ON LATTICE COVERINGS BY SPHERES

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1. The object of this paper is to obtain the most economical lattice coverings of the three-dimensional Euclidean space by equal spheres.

Let K be a body in n dimensions and Λ a lattice such that every point of the n-dimensional space lies in one at least of the bodies obtained from K by applying to it all possible lattice translations. Then we say that Λ is a *covering lattice* for K. If K is bounded, the ratio of the total volume of the bodies K to whole space (in the obvious* sense) is $V(K)/d(\Lambda)$, where $V(K)$ is the volume of K and $d(\Lambda)$ the determinant of Λ. This ratio $V(K)/d(\Lambda)$ is called the density of the lattice covering by K provided by Λ. The lower bound $\theta(K)$ of $V(K)/d(\Lambda)$ taken over all covering lattices Λ for K is called the *density of the most economical coverings by K* and the coverings with this density are called the most economical lattice coverings. Their existence for convex bodies was proved by Hlawka.†

The problem of most economical or thinnest lattice coverings is connected with an interesting type of Diophantine inequalities, an example of which is provided by Theorem 2.

The analogous problem of closed lattice packings has been studied for a long time mainly because of its connection with another type of Diophantine inequalities. But the problem of thinnest coverings has only recently started attracting the attention it deserves and within the last few years a number of interesting results have been obtained. However, the value of $\theta(K)$ is known only for convex bodies in two dimensions, where the problem is equivalent to finding the largest symmetrical hexagon contained in K.‡ The results of this paper seem to be the first § exact results in dimensions higher than the second.

For spheres in n-dimensions Davenport ‖ has proved that for large n,

$$\theta(K) < (1{\cdot}15)^n,$$

and Bambah and Davenport ¶ have shown that

$$\theta(K) > 4/3 - \epsilon_n, \quad \epsilon_n \to 0 \text{ as } n \to \infty.$$

Our results can be stated as

Theorem 1:—*Let K be a sphere in three-dimensional Euclidean space and $\theta(K)$, the density of the most economical lattice covering by K.*

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* Let X be a cube of side L and N, the number of bodies K having a point in common with X. Then the ratio is defined as $\lim_{L\to\infty} NV(K)/L^n$, where $V(K)$ is the volume of K.

† *Monat. Math.*, **53** (1949), 81–131.

‡ See Fáry, I., *Bull. de la Societe Math. de France*, **78** (1950), 152–161.

§ Except, of course, for a number of space filling polyhedra.

‖ *Rend. di Palermo*, Ser. II, vol. **1** (1952), 1–16.

¶ *Jour. Lond. Math. Soc.*, **27** (1952), 224–229.

Then

$$\theta(K) = J_3 \frac{5\sqrt{5}}{32}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where $J_3 = \frac{4}{3}\pi$ *is the volume of a unit sphere.*

Further, a lattice Λ *provides the most economical lattice covering* * *by* K *if and only if by a suitable choice of orthogonal axes,* Λ *is generated by the points*

$$\frac{2r}{\sqrt{5}}(-1, 1, 1), \frac{2r}{\sqrt{5}}(1, -1, 1), \frac{2r}{\sqrt{5}}(1, 1, -1),$$

and K *has the equation*

$$x^2+y^2+z^2 = r^2.$$

It can be easily shown † that Theorem 1 is equivalent to

THEOREM 2:—*Let* $f(x, y, z) = ax^2+by^2+cz^2+2ryz+2szx+2txy$ *be a positive definite quadratic form with real coefficients and determinant* $D>0$. *Then there exist real numbers* x_0, y_0, z_0, *such that for all integers* x, y, z, *we have*

$$f(x+x_0, y+y_0, z+z_0) \geqq \left(\frac{125}{1024} D\right)^{\frac{1}{3}}, \qquad \ldots \quad \ldots \quad \ldots \quad (2)$$

he sign of equality being necessary if and only if

$$f \sim \rho(3x^2+3y^2+3z^2-2yz-2zx-2xy), \qquad \ldots \quad \ldots \quad \ldots \quad (3)$$

and $\rho > 0$ *is any real number.*‡

It will follow from our proof that

THEOREM 3:—*Let* Π *be the largest convex space-filling* § *polyhedron that is contained in* K. *Then* Π *has seven pairs of opposite faces and all the vertices of* Π *lie on the surface of* K.

I am grateful to Professor H. Davenport for a very useful correspondence. I am also indebted to Prof. H. S. M. Coxeter for useful comments.

2. Let $f(x, y, z) = ax^2+by^2+cz^2+2ryz+2szx+2txy$ be a positive definite form. We shall denote its determinant by D_f or by D, when there is no danger of confusion. Let (ξ, η, ζ) be any set of real numbers. Denote by $m_f(\xi, \eta, \zeta)$ the minimum of $f(\xi-x, \eta-y, \zeta-z)$ for integers x, y, z. Let m_f be the upper bound of $m_f(\xi, \eta, \zeta)$ for all sets (ξ, η, ζ). Then Theorem 2 is equivalent to saying that

$$m = \underline{bd}\, M_f = \underline{bd}\left(\frac{m_f}{D_f^{\frac{1}{3}}}\right) = \left(\frac{125}{1024}\right)^{\frac{1}{3}} \qquad \ldots \quad \ldots \quad \ldots \quad (4)$$

the lower bound being taken over all forms f, and that m is attained by those forms and those alone which are equivalent to $\rho\,(3\Sigma x^2-2\Sigma yz)$.

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* Prof. Coxeter points out that this lattice is the '*body centred*' *cubic lattice* of crystallographers and that its Dirichlet region is the *truncated octahedron* (so named by Kepler in 1619).

† See, e.g., Davenport, loc. cit.

‡ It is interesting to remark that the critical forms here are reciprocal to the form $\mu(\Sigma x^2+\Sigma yz)$, which are critical in the packing case.

§ I.e., $\theta(\Pi) = 1$.

Outline of Proof

3. We first observe that the expression $M_f = \frac{m_f}{D_f^{\frac{1}{3}}}$ is invariant under integral uni-modular linear substitutions on the variables x, y, z. Consequently we can suppose, without loss of generality, that f is reduced in the sense of Seeber *, i.e. the coefficients a, b, c, r, s, t satisfy the relations

$$0 < a \leqq b \leqq c, \qquad \ldots \qquad (6)$$

$$2|s|,\ 2|t| \leqq a, \quad 2|r| \leqq b \qquad \ldots \qquad (7)$$

$$r, s, t \text{ are all negative or all non-negative}, \qquad \ldots \qquad (8)$$

and

$$a+b+2r+2s+2t \geqq 0. \qquad \ldots \qquad (9)$$

We denote by Λ_0 the fundamental lattice of points with integer co-ordinates.

We shall also denote by Π_f the set of all points $X = (x, y, z)$ for which $f(X) \leqq f(X-A)$, for all points A of Λ_0. Then it is clear that the bodies obtained by giving all possible lattice translations to Π_f together fill the whole space without gaps or overlappings. Also Π_f being the intersection (common part) of all half spaces $f(X) \leqq f(X-A)$ is convex. Consequently, by a well-known theorem of Minkowski, Π_f is a symmetrical convex polyhedron with at most seven pairs of opposite faces. By the definition of Π_f it is clear that

$$m_f = \max_{X \epsilon \Pi_f} f(X) = \max_{X \text{ a vertex of } \Pi_f} f(X), \qquad \ldots \qquad (10)$$

since the ellipsoid $f(x, y, z) \leqq m_f$ is convex and Π_f is the convex cover of its vertices.

As we know that every convex body has at least one lattice which provides a covering with the least density, it follows that $\underline{bd}\ m_f/D_f^{\frac{1}{3}}$ is actually attained for at least one form $\Sigma(ax^2+2ryz)$.

Our proof will run as follows:

We shall first construct the polyhedron Π_f for the form $3\Sigma x^2 - 2\Sigma yz$ and show that for all vertices V of Π_f, $f(V) \equiv \frac{5}{4}$ from which it easily follows that for

$$f = 3\Sigma x^2 - 2\Sigma yz,\ M_f = m_f/D_f^{\frac{1}{3}} = \left(\frac{125}{1024}\right).$$

We then divide the reduced positive definite quadratic forms into a number of types. For each f in every type we show that either at a point P of Π_f, $f(P)/D^{\frac{1}{3}}$ is greater than $(125/1024)^{\frac{1}{3}}$ or we show that by suitable variations in the coefficients a, b, c, r, s, t, we get a neighbouring form f' of f with $m_{f'}/D_{f'}^{\frac{1}{3}}$ smaller than $m_f/D_f^{\frac{1}{3}}$ unless $f = 3\Sigma x^2 - 2\Sigma yz$. In the first case we often proceed as follows: We first establish that for forms of a certain set Σ, that contains f, and for P, a point of Π_f defined in a certain way in Σ, the minimum of $f(P)/D_f^{\frac{1}{3}}$ for all $f \epsilon \Sigma$ is actually attained at some form of Σ. Then for any f in Σ we show that for a suitable neighbour-

* See, e.g., Dickson, L. E., Studies in the Theory of Numbers, Chicago (1930), pp. 163–165, noting that if $rst = 0$, we can take r, s, t non-negative by a transformation $x = \Sigma_1 x$, $y = \Sigma_2 y$, $z = \Sigma_3 z$, $\Sigma_i = \pm 1$.

ing form $f'\epsilon\Sigma$, $f'(P)/D_{f'}^{\frac{1}{3}}$ is smaller than $f(P)/D_f^{\frac{1}{3}}$ unless f satisfies certain conditions. After obtaining a sufficient number of such conditions we show that for f satisfying these conditions $f(P)/D_f^{\frac{1}{3}}$ is greater than $(125/1024)^{\frac{1}{3}}$. In short the whole method can be described as that of variation of parameters a, b, c, r, s, t.

Proof of the Theorem

4. As explained earlier, the polyhedra Π_f play an important rôle in our investigation. It will be convenient to state certain properties of Π_f which are of use in their construction.

Some Properties of Π_f: The equation of any face of Π_f is of the type $f(X) = f(X-A)$, where A is a suitable point of Λ_0, the fundamental lattice.* If $f(X) = f(X-A)$ is a face of Π_f, then so also is $f(X) = f(X+A)$. Further, $\frac{1}{2}A$ is an inner point of the face $f(X) = f(X-A)$. If we denote the infinite strip bounded by planes $f(X) = f(X-A)$ and $f(X) = f(X+A)$ by S_A, then Π_f is the intersection of a finite number of these strips S_A where A are points of Λ_0. Also if A and B are any two points of Λ_0 and $\frac{1}{2}B$ is not an inner point of S_A, then S_B does not play any part in the construction of Π_f, i.e. Π_f can be regarded as the intersection of a number of S_A, all different from S_B.

5. We shall first dispose of the fact that for $f = 3\Sigma x^2 - 2\Sigma yz$, M_f has the value $(125/1024)^{\frac{1}{3}}$.

