ON THE PROBLEM OF TRANSFER OF RADIATION IN A SPHERICALLY SYMMETRICAL STELLAR ATMOSPHERE FOR THE NON-CONSERVATIVE CASE

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(Communicated by N. R. Sen, F.N.I.)

(*Received April 27 ; read October 9, 1953*)

1. The problem of transfer of radiation in the extended atmospheres of stars like supergiants in which the curvatures of the layers of the atmosphere have significant effect on the transfer problem has been considered by Kosirev (1934) and Chandrasekhar (1934). In recent years Chandrasekhar has developed a method of approximating the solution of the integro-differential equation of the transfer problem by a finite system of linear equations in which the Gaussian weights in the theory of evaluation of integrals play a prominent rôle. Chandrasekhar (1945) has given a first approximation solution of the problem of conservative isotropic scattering in an extended atmosphere by his method. In the present paper we have considered the solution of the problem for the non-conservative isotropic cases for an extended atmosphere with spherical symmetry by Chandrasekhar's method. The approximation is of the first order and two different sets of boundary conditions have been considered. In the first problem, the external boundary condition has been taken to be one of zero intensity both of incoming and outgoing radiation at the surface, while the internal boundary is a photospheric surface with a given outward flux of radiation (in comparison with which inward flux there, is considered negligible). In a second problem, the incoming radiation at the external boundary is supposed to be weak but not zero, so that the star is supposed to be embedded in a weak radiation bath with symmetry round the centre, while the inner boundary condition is the same as before. The external boundary in the second problem is artificially fixed by the condition that here the inward and outward intensities are equal. In the second case the external flux of radiation as a fraction of the flux at the internal boundary has been calculated and plotted against optical depth. It shows that when scattering is large, compared to absorption, large value of the ratio is attained by $\tau_1 = 2$ (and well below $\tau_1 = 2$ when scattering is very large) and for very large absorption a curve of entirely different character is obtained.

2. The equation of transfer appropriate to the problem of spherically symmetric atmosphere is given by

$$\mu \frac{\partial I(r, \mu)}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial I(r, \mu)}{\partial \mu} = -k\rho[I(r, \mu) - \mathcal{J}(r, \mu)] \quad \ldots \quad \ldots \quad (1)$$

where $\mathcal{J}(r, \mu)$ is the source-function and in the non-conservative isotropic case, it is given by

$$\mathcal{J}(r, \mu) = \tfrac{1}{2}\, \omega_0 \int_{-1}^{+1} I(r, \mu')\, d\mu' \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where $\mu = \cos \vartheta$, ϑ being the inclination to the radius vector, and ω_0, the albedo for single scattering, represents the fraction of light lost from the pencil by way of scattering. Hence $(1-\omega_0)$ represents the remaining portion, which has been transformed into other forms of energy or of radiations of other wavelengths. Now replacing the integrals by Gaussian sums by the method suggested by Chandrasekhar (Radiative Transfer, p. 364), the equation of transfer (1) is reduced to an equivalent system of linear equations in a finite approximation, which can be combined into the equation given below.

$$\frac{d}{dr}\Sigma a_i\mu_i P_l'(\mu_i)I_i+\frac{l(l+1)}{r}\Sigma a_i P_l(\mu_i)I_i = -k\rho\Sigma a_i P_l'(\mu_i)I_i+\tfrac{1}{2}k\rho\omega_0\Sigma a_j I_j \Sigma a_i P_l'(\mu_i) \quad \ldots \quad (3)$$

where $(i = \pm 1, \pm 2, \ldots\ldots\ldots\ldots \pm n)$ and $(l = 1, 2, \ldots\ldots\ldots\ldots 2n)$.

Using known expressions for Legendre's polynomials, equation (3) for $l = 1$ and $l = 2$ becomes respectively

$$\frac{d}{dr}\Sigma a_i\mu_i I_i+\frac{2}{r}\Sigma a_i\mu_i I_i = -k\rho(1-\omega_0)\Sigma a_i I_i \quad \ldots \quad \ldots \quad (4)$$

and

$$\frac{d}{dr}\Sigma a_i\mu_i^2 I_i+\frac{1}{r}\Sigma a_i(3\mu_i^2-1)I_i = -k\rho\Sigma a_i\mu_i I_i \quad \ldots \quad \ldots \quad (5)$$

