the g-circles pertaining to sets (2) to (7) provide at least one g-circle for the n-set. This follows from the fact that no circle can cut the circle through p_1, p_2, p_3 and p_n in more than two of these four points. This proves the theorem.

3. If $g(n)$ denotes the least number of g-circles pertaining to a set of n non-concyclic points, whatever their configuration, then in all probability $g(n) \geqslant 4$. I am, however, not yet able to prove it.

References.

Dirac, G. A. (1951). Collinearity Properties of Sets of Points. *Quar. J. of Maths.*, Oxford (2), 2, 221–7.

In view of a result in the paper by Dirac, the words 'in a plane' in Theorem A can be omitted.

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