The polyhedron Π_f can be easily seen to be the intersection of strips S_A for different A in the following set †:

$$(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1,),\ (0, 1, 1),\ (1, 0, 1),\ (1, 1, 0),\ (1, 1, 1).$$

The fourteen faces of Π_f have the following equations †:

$$\begin{aligned} 6x-2y-2z &= \pm 3 \\ -2x+6y-2z &= \pm 3 \\ -2x-2y+6z &= \pm 3 \\ -4x+4y+4z &= \pm 4 \\ 4x-4y+4z &= \pm 4 \\ 4x+4y-4z &= \pm 4 \\ 2x+2y+2z &= \pm 3. \end{aligned}$$

The twenty-four vertices † of Π_f are the following together with their images in the origin:

$(\frac{1}{2}, -\frac{1}{4}, \frac{1}{4})$, $(\frac{1}{4}, \frac{1}{2}, -\frac{1}{4})$, $(-\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{4}, -\frac{1}{4})$, $(-\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$, $(\frac{1}{4}, -\frac{1}{4}, \frac{1}{2})$, $(\frac{3}{4}, \frac{1}{2}, \frac{1}{4})$, $(\frac{1}{4}, \frac{3}{4}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{4}, \frac{3}{4})$, $(\frac{3}{4}, \frac{1}{4}, \frac{1}{2})$, $(\frac{1}{2}, \frac{3}{4}, \frac{1}{4})$, $(\frac{1}{4}, \frac{1}{2}, \frac{3}{4})$.

It can be easily verified that the value of f at each of these points is 5/4. (In fact one has to verify it only for two points, one from the first six and one from the rest.) Since the determinant $D_f = 16$, it follows that

$$M_f = (125/1024)^{\frac{1}{3}}.$$

We now divide our proof into two main parts. In the first part we deal with form f with non-negative r, s, t, and in the second with forms of negative r, s, t.

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* The lattice of points with integer co-ordinates.

† For fuller explanation see § 13.

Part I

6. In this part we shall throughout suppose that the coefficients a, b, c, r, s, t of f satisfy the following conditions:

$$0 < a \leqq b \leqq c; \qquad (11)$$

$$0 \leqq 2s, \quad 2t \leqq a; \quad 0 \leqq 2r \leqq b. \qquad (12)$$

We shall very often denote f by the symbol (a, b, c, r, s, t).

In the following table we write down the equations of a number of faces $f(X) = f(X-A)$ of strips S_A. These S_A will be enough to define Π_f.

Table I

A	*Faces of* S_A	*Symbols for faces*
$\pm(1, 0, 0)$	$2ax+2ty+2sz = \pm a$	$\pm F_1$
$\pm(0, 1, 0)$	$2tx+2by+2rz = \pm b$	$\pm F_2$
$\pm(0, 0, 1)$	$2sx+2ry+2cz = \pm c$	$\pm F_3$
$\pm(1, -1, 0)$	$2(a-t)x-2(b-t)y-2(r-s)z = \pm(a+b-2t)$	$\pm F_4$
$\pm(0, 1, -1)$	$-2(s-t)x+2(b-r)y-2(c-r)z = \pm(b+c-2r)$	$\pm F_5$
$\pm(-1, 0, 1)$	$-2(a-s)x-2(t-r)y+2(c-s)z = \pm(c+a-2s)$	$\pm F_6$
$\pm(1, 1, -1)$	$2(a+t-s)x+2(b+t-r)y-2(c-r-s)z = \pm(a+b+c+2t-2r-2s)$	$\pm F_7$
$\pm(-1, 1, 1)$	$-2(a-s-t)x+2(b+r-t)y+2(c+r-s)z = \pm(a+b+c+2r-2s-2t)$	$\pm F_8$
$\pm(1, -1, 1)$	$2(a+s-t)x-2(b-t-r)y+2(c+s-r)z = \pm(a+b+c+2s-2t-2r)$	$\pm F_9$.

We shall in future refer to these faces by the symbols in the last column.

With the help of (11) and (12) it is easy to verify that for any B of Λ_0 which does not appear in the first column the point $\frac{1}{2}B$ is not an inner point of one at least of the strips S_A corresponding to the A of Table I, e.g. for integers $p, q, r > 0$, $\frac{1}{2}(-p, q, r)$ is not an inner point of S_A bounded by $\pm F_8$, etc. Consequently Π_f is the intersection of the nine strips S_A bounded by the planes $\pm F_i$, $i = 1, \ldots, 9$.

We shall now divide these forms f into three types, (i), (ii), and (iii) according to the relative magnitude of r, s, t which shall be studied separately.

(i) $0 \leqq t \leqq s, r$,

(ii) $0 \leqq s \leqq t, r$,

(iii) $0 \leqq r \leqq s, t$.

Forms of type (i)

7. Let P be the point of intersection of planes F_1, F_2 and F_3 and Q_1 that of F_1, F_3 and $-F_5$. Then it can be easily verified that both P and Q_1 lie in each of the nine S_A appearing in Table I. Consequently both P and Q_1 are points of Π_f and

$$m_f \geq \max.\,(f(P), f(Q_1)).$$

Now

$$f(x, y, z) = \frac{1}{c}(cz+sx+ry)^2+\frac{1}{c(ac-s^2)}\left\{c(ax+ty+sz)-s(cz+sx+ry)\right\}^2$$
$$+\frac{D^2y^2}{(ac-s^2)D}, \quad \ldots \quad (13)$$

where $D = D_f$ is the determinant of f.
Therefore,

$$4(ac-s^2)D\{f(Q_1)-f(P)\} = \begin{vmatrix} a & a & s \\ t & -b+2r & r \\ s & c & c \end{vmatrix}^2 - \begin{vmatrix} a & a & s \\ t & b & r \\ s & c & c \end{vmatrix}^2$$

$$= \begin{vmatrix} a & 0 & s \\ t & 2(-b+r) & r \\ s & 0 & c \end{vmatrix} \begin{vmatrix} a & 2a & s \\ t & 2r & r \\ s & 2c & c \end{vmatrix}$$

$$= 4(b-r)\ (a-s)\ (ac-s^2)\ (ct-rs)$$

and consequently,

$$\text{if } ct = rs, \text{ then } f(P) = f(Q_1). \quad \ldots \quad (14)$$

8. Define the number m_1 as

$$m_1 = \underline{bd}\, f(P)/D_f^{\frac{1}{3}} \quad \ldots \quad (15)$$

where the $\underline{bd}$ is taken over all forms (a, b, c, r, s, t) of type (i) with the additional property

$$ct = rs. \quad \ldots \quad (16)$$

Then we prove, for all f of type (i),

LEMMA 1. $$M_f = m_f/D_f^{\frac{1}{3}} \geq m_1. \quad \ldots \quad (17)$$

Proof: We distinguish three cases :—

(i) $ct > rs$, (ii) $ct = rs$, (iii) $ct < rs$.

(i) Suppose $ct > rs$. .. .. (18)

Then, by (13), we have

$$m_f \geq f(Q_1) = \frac{c}{4}+\frac{c(a-s)^2}{4(ac-s^2)}+\frac{\Phi_1^2}{4(ac-s^2)D_f}, \quad \ldots \quad (19)$$

where

$$\Phi_1 = \begin{vmatrix} a & -a & s \\ t & b-2r & r \\ s & -c & c \end{vmatrix}$$

$$= abc-acr+act-ars-cst-bs^2+2rs^2 > 0, \quad \ldots \quad (20)$$

(since $bs^2+cst+arc \leq \frac{1}{4}abc+\frac{1}{4}abc+\frac{1}{2}abc = abc$ and $ct > rs$), and

$$D_f = abc+2rst-ar^2-bs^2-ct^2. \quad \ldots \quad (21)$$

Now
$$\frac{\partial \Phi_1}{\partial t} = c(a-s) > 0, \quad \ldots \quad (22)$$
and
$$\frac{\partial D_f}{\partial t} = 2(rs-ct) < 0. \quad \ldots \quad (23)$$

Therefore, by (19), (20), (22) and (23) we can decrease $f(Q_1)/D_f^{\frac{1}{3}}$ by decreasing t until $ct = rs$, so that for $f' = (a, b, c, r, s, t')$ with $ct' = rs$, we have, using (14),
$$M_f \geq f(Q_1)/D_f^{\frac{1}{3}} \geq f'(Q_1)/D_{f'}^{\frac{1}{3}} = f'(P)/D_{f'}^{\frac{1}{3}} \geq m_1 \quad \ldots \quad (24)$$

(ii) Obvious.

(iii) Suppose $ct < rs$. $\quad \ldots \quad (25)$

By (13), we have
$$m_f \geq f(P) = \frac{c}{4} + \frac{c(a-s)^2}{4(ac-s^2)} + \frac{\Psi_1^2}{4(ac-s^2)D_f}, \quad \ldots \quad (26)$$
where
$$\Psi_1 = \begin{vmatrix} a & a & s \\ t & b & r \\ s & c & c \end{vmatrix}$$
$$= abc - acr - act + ars + cst - bs^2$$
$$> 0 \quad \ldots \quad (27)$$
(since $abc - acr - act + ars + cst - bs^2$
$$= \tfrac{1}{2}c(b-2t)\,(a-2s) + \tfrac{1}{2}ca(b-2r) + \tfrac{1}{2}bs(c-2s) + ars$$
$$\geq ars > act \geq 0).$$

Now
$$\frac{\partial \psi_1}{\partial t} = -c(a-s) < 0. \quad \ldots \quad (28)$$
and
$$\frac{\partial D}{\partial t} = 2(rs-ct) > 0. \quad \ldots \quad (29)$$

Therefore, by (26)–(29) we can decrease $f(P)/D_f^{\frac{1}{3}}$ by increasing t until $ct = rs$, so that writing $f' = (a, b, c, r, s, t')$ where $ct' = rs$ we get
$$M_f \geq f(P)/D_f^{\frac{1}{3}} \geq f'(P)/D_{f'}^{\frac{1}{3}} \geq m_1. \quad \ldots \quad (30)$$

9. After Lemma 1, in order to prove Theorem 2 for forms of type (i) it will suffice to show that
$$m_1 > \left(\frac{125}{1024}\right)^{\frac{1}{3}} \quad \ldots \quad (31)$$

By (26) and (27) it is easy to see that for forms f with $ct = rs$,

$$F_1(P) = 4D_f \cdot f(P) = abc(\Sigma a - 2\Sigma r) - \Sigma a^2r^2 + 2\Sigma bcst$$
$$= abc(a+b+c-2r-2s) - a^2r^2 - b^2s^2 - r^2s^2 + 2brs^2 + 2ar^2s, \quad .. \quad (32)$$

and

$$D_f = abc + 2rst - ar^2 - bs^2 - ct^2$$
$$= abc + \frac{s^2r^2}{c} - ar^2 - bs^2. \quad .. \quad .. \quad .. \quad .. \quad (33)$$

Further, using the expressions for $F_1(P)$ and D_f free of t, we obtain

$$m_1 = \underline{bd}\ F_1(P)/4D_f^{\frac{4}{3}}, \quad .. \quad .. \quad .. \quad .. \quad (34)$$

where the $\underline{bd}$ is taken over all sets of real numbers (a, b, c, r, s) satisfying *

$$0 \leq a \leq b \leq c\,;\ \ 0 \leq 2s \leq a\,;\ \ 0 \leq 2r \leq b\,. \quad .. \quad .. \quad (35)$$

If there is no danger of confusion we shall often use the symbol f for the set (a, b, c, r, s) in the following section.