Now putting $\quad \tfrac{1}{2}\Sigma a_i I_i = J,\ 2\Sigma a_i\mu_i I_i = F,$ and $\tfrac{1}{2}\Sigma a_i\mu_i^2 I_i = K \quad \ldots \quad \ldots \quad (6)$

and remembering that in the first approximation the radiation field is divided into an outward stream I_{+1} and an inward stream I_{-1} and that $a_{+1} = a_{-1} = 1$,

$\mu_{+1} = -\mu_{-1} = \dfrac{1}{\sqrt{3}}$ and μ_i's are the zeros of $P_2(\mu_i)$ $\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$

$$\text{and } \therefore\ \Sigma a_i P_2(\mu_i)I_i = 0 \text{ or } 3K-J = 0 \text{ or } J = 3K \quad \ldots \quad \ldots \quad (8)$$

we see that the equations (4) and (5) take the forms

$$\frac{dF}{dr}+\frac{2F}{r} = -12k\rho(1-\omega_0)K \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

and

$$4\frac{dK}{dr} = -k\rho F \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

It is well known that in the astrophysical contexts $k\rho$ varies as an inverse power of r (greater than unity)

$$\therefore\ k\rho = Cr^{-n}, \text{ where } C \text{ is a constant} \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

The optical thickness τ measured from $r = \infty$ inwards has the value

$$\tau = \int_r^\infty k\rho dr = \frac{Cr^{-n+1}}{n-1} \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

From (11) and (12)

$$k\rho r = (n-1)\tau \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

$$\text{and } d\tau = -k\rho\, dr \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

From (10)

$$4\frac{dK}{d\tau} = F \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

Eliminating F from equation (9)

$$\frac{d^2K}{d\tau^2}-\frac{2}{(n-1)\tau}\cdot\frac{dK}{d\tau}-3(1-\omega_0)K=0 \quad \ldots \quad \ldots \quad (16)$$

Putting $K=z^{\nu}f(z)$, where $z=q\tau$, $q=\sqrt{3(1-\omega_0)}$ and $\nu=\dfrac{n+1}{2(n-1)}$ .. (17)

and substituting in equation (16), we get

$$z^2\frac{d^2f}{dz^2}+z\frac{df}{dz}-(z^2+\nu^2)f=0 \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

This is Bessel's equation for imaginary arguments and its solutions are known. The fundamental solutions are $I_\nu(z)$ and $K_\nu(z)$. The general solution is

$$f(z)=C_1I_\nu(z)+C_2K_\nu(z) \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

3. If the atmosphere is supposed to extend to $r=\infty$, C_1 and C_2 can be evaluated under the following boundary conditions:

(i) Vanishing of both outward and inward intensities and hence of both flux and K-integral at the outer boundary of the atmosphere denoted by $\tau=0$, $z=0$;

(ii) the existence of a definite outward flux at the lower bound of the photosphere, denoted by $\tau=\tau_1$, $z=z_1$.

That is, from the inner bound is emerging a given flux and overlying this boundary, there exists an envelope scattering according to a constant albedo for single scattering ($\omega_0=$ Constant), whose value is less than unity. The atmosphere is supposed to be so extensive that the flux and the K-integral simultaneously vanish at the outer boundary of the atmosphere.

In order that K-integral may vanish at $\tau=0$ or $z=0$, $z^\nu f(z)\to 0$ as $z\to 0$ by (17). But it is found that while $z^\nu I_\nu(z)\to 0$ for $z\to 0$, $z^\nu K_\nu(z)$ tends to a limiting value (not equal to zero) as $z\to 0$. Hence for K to be zero at $z=0$, we put $C_2=0$ and

$$f(z)=C_1\,I_\nu(z) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

At $\tau=\tau_1$ or $z=z_1$, the intensity of the inward radiation is negligible in comparison with the intensity of the outward radiation. Let the outward flux at this surface be $F(z_1)$.

The flux and the K-integral at the inner boundary are given in the first approximation by the well-known relations

$$F(z)=\frac{2}{\sqrt{3}}I_{+1}(z) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

and

$$K(z_1)=\frac{I_{+1}(z_1)}{6}=\frac{1}{4\sqrt{3}}F(z_1) \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

[The present $I_{+1}(z_1)$ and $K(z_1)$ are not to be confused with $I_\nu(z)$ and $K_\nu(z)$, the solutions of Bessel Equation in (19)]

But from (17),

$$f(z_1)=\frac{K(z_1)}{z_1^\nu}=\frac{F(z_1)}{4\sqrt{3}\,z_1^\nu} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

$$\therefore\quad C_1=\frac{f(z_1)}{I_\nu(z_1)}=\frac{F(z_1)}{4\sqrt{3}\,I_\nu(z_1)\,z_1^\nu} \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

$$\therefore \; f(z) = \frac{F(z_1)}{4\sqrt{3}\, I_\nu(z_1)\, z_1^\nu} I_\nu(z) \qquad \ldots \quad \ldots \quad \ldots \quad (25)$$