Since $\quad f(P) > \frac{c}{4} \geq \frac{1}{4}(abc)^{\frac{1}{3}} \geq \frac{1}{4}D_f^{\frac{1}{3}}$, and $a \geq \frac{1}{bc}D_f \geq \frac{1}{16f^2(P)}D_f$,

it can be shown easily by using Mahler's theorem on bounded lattices (*P.R.S.*, 187, 151–187, Th. 2) that m_1 is attained for some sets (a, b, c, r, s) satisfying (35). We shall call such sets m_1-sets.

10. We now prove a number of lemmas leading to the proof of Theorem 2 for all forms of type (*i*).

Lemma 2: *For an m_1-set we must have $a = 2s$.*

Proof: If not, let f: (a, b, c, r, s) be an m_1-set with $a > 2s$.
Then for small $\delta s > 0$, $\delta c \geqq 0$, the neighbouring set $f' = f + \delta f$ satisfies (35). Let δs and δc satisfy

$$\left(ab - \frac{r^2s^2}{c^2}\right)\delta c - 2s\left(b - \frac{r^2}{c}\right)\delta s = 0\,. \quad .. \quad .. \quad .. \quad (36)$$

Then (32), (33) and (36) give

$$\delta D_f = 0,$$

and

$$\delta F_1(P) = \delta c(a^2b + ab^2 + 2abc - 2abr - 2abs) + \delta s(-2abc - 2b^2s - 2r^2s + ar^2 + 4brs)$$
$$= -\frac{2}{abc^2 - r^2s^2}\delta s\,\{-abc(bc-r^2)(a-2s)(c-s) - ar^2s(c-s)(bc-r^2)$$
$$- r^2s(b-r)^2(ac-s^2)\}$$
$$< 0,$$

so that f' has smaller $F_1(P)/4D_f^{\frac{4}{3}}$ than f and we get a contradiction that proves the lemma.

Lemma 3: *An m_1-set must have $b = 2r$.*

Proof: Suppose f is an m_1-set with $b > 2r$.

* Since $c \geq 2r$, $2s$, the conditions $t \leq r$, s; $0 \leq 2t \leq a$ are contained in $ct = rs$, $0 \leq 2s \leq a$.

Then for small $\delta c \geqq 0$, $\delta r > 0$, the neighbouring set $f' = f + \delta f$ satisfies (35). Let δc and δr be related by

$$\left(ab - \frac{r^2 s^2}{c^2}\right)\delta c - \frac{2r}{c}(ac - s^2)\delta r = 0\,.$$

Then

$$\delta D_f = 0,$$

and

$$\begin{aligned}\delta F_1(P) &= \delta c(a^2 b + ab^2 + 2abc - 2abr - abs) + \delta r(-2abc - 2a^2 r - 2rs^2 + 4ars + 2bs^2)\\ &= -\frac{2}{abc^2 - r^2 s^2}\,\delta r\,\{abc(ac-s^2)(c-r)(b-2r) + brs^2(ac-s^2)(c-r)\\ &\qquad + rs^2(bc-r^2)(a-s)^2\}\\ &< 0,\end{aligned}$$

and we easily obtain a contradiction that proves the lemma.

Lemma 4: $m_1 > \left(\frac{125}{1024}\right)^{\frac{1}{3}}$, *and Theorem 2 is true for all forms of type* (i).

Proof: From Lemmas 2, 3 for an m_1-form we have

$$F_1(P) = abc^2 - \frac{a^2 b^2}{16}\,,$$

and

$$\begin{aligned}D_f &= abc + \frac{a^2 b^2}{16c} - \frac{ab}{4}(a+b)\\ &\leqq abc + \frac{a^2 b^2}{16c} - \frac{1}{2}ab\sqrt{ab}\\ &= \frac{ab}{16c}\left(4c - \sqrt{ab}\right)^2.\end{aligned}$$

Therefore, writing $\xi = c/\sqrt{ab}$, we have

$$\frac{F_1(P)}{4D_f^{4/3}} = \left(\frac{1}{4}\right)^{\frac{1}{3}} \frac{\xi^{4/3}(4\xi+1)}{(4\xi-1)^{5/3}} = K(\xi) \text{ say,}$$

and

$$m_1 = \min_{\xi \geqq 1} K(\xi)\,.$$

Differentiating logarithmically

$$\frac{1}{K}\frac{dK}{d\xi} = \frac{4(8\xi^2 - 8\xi - 1)}{3(16\xi^2 - 1)} = 0 \text{ for } \xi = \frac{2+\sqrt{6}}{4} = \xi_0 > 1\,.$$

Since $\frac{dK}{d\xi} < 0$ for $1 \leqq \xi < \xi_0$ and $\frac{dK}{d\xi} > 0$ for $\xi > \xi_0$, it follows that

$$m_1 = K(\xi_0),$$

so that

$$m_1^3 = \frac{1}{1024}(2+\sqrt{6})^4(3+\sqrt{6})^3(1+\sqrt{6})^{-5} > \frac{125}{1024}$$

since

$$(2+\sqrt{6})^4(3+\sqrt{6})^3 = 31{,}716+12{,}948\sqrt{6},$$

and

$$125(\sqrt{6}+1)^5 = 30{,}125+12{,}625\sqrt{6}.$$

Forms of type (ii)

11. We now consider forms of type (ii). For these forms we have

$$0 \leq a \leq b \leq c; \quad 0 \leq 2s, \quad 2t \leq a; \quad 0 \leq 2r \leq b; \quad s \leq t, r. \qquad (37)$$

It can be verified easily that points P and Q_2 defined as the intersections of planes F_1, F_2, F_3 and F_2, F_3, F_4 respectively lie in each of the nine S_A in Table I and hence are points of Π_f.

By repeating word by word the discussion in §§ 7, 8 replacing (a, b, c), (r, s, t), (x, y, z), and Q_1 by (c, a, b), (t, r, s), (z, x, y) and Q_2 respectively wherever they occur it is possible to prove

LEMMA 5: *Let f: (a, b, c, r, s, t) be a form which satisfies conditions* (37). *Let f' be the form (a, b, c, r, s', t), where $bs' = rt$. Then*

$$m_f/D_f^{\frac{1}{3}} \geq f'(P)/D_{f'}^{\frac{1}{3}}.$$

We next prove

LEMMA 6: *Let f be a form (a, b, c, r, s, t), which satisfies* (37) *and for which $bs = rt$. Let f' be the form $(a, b, c, r', s', t') = \left(a, b, c, r, t, \frac{b}{c}s\right)$. Then*

$$f(P)/D_f^{\frac{1}{3}} \geq f'(P)/D_{f'}^{\frac{1}{3}} \geq m_1.$$

Proof: Since P is the intersection of planes F_1, F_2, and F_3 it is easy to verify (by permuting (a, b, c) to (c, a, b) in (26), (27) for example), that

$$f(P) = \frac{b}{4}+\frac{b(c-r)^2}{4(bc-r^2)}+\frac{\Psi^2}{4(bc-r^2)D_f}, \qquad (38)$$

where

$$\Psi = abc-bct-bcs+ctr+brs-ar^2. \qquad (39)$$

Also

$$f'(P) = \frac{b}{4}+\frac{b(c-r)^2}{4(bc-r^2)}+\frac{\Psi'^2}{4(bc-r^2)D_{f'}}, \qquad (40)$$

where

$$\begin{aligned}\Psi' &= abc-bct'-bcs'+ct'r'+br's'-ar'^2\\ &= abc-b^2s-bct+brs+brt-ar^2. \qquad (41)\end{aligned}$$

Since

$$\begin{aligned}D_{f'}-D_f &= abc+2r's't'-ar'^2-bs'^2-ct'^2-(abc+2rst-ar^2-bs^2-ct^2)\\ &= 2\frac{b}{c}rst-bt^2-\frac{b^2}{c}s^2-2rst+bs^2+ct^2\\ &= \frac{1}{c}(c-b)\ (ct^2+bs^2-2rst)\\ &\geq 0\end{aligned}$$

(Since $bs = rt$ and $rst \leq rt^2 < ct^2$), and

$$\Psi - \Psi' = b^2s - bcs + crt - brt$$

$$= (b-c)(bs-rt) = 0,$$

it follows from (38)—(41) that

$$f'(P)/D_{f'}^{\frac{1}{3}} \leq f(P)/D_f^{\frac{1}{3}}.$$

As $ct' = bs = rt = r's'$, the second inequality also follows.

Lemmas 4, 5 and 6 together give a proof of Theorem 2 for forms of type (ii) also.

Forms of type (iii)

12. We now consider forms of type (iii), namely those forms (a, b, c, r, s, t) which satisfy

$$0 \leq a \leq b \leq c\,; \quad 0 \leq 2s, 2t \leq a\,; \quad 0 \leq 2r \leq b\,; r \leq s, t. \qquad \ldots \quad (42)$$

It can be verified that points P and Q_3 defined as the intersections of planes F_1, F_2, F_3 and F_1, F_2, F_6 respectively are vertices of Π_f.

By repeating word by word the discussion in §§ 7-8 replacing (a, b, c), (r, s, t), (x, y, z) and Q_1 by (b, c, a), (s, t, r), (y, z, x) and Q_3 respectively one can prove

LEMMA 7: *Let* $f: (a, b, c, r, s, t)$ *be a form of type* (iii). *Let* f' *be the form* (a, b, c, r', s, t) *where* $ar' = st$. *Then*

$$m_f/D_f^{\frac{1}{3}} \geq f'(P)/D_{f'}^{\frac{1}{3}}.$$

We next prove

LEMMA 8: *Let* f *be a form* (a, b, c, r, s, t) *of type* (iii) *which satisfies the further condition* $ar = st$. *Let* f' *be the form* $(a, b, c, r', s', t') = (a, b, c, s, \frac{a}{b}\, r, t)$. *Then*

$$f(P)/D_f^{\frac{1}{3}} \geq f'(P)/D_{f'}^{\frac{1}{3}} \geq m_1.$$

Proof: One can easily verify that

$$f(P) = \frac{a}{4} + \frac{a(b-t)^2}{4(ab-t^2)} + \frac{\Phi^2}{4(ab-t^2)D_f}, \quad \ldots \qquad \ldots \qquad \ldots \quad (43)$$

where

$$\Phi = abc - abs - abr + bst + art - ct^2, \quad \ldots \qquad \ldots \qquad \ldots \quad (44)$$

and

$$f'(P) = \frac{a}{4} + \frac{a(b-t)^2}{4(ab-t^2)} + \frac{\Phi'^2}{4(ab-t^2)D_{f'}}, \quad \ldots \qquad \ldots \qquad \ldots \quad (45)$$

where

$$\Phi' = abc - a^2r - abs + art + ast - ct^2. \quad \ldots \qquad \ldots \qquad \ldots \quad (46)$$

As in Lemma 6, it is easily verified that $D_{f'} \geq D_f$, and $\Phi' = \Phi$, and the first inequality follows from (43)–(46). The second inequality follows easily from Lemma 6.

Lemmas 7 and 8 together prove Theorem 2 for forms of this type also. This completes the proof of Theorem 2 for all forms with non-negative r, s, t.