Hence at any depth z,

$$K(z) = z^\nu f(z) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

$$J(z) = \tfrac{1}{3} K(z)$$

The source function

$$\mathcal{J}(z) = \frac{\omega_0}{3} z^\nu f(z) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

$$F(z) = 4q \frac{dK(z)}{dz} = q \frac{F(z_1)}{\sqrt{3}\, z_1^\nu I_\nu(z_1)} z^\nu I_{\nu-1}(z) \qquad \ldots \quad \ldots \quad (28)$$

Thus for given values of q, z_1 and $F(z_1)$ the K-integral, the flux, the energy density and the source-function can be evaluated.

4. A special case may now be considered, where the outer boundary is taken to be the surface, which looks equally bright from both sides. On this surface the net flux is zero, not because the inward and outward intensities simultaneously vanish, but because the inward and outward intensities are equal. The atmosphere under this circumstance is finite but yet sufficiently large to validate the approximation

$$\tau = \int_r^R k\rho \, dr \approx \frac{Cr^{-n+1}}{n-1}$$

($r = R$ denoting the outer boundary is extremely large) and hence

$$k\rho r = (n-1)\tau \text{ and } d\tau = -k\rho \, dr$$

To realise the above conditions for this configuration the measurement of optical depth is made from a surface somewhat lower than the boundary fixed in the previous case, where the above condition is supposed to obtain. Or, as an alternative, we can consider the star to be in a weak radiation bath. The boundary is defined as the surface on which outward flow of radiation is balanced by the inward stream due to the surrounding field. As the atmosphere is very extensive, both the intensities will be very feeble. So the term containing reduced incident radiation in the transfer equation can be ignored and the original equation of transfer adapted to the present case.

Under the circumstances mentioned above, the constants in the general equation (19) may be evaluated under the following boundary conditions :—

(i) at the outer boundary denoted by $\tau = 0$, $z = 0$, $K = K(0) =$ Constant and $F = F(0) = 0$.

(ii) at the inner boundary denoted by $\tau = \tau_1$, $z = z_1$, flux $F(z) = F(z_1)$. This flux is always outwards as the inward flux at the lower bound of the photosphere may be neglected in comparison with the outward one.

From (17) and (19)

$$K(z) = z^\nu f(z) = C_1 z^\nu I_\nu(z) + C_2 z^\nu K_\nu(z) \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

Applying the first boundary condition, the K-integral at $\tau = 0$ or $z = 0$ is given by

$$K(0) = C_2 \times \lim_{z \to 0} \left(z^\nu K_\nu(z)\right) \qquad \ldots \quad \ldots \quad \ldots \quad (31)$$

If n is taken to be equal to 2,

$$\nu = \frac{n+1}{2(n-1)} = \tfrac{3}{2} \quad \ldots \quad (32)$$

$$\lim_{z \to 0} \; z^{\frac{3}{2}} K_{\frac{3}{2}}(z) = \sqrt{\frac{\pi}{2}} \quad \ldots \quad (33)$$

$$C_2 = K(0) \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \quad \ldots \quad (34)$$

Using the second boundary condition, C_1 can be evaluated. If at the lower boundary given by $\tau = \tau_1$ or $z = z_1$, $K(z) = K(z_1)$

$$C_1 = \frac{K(z_1)}{z_1^\nu I_\nu(z_1)} - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} K(0) \frac{K_\nu(z_1)}{I_\nu(z_1)} \quad \ldots \quad (35)$$

Hence

$$f(z) = \left\{ \frac{K(z_1)}{z_1^\nu I_\nu(z_1)} - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} K(0) \frac{K_\nu(z_1)}{I_\nu(z_1)} \right\} I_\nu(z) + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} K(0)\, K_\nu(z) \quad \ldots \quad (36)$$