PART II

13. Let f be a form for which $M_f = m$ (see (4)). Then we shall call f an 'm-form'.

Theorem 2 is obviously equivalent to the statement 'All reduced m-forms are multiples of $3\Sigma x^2 - 2\Sigma yz$.' We have seen in Part I that the reduced forms with non-negative r, s, t cannot be m-forms. Therefore, we have only to establish that if a reduced form with negative r, s, t is an m-form, then it must be a multiple of $3\Sigma x^2 - 2\Sigma yz$. This we proceed to do in this part.

It will be convenient to write

$$r = -\rho, \quad s = -\sigma, \quad \text{and } t = -\tau, \text{ where } \rho, \sigma, \tau > 0,$$

and to use the symbol $(a, b, c, \rho, \sigma, \tau)$ for the form $(a, b, c, -\rho, -\sigma, -\tau)$. We note that the real numbers $a, b, c, \rho, \sigma, \tau$ satisfy the following:

$$0 < a \leqq b \leqq c; \quad 0 < 2\sigma,\ 2\tau \leqq a; \quad 0 < 2\rho \leqq b;$$

and

$$a+b-2\rho-2\sigma-2\tau \geqq 0. \qquad (47)$$

Our first object is to obtain an expression for m_f in terms of the values of f at a finite number of points. For this purpose we have to distinguish between two cases (i) $a > \sigma+\tau$, (ii) $a = \sigma+\tau$; we will be able to drop this distinction later.

(i) Suppose $a > \sigma+\tau$. Then we shall show that Π_f has fourteen faces whose equations are given in the following table. As we know, every face of Π_f has an equation of the form $f(X) = f(X-A)$, $A \in \Lambda_0$. In the first column we give the points A, corresponding to the faces of Π_f given in column 2 and in the third column we give symbols $\pm F_i$ by which we shall refer to these faces.

TABLE II

A	*Face*	*Symbol*
$\pm(1, 0, 0)$	$2ax-2\tau y-2\sigma z = \pm a$	$\pm F_1$
$\pm(0, 1, 0)$	$-2\tau x+2by-2\rho z = \pm b$	$\pm F_2$
$\pm(0, 0, 1)$	$-2\sigma x-2\rho y+2cz = \pm c$	$\pm F_3$
$\pm(1, 1, 0)$	$2(a-\tau)x+2(b-\tau)y-2(\rho+\sigma)z = \pm(a+b-2\tau)$	$\pm F_{10}$
$\pm(0, 1, 1)$	$-2(\sigma+\tau)x+2(b-\rho)y+2(c-\rho)z = \pm(b+c-2\rho)$	$\pm F_{11}$
$\pm(1, 0, 1)$	$2(a-\sigma)x-2(\tau+\rho)y+2(c-\sigma)z = \pm(c+a-2\sigma)$	$\pm F_{12}$
$\pm(1, 1, 1)$	$2(a-\sigma-\tau)x+2(b-\tau-\rho)y+2(c-\rho-\sigma)z = \pm(a+b+c-2\Sigma\rho)$	$\pm F_{13}$

With the help of inequalities (47) it is easy to verify that for any $B \in \Lambda_0$ which does not appear in the first column of the above table, the point $\frac{1}{2}B$ is not an inner point of at least one S_A bounded by the planes $\pm F_i$, $i = 1, 2, 3, 10, 11, 12, 13$. Also for each A' of the above table $\frac{1}{2}A'$ lies in the interior of all the S_A with $A \neq A'$. From this it easily follows that Π_f is bounded by the planes occurring in the above table.

In the next table we list the vertices of Π_f. In the first column we write symbols by which we shall refer to these vertices, in the second the planes on which these vertices lie and in the third the values at the vertex of the expressions

$$2\xi_1 = 2(ax-\tau y-\sigma z),\ 2\xi_2 = 2(-\tau x+by-\rho z) \text{ and } 2\xi_3 = 2(-\sigma x-\rho y+cz).$$

TABLE III

Vertex	*Faces*	*Values of* $2\xi_1, 2\xi_2, 2\xi_3$
$\pm V_1^{(1)}$	$\pm F_1, \mp F_2, \pm F_{12}$	$\pm a, \mp b, \pm (c-2\sigma)$
$\pm V_1^{(2)}$	$\pm F_1, \mp F_3, \pm F_{10}$	$\pm a, \pm (b-2\tau), \mp c$
$\pm V_1^{(3)}$	$\pm F_3, \pm F_{12}, \pm F_{13}$	$\pm (a-2\sigma), \pm (b-2\rho-2\tau), \pm c$
$\pm V_1^{(4)}$	$\pm F_2, \pm F_{10}, \pm F_{13}$	$\pm (a-2\tau), \pm b, \pm (c-2\rho-2\sigma)$
$\pm V_2^{(1)}$	$\pm F_2, \mp F_3, \pm F_{10}$	$\pm (a-2\tau), \pm b, \mp c$
$\pm V_2^{(2)}$	$\pm F_2, \mp F_1, \pm F_{11}$	$\mp a, \pm b, \pm (c-2\rho)$
$\pm V_2^{(3)}$	$\pm F_1, \pm F_{10}, \pm F_{13}$	$\pm a, \pm (b-2\tau), \pm (c-2\rho-2\sigma)$
$\pm V_2^{(4)}$	$\pm F_3, \pm F_{11}, \pm F_{13}$	$\pm (a-2\sigma-2\tau), \pm (b-2\rho), \pm c$
$\pm V_3^{(1)}$	$\pm F_3, \mp F_1, \pm F_{11}$	$\mp a, \pm (b-2\rho), \pm c$
$\pm V_3^{(2)}$	$\pm F_3, \mp F_2, \pm F_{12}$	$\pm (a-2\sigma), \mp b, \pm c$
$\pm V_3^{(3)}$	$\pm F_2, \pm F_{11}, \pm F_{13}$	$\pm (a-2\sigma-2\tau), \pm b, \pm (c-2\rho)$
$\pm V_3^{(4)}$	$\pm F_1, \pm F_{12}, \pm F_{13}$	$\pm a, \pm (b-2\tau-2\rho), \pm (c-2\sigma)$.

It is easy to see, either by using the symmetry about the point $\frac{1}{2}A$ of any face $f(X) = f(X-A)$, or by direct calculations that the values of $f(x, y, z)$ at vertices with the same subscript are the same. Consequently, writing

$$\left.\begin{aligned} f_1 &= f(\pm V_1^{(i)}) \\ f_2 &= f(\pm V_2^{(i)}) \\ f_3 &= f(\pm V_3^{(i)}) \end{aligned}\right\} i = 1, 2, 3, 4.$$

and

we obtain

$$m_f = \max (f_1, f_2, f_3). \quad \ldots \quad (48)$$

Next suppose $a = \sigma + \tau$. Then it can be verified that Π_f is bounded by six pairs of planes, namely $\pm F_1$, $\pm F_2$, $\pm F_3$, $\pm F_{10}$, $\pm F_{12}$, and $\pm F_{13}$. Also Π_f has sixteen vertices, namely those of Table III except for $\pm V_2^{(2)}$, $\pm V_2^{(4)}$, $\pm V_3^{(1)}$ and $\pm V_3^{(3)}$. But the equation (48) continues to hold.

Therefore, we can throughout suppose that

$$m_f = \max (f_1, f_2, f_3)$$

and f_1, f_2, f_3 are defined by the points of Table III.

By straightforward calculations we obtain

$$F_1 = 4D_f \cdot f_1 = abc(\Sigma a - 2\Sigma\rho) - \Sigma a^2\rho^2 + 2\Sigma bc\sigma\tau - 4bc\sigma\tau + 4b\sigma^2\tau + 4c\sigma\tau^2 - 4a\rho\sigma\tau - 4\sigma^2\tau^2, \quad \ldots \quad (49)$$

$$F_2 = 4D_f \cdot f_2 = abc(\Sigma a - 2\Sigma\rho) - \Sigma a^2\rho^2 + 2\Sigma bc\sigma\tau - 4ca\tau\rho + 4c\tau^2\rho + 4a\tau\rho^2 - 4b\rho\sigma\tau - 4\tau^2\rho^2, \quad \ldots \quad (50)$$

and

$$F_3 = 4D_f \cdot f_3 = abc(\Sigma a - 2\Sigma\rho) - \Sigma a^2\rho^2 + 2\Sigma bc\sigma\tau - 4ab\rho\sigma + 4a\rho^2\sigma + 4b\rho\sigma^2 - 4c\rho\sigma\tau - 4\rho^2\sigma^2. \quad \ldots \quad (51)$$

We next observe

$$F_1-F_2 = 4\tau(c-\rho-\sigma)\,(a\rho+\sigma\tau-b\sigma-\rho\tau), \qquad \ldots \quad (52)$$

$$F_2-F_3 = 4\rho(a-\sigma-\tau)\,(b\sigma+\rho\tau-c\tau-\rho\sigma), \qquad \ldots \quad (53)$$

and

$$F_3-F_1 = 4\sigma(b-\tau-\rho)\,(c\tau+\rho\sigma-a\rho-\sigma\tau). \qquad \ldots \quad (54)$$

14. In this section we prove a number of lemmas to show that for an m-form we must have m_f equal to at least two of f_1, f_2 and f_3.

LEMMA 9. *For an m-form we cannot have*

$$f_1 > f_2, f_3.$$

Proof. Suppose f: $(a, b, c, \rho, \sigma, \tau)$ is an m-form with $f_1 > f_2, f_3$. Then we must also have $F_1 > F_2, F_3$, and therefore, by (52)–(54), observing that

$$c \geqq \rho+\sigma, \quad b \geqq \tau+\rho, \quad a \geqq \sigma+\tau$$

we get

$$a\rho+\sigma\tau > b\sigma+\rho\tau,\ c\tau+\rho\sigma.$$

In particular

$$\sigma(b-\tau) < \rho(a-\tau),$$

and therefore, $\sigma < \rho$.

Also

$$\sigma < \frac{a\rho}{b}+\frac{\tau}{b}(\sigma-\rho) < \frac{a\rho}{b} \leqq \tfrac{1}{2}a.$$

Therefore, $a > 2\sigma$.

In view of the above for small $\delta\sigma > 0$, $\delta\rho < 0$, $\delta\rho+\delta\sigma = 0$, the neighbouring form $f' = f+\delta f$ satisfies inequalities (47).

Now

$$m_f = f_1 > f_2, f_3.$$

Therefore, for small $\delta\sigma$, $\delta\rho$

$$m_{f'} = f_1' > f_2', f_3'.$$

As

$$\begin{aligned} f(x, y, z) &= ax^2+by^2+cz^2-2\rho yz-2\sigma zx-2\tau xy \\ &= \frac{1}{a}(ax-\tau y-\sigma z)^2+\frac{1}{a(ab-\tau^2)}\{(ab-\tau^2)y-(a\rho+\sigma\tau)z\}^2+\frac{D_f z^2}{(ab-\tau^2)} \\ &= \frac{1}{a}\xi_1^2+\frac{1}{a(ab-\tau^2)}(a\xi_2+\tau\xi_1)^2+\frac{1}{4(ab-\tau^2)D_f}(2D_f z)^2, \end{aligned}$$

it follows that

$$m_f = f_1 = f\big(V_1^{(1)}\big) = \frac{a}{4}+\frac{a(b-\tau)^2}{4(ab-\tau^2)}+\frac{\Phi_1^2}{4(ab-\tau^2)D_f}, \qquad \ldots \quad (55)$$

where

$$\Phi_1 = \begin{vmatrix} a & -\tau & a \\ -\tau & b & -b \\ -\sigma & -\rho & c-2\sigma \end{vmatrix} = (abc-ab\sigma-ab\rho-b\sigma\tau+a\rho\tau+2\tau^2\sigma-c\tau^2)$$

$$= (ab-\tau^2)\,(c-\rho-\sigma)+\tau(a\rho+\sigma\tau-b\sigma-\rho\tau) > 0, \qquad \ldots \quad (56)$$

and

$$D_f = abc-2\rho\sigma\tau-a\rho^2-b\sigma^2-c\tau^2 > 0. \quad \ldots \quad \ldots \quad \ldots \quad (57)$$

Now

$$\begin{aligned}\delta\Phi_1 &= (2\tau^2-ab-b\tau)\delta\sigma+(a\tau-ab)\delta\rho\\ &= -\tau(a+b-2\tau)\,\delta\sigma < 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (58)\end{aligned}$$

and

$$\begin{aligned}\delta D_f &= -2(\rho\tau+b\sigma)\delta\sigma-2(\sigma\tau+a\rho)\delta\rho\\ &= 2(a\rho+\sigma\tau-b\sigma-\rho\tau)\delta\sigma > 0. \quad \ldots \quad \ldots \quad \ldots \quad (59)\end{aligned}$$

Therefore f' has a smaller $m_{f'}/D_{f'}^{\frac{1}{3}}$ than f and we get a contradiction that proves the lemma.

LEMMA 10: *For an m-form we cannot have*

$$f_2 > f_3, f_1.$$

Proof: Suppose f is an m-form with

$$f_2 > f_3, f_1.$$

Then $F_2 > F_3$, F_1 and by (53), we have

$$b\sigma+\rho\tau > c\tau+\rho\sigma,$$

and

$$\tau < \frac{\sigma(b-\rho)}{c-\rho} \leqq \sigma \leqq \tfrac{1}{2}\,a,$$

i.e. $a > 2\tau$.

Now the proof follows by applying a cyclic permutation* to the proof of Lemma 9 from the sentence 'In view of the above for small $\delta\sigma > 0$, $\delta\rho < 0, \ldots$' onwards.

LEMMA 11: *For an m-form we cannot have*

$$f_3 > f_1, f_2.$$

Proof: Suppose f is an m-form with

$$f_3 > f_1, f_2\,.$$

Then

$$c\tau+\rho\sigma > a\rho+\sigma\tau,\ b\sigma+\rho\tau\,.$$

We distinguish between the following two cases:

(i) $b > 2\rho$,

(ii) $b = 2\rho$.

(i) We get a contradiction by applying a cyclic permutation to the proof of Lemma 10 from the step $a > 2\tau$ onwards.

(ii) $b = 2\rho$. Since $a+b \geqq 2\rho+2\sigma+2\tau$, it follows that $a > 2\sigma$. Consequently, for small $\delta\sigma > 0$, $\delta\tau < 0$, $\delta\sigma+\delta\tau = 0$, the form f': $(a, b, c, \rho, \sigma+\delta\sigma, \tau+\delta\tau)$ satisfies (47). Also

$$m_{f'} = f'_3 > f'_1, f'_2\,.$$

* I.e. by replacing in the proof (a, b, c), (ρ, σ, τ), (f_1, f_2, f_3), (F_1, F_2, F_3) and (x, y, z) by (b, c, a), (σ, τ, ρ), (f_2, f_3, f_1), (F_2, F_3, F_1) and (y, z, x) respectively.

Now

$$f(x, y, z) = \frac{1}{c}(cz-\sigma x-\rho y)^2+\frac{1}{(bc-\rho^2)c}\{(bc-\rho^2)y-(c\tau+\rho\sigma)x\}^2+\frac{D_f x^2}{(bc-\rho^2)}$$

$$= \frac{1}{c}\xi_3^2+\frac{1}{c(bc-\rho^2)}(c\xi_2+\rho\xi_3)^2+\frac{1}{4D_f(bc-\rho^2)}(2D_f x)^2.$$

Therefore,

$$f_3 = f\left(V_3^{(2)}\right) = \frac{c}{4}+\frac{c(b-\rho)^2}{4(bc-\rho^2)}+\frac{\Phi^2}{4D_f(bc-\rho^2)},$$

where

$$\Phi = \begin{vmatrix} a-2\sigma & -\tau & -\sigma \\ -b & b & -\rho \\ c & -\rho & c \end{vmatrix} = (abc-bc\sigma-bc\tau+c\rho\tau-b\rho\sigma-a\rho^2+2\rho^2\sigma)$$

$$= (bc-\rho^2)(a-\sigma-\tau)+\rho(c\tau+\rho\sigma-b\sigma-\rho\tau) > 0, \qquad (60)$$

and

$$D_f = abc-2\rho\sigma\tau-a\rho^2-b\sigma^2-c\tau^2 > 0. \qquad (61)$$

As

$$\delta\Phi = \delta\sigma(-bc-b\rho+2\rho^2)+\delta\tau(-bc+c\rho)$$
$$= -\delta\sigma(b+c-2\rho)$$
$$< 0,$$

and

$$\delta D_f = \delta\sigma(-2\rho\tau-2b\sigma)+\delta\tau(-2\rho\sigma-2c\tau)$$
$$= 2\delta\sigma(c\tau+\rho\sigma-b\sigma-\rho\tau)$$
$$> 0,$$

it follows that f' has a smaller $m_f/D_f^{\frac{1}{3}}$ than f, which is a contradiction that leads to the lemma.

From Lemmas 9–11 it follows that for an m-form m_f is equal to at least two of f_1, f_2 and f_3. The object of the next three sections will be to prove that for an m-form we must have $f_1 = f_2 = f_3$.

15. The object of this section will be to prove that an m-form cannot have

$$f_1 = f_2 > f_3.$$

We shall throughout this section suppose that f: $(a, b, c, \rho, \sigma, \tau)$ is an m-form with $f_1 = f_2 > f_3$ and get a contradiction.

LEMMA 12: *The form f must have*

$$a = 2\sigma+\tau, \quad b = 2\rho+\tau. \qquad (62)$$

Proof: Since $f_1 = f_2 > f_3$, we clearly have

$$F_1 = F_2 > F_3,$$

and it follows from (52) and (53) that

$$a\rho+\sigma\tau = b\sigma+\rho\tau > c\tau+\rho\sigma. \qquad (63)$$

We now observe that

$$b \geqq 2\rho+\tau, \qquad (64)$$

If not, suppose

$$b < 2\rho+\tau.$$

Then,

$$a > 2\sigma+\tau$$

and

$$\sigma(2\rho+\tau)+\rho\tau > b\sigma+\rho\tau = a\rho+\sigma\tau > \rho(2\sigma+\tau)+\sigma\tau,$$

which is a contradiction that proves (64).

From (63) and (64) it follows that

$$a \geqq 2\sigma+\tau \qquad (65)$$

Also the signs of equality and inequality in (64) and (65) go together.

Now suppose

$$a > 2\sigma+\tau,\ b > 2\rho+\tau. \qquad (66)$$

Then

$$a+b > 2\rho+2\sigma+2\tau,$$

so that for small $\delta\rho > 0$, $\delta\sigma > 0$, and $\delta c > 0$, the neighbouring form f': $(a,\ b,\ c+\delta c,\ \rho+\delta\rho,\ \sigma+\delta\sigma,\ \tau)$ satisfies the inequalities (47). Let $\delta\sigma$, $\delta\rho$, and δc be connected by the relations

$$(a-\tau)\delta\rho = (b-\tau)\delta\sigma, \qquad (67)$$

and

$$(ab-\tau^2)\delta c = 2(a\rho+\sigma\tau)\,(\delta\rho+\delta\sigma). \qquad (68)$$

Now.

$$a(\rho+\delta\rho)+(\sigma+\delta\sigma)\tau-b(\sigma+\delta\sigma)-(\rho+\delta\rho)\tau = 0.$$

Therefore,

$$m_{f'} = f_1' = f_2' > f_3'.$$

By giving cyclic permutation twice to the expression for f_3 in Lemma 11, (part ii), or by direct computation,

$$f_2 = (V_2^{(2)}) = \frac{b}{4}+\frac{b(a-\tau)^2}{4(ab-\tau^2)}+\frac{\Phi^2}{4D_f(ab-\tau^2)}.$$

where

$$\begin{aligned}\Phi &= (abc-ab\rho-ab\sigma+b\sigma\tau-a\rho\tau-c\tau^2+2\rho\tau^2)\\ &= (ab-\tau^2)\,(c-\rho-\sigma)+\tau(b\sigma+\rho\tau-a\rho-\sigma\tau)\\ &> 0, \qquad (69)\end{aligned}$$

since

$$c \geqq \tfrac{1}{2}(a+b) > \rho+\sigma+\tau.$$

Now

$$\begin{aligned}\delta\Phi &= (ab-\tau^2)\,\delta c-(ab+a\tau-2\tau^2)\,\delta\rho-(ab-b\tau)\delta\sigma\\ &= \frac{1}{(a-\tau)}\,\delta\sigma\,\{2(a\rho+\sigma\tau)\,(a+b-2\tau)-(ab+a\tau-2\tau^2)\,(b-\tau)-b(a-\tau)^2\}\\ &= \frac{1}{(a-\tau)}\,\delta\sigma(a+b-2\tau)\,(2a\rho+2\sigma\tau-ab+\tau^2)\\ &< 0,\end{aligned}$$

since, by (66),

$$ab-2a\rho-2\sigma\tau-\tau^2 = a(b-2\rho)-\tau(2\sigma+\tau) > 0.$$

Also

$$\delta D_f = (ab-\tau^2)\delta c-2(a\rho+\sigma\tau)\delta\rho-2(b\sigma+\rho\tau)\delta\sigma$$
$$= (ab-\tau^2)\delta c-2(a\rho+\sigma\tau)(\delta\rho+\delta\sigma)$$
$$= 0.$$

Therefore, f' has a smaller $m_f/D_f^{\frac{1}{3}}$ than f which is a contradiction that disproves (66) and the lemma follows from (64), (65) and the remark before (66).