The value of K-integral at any depth is given by

$$\frac{K(z)}{K(0)} = \left\{ \frac{K(z_1)}{K(0)} \cdot \frac{1}{z_1^\nu I_\nu(z_1)} - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{K_\nu(z_1)}{I_\nu(z_1)} \right\} z^\nu I_\nu(z) + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} K_\nu(z) z^\nu \quad \ldots \quad (37)$$

$$\therefore \quad \frac{K(z)}{K(0)} = A z^\nu I_\nu(z) + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} K_\nu(z) z^\nu \quad \ldots \quad (38)$$

where

$$A = \frac{K(z_1)}{K(0)} \cdot \frac{1}{z_1^\nu I_\nu(z_1)} \cdot - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cdot \frac{K_\nu(z_1)}{I_\nu(z_1)} \quad \ldots \quad (39)$$

and

$$J(z) = \tfrac{1}{3} K(z), \quad \mathcal{J}(z) = \frac{\omega_0}{3} K(z) \quad \ldots \quad (40)$$

And $F(z)$ at any depth is given by

$$\frac{F(z)}{K(0)} = 4q \frac{d}{dz} \left(\frac{K(z)}{K(0)} \right) = 4q \left[A z^\nu I_{\nu-1}(z) - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} z^\nu K_{\nu-1}(z) \right] \quad \ldots \quad (41)$$

Thus knowing the values of ω_0, z_1, and $\frac{K(z_1)}{K(0)}$, we can evaluate $\frac{K(z)}{K(0)}$, $\frac{F(z)}{K(0)}$ $\frac{J(z)}{K(0)}$ and $\frac{\mathcal{J}(z)}{K(0)}$ in the present case.

5. The boundary condition at the outer boundary allows us to divide the total flux there into two equal fluxes $F_{+1}(0)$ and $F_{-1}(0)$ meaning the outward and inward fluxes respectively at $\tau = 0$ and $z = 0$. Now, expressing the K-integrals and fluxes at the inner and outer boundary in terms of inward and outward intensities, it is possible for us to find out an expression for the ratio of the outward fluxes at the inner and outer surface in terms of q and z_1. Fixing the value of q, it is possible to draw up a table showing the relation between $\frac{F(z_1)}{F_{+1}(0)}$ and z_1 and plot $\frac{F(z_1)}{F_{+1}(0)}$ against z_1 in a curve.

We know that

$$K(z_1) = \tfrac{1}{2} \Sigma a_i \mu_i^2 I_i = \tfrac{1}{6} \left(I_{+1}(z_1) + I_{-1}(z_1) \right) \approx \tfrac{1}{6} I_{+1}(z_1) \quad \ldots \quad (42)$$

$$F(z_1) = 2\, \Sigma a_i \mu_i I_i = \frac{2}{\sqrt{3}} \left(I_{+1}(z_1) - I_{-1}(z_1) \right) \approx \frac{2}{\sqrt{3}} I_{+1}(z_1) \quad \ldots \quad (43)$$

as at the inner boundary inward intensity is negligible in comparison with the outward.

$$\therefore K(z_1) = \frac{\sqrt{3}}{12} F(z_1) \quad \ldots \quad (44)$$

At the outer boundary total flux is equal to zero, and the K-integral is $K(0)$.

$$\left.\begin{aligned} \therefore F(0) &= \frac{2}{\sqrt{3}}\left(I_{+1}(0) - I_{-1}(0)\right) = 0 \\ \therefore I_{+1}(0) &= I_{-1}(0) \end{aligned}\right\} \quad \ldots \quad (45)$$

$$K(0) = \tfrac{1}{6}\left(I_{+1}(0) + I_{-1}(0)\right) = \tfrac{1}{3} I_{+1}(0) \quad \ldots \quad (46)$$

$$\text{and } F_{+1}(0) = \frac{2}{\sqrt{3}} I_{+1}(0),\ K(0) = \frac{F_{+1}(0)}{2\sqrt{3}} \quad \ldots \quad (47)$$

$$\therefore \frac{K(z_1)}{K(0)} = \tfrac{1}{2}\frac{F(z_1)}{F_{+1}(0)},\ \text{and} \frac{F(z_1)}{K(0)} = 2\sqrt{3}\,\frac{F(z_1)}{F_{+1}(0)} \quad \ldots \quad (48)$$

Now putting $z = z_1$ in equation (41)

$$\frac{F(z_1)}{F_{+1}(0)} = \frac{4qz_1^{\nu-1}}{(2\pi)^{\frac{1}{2}}[qI_{\nu-1}(z_1) - \sqrt{3}\,I_\nu(z_1)]} \quad \ldots \quad (49)$$

The values of this ratio for different values of τ_1 and hence z_1 are shown in the following table for different values of ω_0. Three cases have been considered, namely $\omega_0 = 90\%$, $\omega_0 = 67\%$, $\omega_0 = 10\%$