LEMMA 13. *The form f cannot have*

$$b = 2\rho+\tau, \quad a = 2\sigma+\tau. \quad \ldots \quad (67)$$

Proof: Suppose f satisfies (67). Since $f_1 = f_2 > f_3$, we have

$$a\rho+\sigma\tau = b\sigma+\rho\tau > c\tau+\rho\sigma. \quad \ldots \quad (68)$$

In particular,

$$\sigma(b-\rho) > \tau(c-\rho),$$

so that

$$\tau < \sigma \leqq \tfrac{1}{2} a.$$

Consequently for small $\delta\tau > 0$, $\delta\rho < 0$, $\delta\sigma < 0$ and

$$\delta\tau = -2\delta\rho = -2\delta\sigma, \quad \ldots \quad (69)$$

the form f': $(a, b, c, \rho+\delta\rho, \sigma+\delta\sigma, \tau+\delta\tau)$ satisfies conditions (47). As

$$a(\rho+\delta\rho)+(\sigma+\delta\sigma)(\tau+\delta\tau)-b(\sigma+\delta\sigma)-(\rho+\delta\rho)(\tau+\delta\tau)$$
$$= \delta\rho(a-2\sigma+\tau-b+2\rho-\tau) \qquad \text{(by 69)},$$
$$= 0, \qquad \text{(by 67)},$$

it follows that

$$m_{f'} = f_1' = f_2' > f_3'.$$

Now, by (49),

$$4D_f f_1 = F_1 = abc(a+b+c-2\rho-2\sigma-2\tau)-a^2\rho^2-b^2\sigma^2-c^2\tau^2+2bc\sigma\tau$$
$$+2ca\tau\rho+2ab\rho\sigma-4bc\sigma\tau+4b\sigma^2\tau+4c\tau^2\sigma-4a\rho\sigma\tau-4\sigma^2\tau^2. \quad \ldots \quad (71)$$

Therefore, substituting for $\delta\rho$ and $\delta\sigma$ from (69) and simplifying, we get

$$\delta F_1 = \delta\tau(b-a-2c+4\sigma-2\tau)\,(b\sigma+c\tau-2\sigma\tau-a\rho). \quad \ldots \quad (72)$$

On substituting for a and b from (67) in the above, we obtain

$$\delta F_1 = -2\tau(c-\rho-\sigma+\tau)\,(c-\rho-\sigma)\delta\tau < 0. \quad \ldots \quad (72.1)$$

Also

$$\delta D_f = \delta\tau(a\rho+\sigma\tau+b\sigma+\rho\tau-2c\tau-2\rho\sigma) > 0. \quad \ldots \quad (73)$$

From (70)–(73) it follows that f' has a smaller $m_f/D_f^{\frac{1}{3}}$ than f which is a contradiction that proves the lemma.

Since Lemmas 12 and 13 contradict each other, it follows that

LEMMA 14: *For an m-form we cannot have*

$$f_1 = f_2 > f_3.$$

16. In this section we shall prove by reductio ad absurdum that an m-form cannot have

$$f_2 = f_3 > f_1 .$$

We assume the contrary, i.e. that f: $(a, b, c, \rho, \sigma, \tau)$ is an m-form for which

$$f_2 = f_3 > f_1,$$

and so also

$$F_2 = F_3 > F_1 .$$

Then from (53) and (54) it follows that we have

either (i) $a = 2\sigma = 2\tau, \quad c\tau + \rho\sigma \geqq b\sigma + \rho\tau > a\rho + \sigma\tau,$ .. .. (74)

or (ii) $a > \sigma + \tau, \quad c\tau + \rho\sigma = b\sigma + \rho\tau > a\rho + \sigma\tau .$ .. .. (75)

We now prove a number of lemmas which will lead to a contradiction and hence prove that m-forms cannot have $f_2 = f_3 > f_1$.

Lemma 15: *The form f cannot have*

$$a = 2\sigma = 2\tau,\ c\tau + \rho\sigma > b\sigma + \rho\tau > a\rho + \sigma\tau . \qquad (76)$$

Proof: Suppose f satisfies (76). Since $a = 2\sigma = 2\tau$ and $a + b \geqq 2\rho + 2\sigma + 2\tau$, it follows that $b > 2\rho$. Consequently for small $\delta\rho > 0$, $\delta\tau < 0$, $\delta\rho + \delta\tau = 0$, the form f': $(a, b, c, \rho + \delta\rho, \sigma, \tau + \delta\tau)$ satisfies the inequalities (47). Also for f',

$$m_{f'} = f_3' > f_2' > f_1' .$$

Now applying two 'cyclic permutations' to the proof of Lemma 9 from the relation (55) onwards we get a contradiction that gives the lemma.

Lemma 16: *We cannot have*

$$a = 2\sigma = 2\tau, \quad c\tau + \rho\sigma = b\sigma + \rho\tau > a\rho + \sigma\tau . \qquad (77)$$

Proof: Suppose f satisfies (77). Then it follows from $\sigma = \tau$ that $b = c$. Also, as before, $b > 2\rho$. Therefore, for small $\delta\rho > 0$, $\delta\tau < 0$, $\delta\sigma < 0$, and

$$\delta\rho = -2\delta\sigma = -2\delta\tau, \qquad (78)$$

the neighbouring form f': $(a, b, c, \rho + \delta\rho, \sigma + \delta\sigma, \tau + \delta\tau)$ satisfies (47).

Since

$$c(\tau + \delta\tau) + (\rho + \delta\rho)(\sigma + \delta\sigma) - b(\sigma + \delta\sigma) - (\rho + \delta\rho)(\tau + \delta\tau) = 0,$$

it follows that

$$m_{f'} = f_3' = f_2' > f_1' .$$

Applying a cyclic permutation to the proof of Lemma 13 from (70) to (72) we have

$$\delta F_2 = (c - b - 2a + 4\tau - 2\rho)(c\tau + a\rho - 2\rho\tau - b\sigma)\delta\rho .$$

On substituting $a = 2\tau$, $c = b$ and $\tau = \sigma$, the above gives

$$\delta F_2 = 0 . \qquad (79)$$

Also, by (78) and (77),

$$\delta D_f = \delta\rho\,(b\sigma + \rho\tau + c\tau + \rho\sigma - 2a\rho - 2\sigma\tau) > 0 . \qquad (80)$$

From (79) and (80) we obtain a contradiction that proves the lemma.

After Lemmas 15 and 16 we can suppose that f satisfies inequalities (75).

LEMMA 17: *We cannot have*

$$a = 2\tau .$$

Proof: Since $b\sigma+\rho\tau = c\tau+\rho\sigma$ and $b \leqq c$, it follows that $\tau \leqq \sigma$, and since $2\sigma \leqq a$, $a = 2\tau$ would imply $a = \sigma+\tau$, which contradicts (75).

LEMMA 18: *If $b > a$ and $a+b > 2\rho+2\sigma+2\tau$, then $a > 2\sigma$.*

Proof: Suppose $a = 2\sigma$, so that $b > 2\rho+2\tau$. By Lemma 17, $a > 2\tau$. Therefore, for small $\delta a > 0$, $\delta\tau > 0$, the form f': $(a+\delta a, b, c, \rho, \sigma, \tau+\delta\tau)$ satisfies (47). Let δa and $\delta\tau$ be connected by

$$(bc-\rho^2)\,\delta a = 2(c\tau+\rho\sigma)\,\delta\tau . \qquad \ldots \quad \ldots \quad \ldots \quad (81)$$

Now

$$\delta(c\tau+\rho\sigma-b\sigma-\rho\tau) = (c-\rho)\,\delta\tau > 0 .$$

Therefore,

$$m_{f'} = f_3' > f_2' > f_1' .$$

By applying a cyclic permutation to (69) or by direct calculation we obtain

$$f_3 = f(V_3^{(2)}) = \frac{c}{4} + \frac{c(b-\rho)^2}{4(bc-\rho^2)} + \frac{\Phi^2}{4D_f(bc-\rho^2)}$$

where

$$\begin{aligned}\Phi &= (abc-bc\sigma-bc\tau+c\rho\tau-b\rho\sigma-a\rho^2+2\sigma\rho^2)\\ &= (bc-\rho^2)(a-\sigma-\tau)+\rho(c\tau+\rho\sigma-b\sigma-\rho\tau)\\ &> 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (82)\end{aligned}$$

Since $b-\rho-2\tau > \rho$ and $c \geqq a \geqq 2\sigma$, it is clear that

$$\begin{aligned}\delta\,\Phi &= (bc-\rho^2)\delta a+(c\rho-cb)\delta\tau\\ &= -\{c(b-\rho-2\tau)-2\rho\sigma\}\delta\tau\\ &< 0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (83)\end{aligned}$$

Also

$$\delta D_f = (bc-\rho^2)\delta a-2(c\tau+\rho\sigma)\delta\tau = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (84)$$

From (83) and (84) we obtain a contradiction to f having smallest $m_f/D_f^{\frac{1}{3}}$ and that proves the lemma.

LEMMA 19: *We must have one at least of the following:*

(i) $a = b$,

(ii) $a+b = 2\rho+2\sigma+2\tau$.

Proof: Suppose $a < b$ and $a+b > 2\rho+2\sigma+2\tau$.

Then by Lemmas 18 and 17, we have $a > 2\sigma$, 2τ. Therefore, for small $\delta a > 0$, $\delta\sigma > 0$, $\delta\tau > 0$, the form f': $(a+\delta a, b, c, \rho, \sigma+\delta\sigma, \tau+\delta\tau)$ satisfies (47). Let δa, $\delta\sigma$ and $\delta\tau$ be connected by the relations

$$(c-\rho)\delta\tau = (b-\rho)\delta\sigma, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (85)$$

and

$$(bc-\rho^2)\delta a = 2(b\sigma+\rho\tau)(\delta\sigma+\delta\tau). \qquad \ldots \quad \ldots \quad \ldots \quad (86)$$

From

$$c\tau+\rho\sigma = b\sigma+\rho\tau > a\rho+\sigma\tau,$$

it is easy to prove that $c > 2\sigma+\rho$, $b > 2\tau+\rho$,

($c \leqq 2\sigma+\rho => b \geqq a > 2\tau+\rho$ and a contradiction) and the proof can be completed by giving a cyclic permutation to the proof of Lemma 12 from (69) onwards.

LEMMA 20: *We must have*

$$a+b = 2\rho+2\sigma+2\tau.$$

Proof: Suppose $a+b > 2\rho+2\sigma+2\tau$.

Then, by Lemma 19, $a = b$. Since $a\rho+\sigma\tau < b\sigma+\rho\tau$, and $a = b$ it follows that $\rho < \sigma$, and $b > \rho+\sigma+\tau > 2\rho$. Consequently for small $\delta\rho > 0$, $\delta\sigma < 0, f'$: (a, b, c, $\rho+\delta\rho$, $\sigma+\delta\sigma$, τ) satisfies (47). Let $\delta\rho$ and $\delta\sigma$ satisfy the equation

$$(a\rho+\sigma\tau)\delta\rho+(b\sigma+\rho\tau)\delta\sigma = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (87)$$

As $c\tau+\rho\sigma = b\sigma+\rho\tau$, it follows that $\sigma \geqq \tau$, and

$$\delta(c\tau+\rho\sigma-b\sigma-\rho\tau) = (\sigma-\tau)\delta\rho-(b-\rho)\delta\sigma > 0,$$

so that

$$m_{f'} = f_3' > f_2', f_1'.$$

By (51) we have

$$4D_f \,.\, f_3 = F_3 = abc(\Sigma a-2\Sigma\rho)-\Sigma a^2\rho^2+2\Sigma bc\sigma\tau-4ab\rho\sigma+4a\rho^2\sigma$$
$$+4b\rho\sigma^2-4c\rho\sigma\tau-4\rho^2\sigma^2.$$

Using (87), we get

$$\begin{aligned}(b\sigma+\rho\tau)\delta F_3 = & -2\delta\rho\{(\sigma-\rho)(a-\tau)(a^2c+a^2\rho+a\rho\sigma-a\rho\tau-4a\rho\sigma)-2a\rho\sigma\tau(\sigma-\rho)\\ & -2a^2(\sigma-\rho)(\sigma^2+\rho^2+\sigma\rho)+2a(\sigma^2-\rho^2)(a\sigma+\rho\tau+\rho\sigma)\}\\ = & -2\delta\rho(\sigma-\rho)\Psi,\end{aligned}$$

where, using $a = b > \rho+\sigma+\tau$, we have

$$\begin{aligned}\Psi & > (\rho+\sigma)(a^2c+a^2\rho+a\rho\sigma-a\rho\tau-4a\rho\sigma)-2a\rho\sigma\tau-2a^2(\sigma+\rho)^2\\ & \qquad +2a^2\rho\sigma+2a(\sigma+\rho)(a\sigma+\rho\tau+\rho\sigma)\\ & = (\rho+\sigma)(a^2c-a^2\rho+a\rho\tau-a\rho\sigma)+2a\rho\sigma(a-\tau)\\ & > (\rho+\sigma)(a^2c+a\rho\sigma+a\rho\tau-a^2\rho)\\ & > 0.\end{aligned}$$

Therefore

$$\delta F_3 < 0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (88)$$

Also, by (81),

$$\delta D_f = 0$$

and, as before, we obtain a contradiction that proves the lemma.