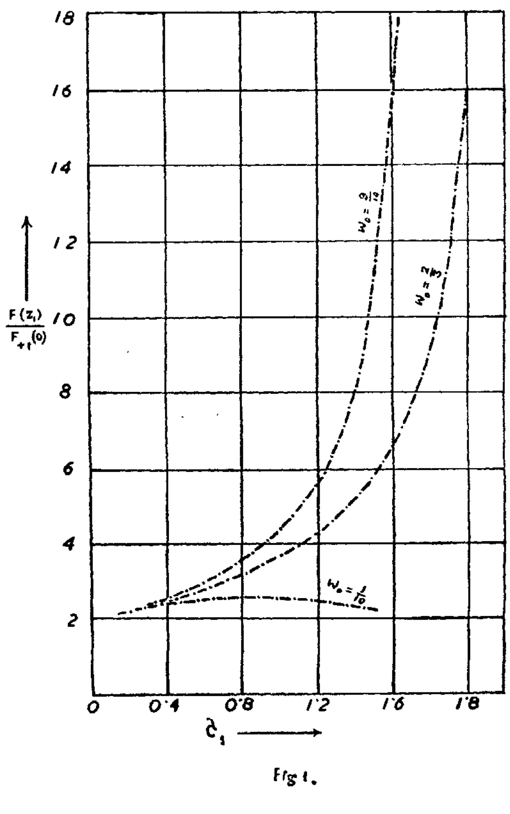

Fig 1.

TABLE

z_1	$\omega_0 = 90\%$		$\omega_0 = 67\%$		$\omega_0 = 10\%$	
	τ_1	$\frac{F(z_1)}{F_{+1}(0)}$	τ_1	$\frac{F(z_1)}{F_{+1}(0)}$	τ_1	$\frac{F(z_1)}{F_{+1}(0)}$
0·2	0·36	2·49	0·2	2·25	0·12	2·13
0·3	0·55	2·86	0·3	2·38	0·18	2·20
0·4	0·73	3·32	0·4	2·52	0·24	2·26
0·5	0·91	4·00	0·5	2·68	0·30	2·32
0·6	1·10	4·94	0·6	2·85	0·36	2·38
0·7	1·28	6·44	0·7	3·02	0·43	2·42
0·8	1·46	9·42	0·8	3·24	0·49	2·47
0·9	1·64	17·61	0·9	3·46	0·55	2·51
1·0	1·83	156·5	1·0	3·71	0·61	2·54
1·1	....	....	1·1	3·98	0·67	2·56
1·2	....	....	1·2	4·33	0·73	2·59
1·3	....	....	1·3	4·74	0·79	2·60
1·4	....	....	1·4	5·23	0·85	2·62
1·5	....	....	1·5	5·86	0·91	2·62
1·6	....	....	1·6	6·62	0·97	2·62
1·7	....	....	1·7	7·62	1·03	2·60
1·8	....	....	1·8	9·26	1·10	2·59
1·9	....	....	1·9	11·66	1·16	2·57
2·0	....	....	2·0	15·81	1·22	2·54
2·14	....	....	....	....	1·30	2·49
2·30	....	....	....	....	1·40	2·43
2·46	....	....	....	....	1·50	2·36

The variation of the ratio of the outward fluxes of radiation at the inner and outer boundaries of an atmosphere with a surface slowly merging into an external radiation field, with optical depth has been plotted in Fig. 1. The atmosphere considered is an extensive one. So that the flux ratios obtained for small z_1 are unreliable. For $\omega_0 = 90\%$, beyond $z_1 = 1 \cdot 1$, the value of the flux ratio increases very fast (tending to infinity). The curves for $\omega_0 = 90\%$ and $\omega_0 = 67\%$ show remarkable similarity of behaviour. But for $\omega_0 = 10\%$, i.e. for small scattering, the curve behaves in a different manner. According to the present theory, the curve in this case is very flat and gradually approaches a limiting value, a large value of flux ratio never being attained in this case.

It is a pleasure to acknowledge my indebtedness to Prof. N. R. Sen for his constant advice and helpful discussions, during the preparation of this work.

Abstract

The equation of transfer of radiation in spherically symmetric atmosphere for non-conservative isotropic scattering has been solved by the method of Chandrasekhar in which integrals are replaced by corresponding Gaussian sums, and first approximation results have been fitted to two boundary conditions, one of no incident radiation and the other of a very weak radiation field penetrating from outside. In the second case the variation of outward flux with optical depth for very large, moderate and weak scattering has been worked out and the results shown in a diagram.

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Issued January 29, 1954.