LEMMA 21: *We must have* $b = c$.

Proof: We remark that

$$b > 2\rho+\tau.$$

For, if not, $a \geqq 2\sigma+\tau$, and

$$a\rho+\sigma\tau \geqq \rho(2\sigma+\tau)+\sigma\tau = \sigma(2\rho+\tau)+\rho\tau \geqq b\sigma+\rho\tau,$$

which is impossible.

Also $b \geqq 2\tau+\rho$,
for if not, $c > a > 2\sigma+\rho$, and $c\tau+\rho\sigma > b\sigma+\rho\tau$, which is impossible.

From $b \geqq 2\tau+\rho$, $b\sigma+\rho\tau = c\tau+\rho\sigma$, it follows that $c \geqq 2\sigma+\rho$. Therefore, since $a+b = 2\rho+2\sigma+2\tau$, we easily see that

$$a < 2\sigma+\tau,\ b > 2\rho+\tau,\ b \geqq 2\tau+\rho,\ c \geqq 2\sigma+\rho. \qquad \text{(89)}$$

Now suppose $b < c$. Then for small $\delta c < 0$, $\delta\sigma < 0$, $f' = f+\delta f$ satisfies (47). Let δc and $\delta\sigma$ be connected by

$$(ab-\tau^2)\delta c = 2(b\sigma+\rho\tau)\delta\sigma. \qquad \text{(90)}$$

Then, by (90),

$$\begin{aligned}(ab-\tau^2)\delta(c\tau+\rho\sigma-b\sigma-\rho\tau) &= -\{ab(b-\rho)-2b\sigma\tau-b\tau^2-\rho\tau^2\}\delta\sigma \\ &\geqq -\delta\sigma\{b\tau(2a-2\sigma-\tau)-\rho\tau^2\},\ \text{since } b \geqq \rho+2\tau, \\ &> -\delta\sigma(b-\rho)\tau^2,\ \text{since } a > \sigma+\tau \\ &> 0.\end{aligned}$$

Consequently,

$$m_{f'} = f_3' > f_2' > f_1'.$$

Now

$$f_3 = f(V_3^{(1)}) = \frac{a}{4}+\frac{a(b-2\rho-\tau)^2}{4(ab-\tau^2)}+\frac{\Phi^2}{4(ab-\tau^2)D_f},$$

where

$$\begin{aligned}\Phi &= \begin{vmatrix} a & -\tau & -a \\ -\tau & b & b-2\rho \\ -\sigma & -\rho & c \end{vmatrix} \\ &= (abc+ab\rho-2a\rho^2-c\tau^2+b\sigma\tau-2\rho\sigma\tau-a\rho\tau-ab\sigma) \\ &= ab(c-\sigma)+a\rho(b-2\rho-\tau)+\tau(b\sigma-c\tau-2\rho\sigma) \\ &= ab(c-\sigma)+a\rho(b-2\rho-\tau)-\rho\tau(\sigma+\tau),\ \text{since } b\sigma+\rho\tau = c\tau+\rho\sigma \\ &> 0 \qquad \text{(by (89))}\end{aligned}$$

and

$$\begin{aligned}\delta\Phi &= (ab-\tau^2)\delta c+(b\tau-2\rho\tau-ab)\delta\sigma \\ &= b\delta\sigma(2\sigma+\tau-a) \\ &< 0,\end{aligned}$$

since $\delta\sigma < 0$ and $a < 2\sigma+\tau$.

Combining the above with

$$\delta D_f = (ab-\tau^2)\delta c-2(b\sigma+\rho\tau)\delta\sigma = 0,$$

we get a contradiction in the usual way and the lemma follows.

LEMMA 22: *We cannot have $b = c$.*

Proof: Suppose $b = c$. Then $\sigma = \tau$ and $a > 2\sigma = 2\tau$. As we saw in Lemma 21, $b > 2\rho$. Therefore, for small $\delta\rho > 0$, $\delta\sigma < 0$, $\delta\tau < 0$ and $\delta\rho = -2\delta\sigma = -2\delta\tau$, the form $f' = f + \delta f$ satisfies (47).

Since

$$c(\tau+\delta\tau)+(\rho+\delta\rho)(\sigma+\delta\sigma)-b(\sigma+\delta\sigma)-(\rho+\delta\rho)(\tau+\delta\tau) = 0,$$

it follows that

$$m_{f'} = f_3' = f_2' > f_1' .$$

Then, as in Lemma 16, equation (78), we get on writing $b = c$, $\sigma = \tau$,

$$\begin{aligned}\delta F_2 &= (c-b-2a+4\tau-2\rho)(c\tau+a\rho-2\rho\tau-b\sigma)\delta\rho \\ &= -2\rho(a+\rho-2\tau)(a-2\tau)\delta\rho \\ &< 0\end{aligned}$$

and

$$\delta D_f = \delta\rho(c\tau+\rho\sigma+b\sigma+\rho\tau-2a\rho-2\sigma\tau) > 0,$$

and we obtain in the usual way a contradiction that proves the lemma.

Since Lemmas 21 and 22 contradict each other, we achieve our object in this section, i.e.

LEMMA 23: *For an m-form, we cannot have*

$$f_2 = f_3 > f_1.$$

17. In this section our object is to prove that an m-form cannot have

$$f_3 = f_1 > f_2,$$

or what is equivalent,

$$c\tau+\rho\sigma = a\rho+\sigma\tau > b\sigma+\rho\tau. \quad \ldots \quad \ldots \quad \ldots \quad (91)$$

As usual, we assume the contrary, i.e. assume that the form f: $(a, b, c, \rho, \sigma, \tau)$ is an m-form which satisfies (91). Then on this assumption prove a number of lemmas that lead to a contradiction.

LEMMA 24: *We must have one at least of the following:*

$$\left.\begin{array}{l}\text{(i) } b = 2\rho, \\ \text{(ii) } b = c, \text{ and} \\ \text{(iii) } a+b = 2\rho+2\sigma+2\tau.\end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (92)$$

Proof: Suppose f does not satisfy any of the relations (92). Then for small $\delta b > 0$, $\delta\rho > 0$, $\delta\tau > 0$, $f' = f + \delta f$ satisfies (47). We can also show that $a > 2\tau+\sigma$, $c > 2\rho+\sigma$ and then the proof of the lemma follows by applying a 'cyclic permutation' twice* to the proof of Lemma 12 from (67) onwards or once to the proof of Lemma 19.

---

* I.e., by replacing in the proof (x, y, z), (a, b, c), (ρ, σ, τ), (f_1, f_2, f_3) and (F_1, F_2, F_3) by (z, x, y), (c, a, b), (τ, ρ, σ), (f_3, f_1, f_2) and (F_3, F_1, F_2) respectively.

LEMMA 25: *We must have $b = c$.*

Proof: We first remark that from

$$a\rho+\sigma\tau = c\tau+\rho\sigma > b\sigma+\rho\tau, \quad a+b \geqq 2\rho+2\sigma+2\tau,$$

it is easy to deduce that

$$a > 2\sigma+\tau, \quad a \geqq \sigma+2\tau, \quad c \geqq 2\rho+\sigma.$$

Suppose $b < c$. Then for small $\delta c < 0$, $\delta\rho < 0$, $f' = f+\delta f$ satisfies (47). Let δc and $\delta\rho$ satisfy

$$(ab-\tau^2)\delta c = 2(a\rho+\sigma\tau)\delta\rho. \qquad \text{.. .. ..} \quad (93)$$

Then

$$\begin{aligned}(ab-\tau^2)\delta(c\tau+\rho\sigma-a\rho-\sigma\tau) &= -\delta\rho\{ab(a-\sigma)-a\tau(2\rho+\tau)-\sigma\tau^2\}\\ &\geqq -\delta\rho\{a\tau(2b-2\rho-\tau)-\sigma\tau^2\}\\ &> 0,\end{aligned}$$

and

$$m_{f'} = f'_3 > f'_1 > f'_2.$$

Now

$$f_3 = f(V_3^{(2)}) = \frac{b}{4}+\frac{b(a-2\sigma-\tau)^2}{4(ab-\tau^2)}+\frac{\Phi^2}{4(ab-\tau^2)D_f},$$

where

$$\begin{aligned}\Phi &= \begin{vmatrix} a & -\tau & a-2\sigma \\ -\tau & b & -\sigma \\ -\sigma & -\rho & c \end{vmatrix}\\ &= (abc-ab\rho-c\tau^2-b\sigma\tau+a\rho\tau+ab\sigma-2\rho\sigma\tau-2b\sigma^2)\\ &= ab(c-\rho)+b\sigma(a-\tau-2\sigma)+\tau(a\rho-c\tau-2\rho\sigma)\\ &= ab(c-\rho)+b\sigma(a-\tau-2\sigma)-\tau\sigma(\rho+\tau)\\ &> 0\end{aligned}$$

since $c > \rho+\tau$, $b > \rho+\tau$, $a > 2\sigma+\tau$.

Again

$$\begin{aligned}\delta\Phi &= (ab-\tau^2)\delta c+(-ab+a\tau-2\sigma\tau)\delta\rho\\ &= -a(b-\tau-2\rho)\delta\rho.\end{aligned}$$

Since $b \neq c$, Lemma 24 gives

$$\text{either } b = 2\rho \text{ or } a+b = 2\rho+2\sigma+2\tau,$$

so that in both cases since $a > 2\sigma+\tau$, we have $b < 2\rho+\tau$. Consequently,

$$\delta\Phi < 0.$$

Combined with

$$\delta D_f = 0,$$

the above gives a contradiction in the usual way and the lemma follows.

LEMMA 26: *We cannot have* $b = c$.

Proof: We first observe that

$$c\tau+\rho\sigma = a\rho+\sigma\tau \qquad (94)$$

implies that $\tau \leqq \rho$. Also we have seen in the beginning of Lemma 25 that

$$a = 2\tau+\sigma+k, \quad c = 2\rho+\sigma+k', \qquad (95)$$

with $k \geqq 0$, $k' \geqq 0$. Substituting from (95) in (94), we have

$$\rho k = \tau k',$$

and since $\tau \leqq \rho$, it follows that

$$k' \geqq k. \qquad (96)$$

Now suppose f satisfies the condition

$$b = c.$$

As $a > 2\sigma+\tau > 2\sigma$, it is clear that for small $\delta\sigma > 0$, $\delta\tau < 0$, $\delta\rho < 0$, $\delta\sigma = -2\delta\tau = -2\delta\rho$, $f' = f+\delta f$ satisfies (47).

Now

$$\begin{aligned} &c(\tau+\delta\tau)+(\rho+\delta\rho)(\sigma+\delta\rho)-a(\rho+\delta\rho)-(\sigma+\delta\sigma)(\tau+\delta\tau) \\ &= \tfrac{1}{2}\delta\sigma[\{a-(2\tau+\sigma)\}-\{c-(2\rho+\sigma)\}] \\ &= \tfrac{1}{2}\delta\sigma(k-k') \\ &\leqq 0, \end{aligned}$$

therefore,

$$m_{f'} = f_1' > f_3' > f_2'.$$

Since

$$\begin{aligned} 4D_f \,.\, f_1 = F_1 = abc(\Sigma a-2\Sigma\rho)-\Sigma a^2\rho^2+2\Sigma bc\sigma\tau-4bc\sigma\tau+4b\sigma^2\tau \\ +4c\tau^2\sigma-4a\rho\sigma\tau-4\sigma^2\tau^2, \end{aligned}$$

it follows, on putting $\delta\tau = \delta\rho = -\tfrac{1}{2}\delta\sigma$, that

$$\delta F_1 = \delta\sigma(a\rho-b\sigma-c\tau+2\sigma\tau)(a+2b-c-4\tau+2\sigma).$$

On putting $c = b$ and $a\rho-c\tau = \rho\sigma-\sigma\tau$, we get

$$\delta F_1 = -\delta\sigma(a+b+2\sigma-4\tau)(b-\rho-\tau)\sigma < 0.$$

Combining the above with

$$\delta D_f = \delta\sigma\,(a\rho+\sigma\tau+c\tau+\rho\sigma-2b\sigma-2\rho\tau) > 0,$$

we get a contradiction which proves the lemma.

Since Lemmas 25 and 26 contradict each other we obtain

LEMMA 27: *An m-form cannot have*

$$f_3 = f_1 > f_2.$$

18. It follows from the discussion in § 14–§ 17 that if $f = (a, b, c, \rho, \sigma, \tau)$ is an m-form, then we must have

$$a\rho+\sigma\tau = b\sigma+\rho\tau = c\tau+\rho\sigma. \qquad (97)$$

It is easy to prove with the help of $a+b \geqq 2\rho+2\sigma+2\tau$, $c \geqq b \geqq a$, that

$$a \geqq 2\sigma+\tau, \quad b \geqq 2\rho+\tau, \quad a \geqq 2\tau+\sigma \,. \qquad (98)$$

We suppose f is an m-form. Then

LEMMA 28: $a+b = 2\rho+2\sigma+2\tau$.

Proof: Suppose $a+b > 2\rho+2\sigma+2\tau$. Then

$$a > 2\sigma+\tau, \quad b > 2\rho+\tau,$$

and for small $\delta c > 0$, $\delta\rho > 0$, $\delta\sigma > 0$, the form $f' = f+\delta f$ satisfies conditions (77). Now let δc, $\delta\rho$, $\delta\sigma$ satisfy the relations

$$(a-\tau)\,\delta\rho = (b-\tau)\,\delta\sigma \qquad (99)$$

and

$$(ab-\tau^2)\,\delta c = 2(b\sigma+\rho\tau)\,(\delta\rho+\delta\sigma)\,. \qquad (100)$$

Then, using (99) and (100),

$$(ab-\tau^2)\,(a-\tau)\,\delta\left\{(a\rho+\sigma\tau)-(c\tau+\rho\sigma)\right\}$$

$$= (ab-\tau^2)\,(a-\tau)\left\{(a-\sigma)\,\delta\rho+(\tau-\rho)\,\delta\sigma-\tau\,\delta c\right\}$$

$$= (ab-\tau^2)\left\{(a-\sigma)\,(b-\tau)+(\tau-\rho)\,(a-\tau)\right\}\delta\sigma$$

$$-\tau\left\{2(b\sigma+\rho\tau)\,(a+b-2\tau)\right\}\delta\sigma$$

$$= \delta\sigma\left\{(ab-\tau^2)^2-2(ab-\tau^2)\,(a\rho-\rho\tau)-2\tau(b\sigma+\rho\tau)\,(a+b-2\tau)\right\}$$

$$(\text{since } a\rho-\rho\tau = b\sigma-\sigma\tau)$$

$$= \delta\sigma\left\{(ab-\tau^2)^2-2(ab-\tau^2)\,(a\rho-\rho\tau)-2\tau(b\sigma+\rho\tau)\,(a+b-2\tau)\right.$$

$$\left.+2\tau^2(a\rho+\sigma\tau-b\sigma-\rho\tau)\right\}$$

$$= \delta\sigma\left\{(b-2\rho-\tau)\,(a^2b-a\tau^2-2b\sigma\tau)+\tau(a-2\sigma-\tau)\,(ab-\tau^2+2b\rho)\right\}$$

$$> 0\,.$$

Also

$$a(\rho+\delta\rho)+(\sigma+\delta\sigma)\tau-b(\sigma+\delta\sigma)-(\rho+\delta\rho)\tau = 0\,.$$

Therefore,

$$m_{f'} = f'_2 = f'_1 > f'_3\,.$$

Repeating the proof of Lemma 12 from (67) onwards we get a contradiction which proves the lemma.

From Lemma 28 it follows that

$$a = 2\sigma+\tau, \quad b = 2\rho+\tau\,. \qquad (101)$$

Consequently from (97) we get

$$c = \rho+\sigma+\frac{\rho\sigma}{\tau}\,. \qquad (102)$$

We now observe that any form $f = (a, b, c, \rho, \sigma, \tau)$ which satisfies (101), (102), and

$$\rho \geqq \sigma \geqq \tau > 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (103)$$

automatically satisfies conditions (47). Consequently we can say that m-forms are those forms which satisfy (101)–(103) and for which $m_f/D^{\frac{1}{3}}$ is minimum.

In other words, we can define m-forms in terms of sets of real numbers ρ, σ, τ which satisfy (103) and for which the expression $F/4D^{\frac{4}{3}}$, with F and D as defined below, takes its minimum value.

$$\begin{aligned}
F &= 4D_f \cdot f_1 = 4D_f \cdot f_2 = 4D_f \cdot f_3 \\
&= abc(\Sigma a - 2\Sigma\rho) - \Sigma a^2\rho^2 + 2\Sigma bc\sigma\tau - 4ab\,\rho\sigma + 4\rho\sigma\,(a\rho + b\sigma - c\tau) - 4\rho^2\sigma^2 \\
&= abc^2 - (a\rho + b\sigma - c\tau)^2 + 4\rho\sigma\,(a\rho + b\sigma - c\tau) - 4\rho^2\sigma^2 \\
&= (2\sigma + \tau)\,(2\rho + \tau)\left(\rho + \sigma + \frac{\rho\sigma}{\tau}\right)^2 - \rho^2\,\sigma^2 \\
&= 2\tau(\rho+\sigma)^3 + \tau^2(\rho+\sigma)^2 + 8\rho\sigma(\rho+\sigma)^2 + (\rho+\sigma)\left(2\rho\sigma\tau + \frac{10\,\rho^2\sigma^2}{\tau}\right) + \frac{4\rho^3\sigma^3}{\tau^2}, \qquad \ldots (104)
\end{aligned}$$

and

$$\begin{aligned}
D &= abc - 2\rho\sigma\tau - a\rho^2 - b\sigma^2 - c\tau^2 \\
&= \frac{1}{\tau}\left\{\tau(\rho+\sigma) + 2\rho\sigma\right\}^2. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (105)
\end{aligned}$$

We shall now refer to the sets ρ, σ, τ corresponding to the m-forms as m-sets.

Lemma 29: *For an m-set* $\rho = \sigma$.

Proof: Suppose $\rho > \sigma$.

Then for small $\delta\rho < 0$, $\delta\sigma > 0$, the set $\rho + \delta\rho$, $\sigma + \delta\sigma$, τ satisfies (103). Let $\delta\rho$ and $\delta\sigma$ satisfy

$$(2\sigma + \tau)\,\delta\rho = -(2\rho + \tau)\,\delta\sigma. \qquad \ldots \quad \ldots \quad \ldots (106)$$

Then

$$\begin{aligned}
(2\sigma+\tau)\,\delta F &= \delta\sigma\Big\{12\tau(\rho+\sigma)^2\,(\sigma-\rho) + 4\tau^2\,(\rho+\sigma)\,(\sigma-\rho) + 32\rho\sigma\,(\rho+\sigma)\,(\sigma-\rho) \\
&\qquad - 8(\rho+\sigma)^2\,(\sigma-\rho)\tau + 2(\sigma-\rho)\left(2\rho\sigma\tau + \frac{10\rho^2\sigma^2}{\tau}\right) + 2(\rho+\sigma)\,(\rho-\sigma)\tau^2 \\
&\qquad + 20(\rho+\sigma)\,\rho\sigma\,(\rho-\sigma) + \frac{12\rho^2\sigma^2}{\tau}\,(\rho-\sigma)\Big\} \\
&= (\sigma-\rho)\left(12\rho^2\sigma + 12\rho\sigma^2 + 4\rho^2\tau + 4\sigma^2\tau + 12\rho\sigma\tau + 2\rho\tau^2 + 2\sigma\tau^2 + \frac{8\rho^2\sigma^2}{\tau}\right)\delta\sigma \\
&< 0,
\end{aligned}$$

and

$$\delta D = 0,$$

so that the set $\rho + \delta\rho$, $\sigma + \delta\sigma$, τ has a smaller $F/4D^{\frac{4}{3}}$ which is a contradiction that proves the lemma.

After the last lemma we can say that the only m-sets are ρ, ρ, τ for which ρ/τ is the value of x in the range $x \geqq 1$, where

$$A(x) = \frac{16\rho^3\tau + 4\rho^2\tau^2 + 32\rho^4 + 4\rho^3\tau + \dfrac{20\rho^5}{\tau} + \dfrac{4\rho^6}{\tau^2}}{4\left\{\dfrac{4\rho^2}{\tau}(\rho+\tau)^2\right\}^{\frac{4}{3}}}$$

$$= \frac{4x^6 + 20x^5 + 32x^4 + 20x^3 + 4x^2}{4\left\{4x^2(1+x)^2\right\}^{\frac{4}{3}}}$$

$$= \frac{1}{4^{\frac{1}{3}}} \cdot \frac{x^2 + 3x + 1}{x^{\frac{2}{3}}(1+x)^{\frac{2}{3}}}$$

attains its minimum value.

Differentiating logarithmically we have

$$\frac{1}{A} \cdot \frac{dA}{dx} = \frac{2x+3}{x^2+3x+1} - \frac{2}{3x} - \frac{2}{3(1+x)} = \frac{(x-1)(2x^2+3x+2)}{3x(x+1)(x^2+3x+1)}$$

and it is easy to see that $x = 1$ gives the minimum. From this it follows that the only m-sets are ρ, ρ, ρ and therefore the only m-forms are $\rho(3, 3, 3, 1, 1, 1)$ which completes the proof of the theorem.

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