SOME INFINITE INTEGRALS INVOLVING BESSEL FUNCTIONS

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1. The object of this paper is to evaluate a few infinite integrals involving modified Bessel functions of the second kind by the methods of Operational Calculus. Some of the results obtained are believed to be new and interesting.

As usual, the notation $\phi(p) \doteqdot f(x)$ means that

$$\phi(p) = p\int_0^\infty e^{-px} f(x)\,dx$$

when the integral is convergent and $R(p) > 0$.

We shall use the following

THEOREM. If

$$f(x) \doteqdot \phi(p) \text{ and } p^{2-2\nu} e^{\frac{a}{p}} f(p) \doteqdot g(x)$$

then

$$\phi(p) = 2pa^\nu \int_0^\infty (p+x)^{-\nu} K_{2\nu}\{2\sqrt{a(p+x)}\}\, g(x)\,dx \quad \ldots \quad \ldots \quad (1)$$

provided that $g(x) = 0\,(x^\mu)$ for small x, where $R(\mu) > -1$, and for large x, $g(x)$ may utmost be of the order of $x^\lambda e^{2\sqrt{ax}}$ if $R(\nu-\lambda-\frac{3}{4}) > 0$, $R(p) > 0$, $R(a) > 0$.

PROOF. Since

$$p^{2-2\nu} e^{\frac{a}{p}} f(p) \doteqdot g(x)$$

and (McLachlan and Humbert, 1950, p. 16)

$$e^{-kx-\frac{a}{x}} x^{2\nu-1} \doteqdot 2p\left(\frac{p+k}{a}\right)^{-\nu} K_{2\nu}\{2\sqrt{a(p+k)}\},\ R(k) > 0,\ R(a) > 0$$

we have, on applying Parseval-Goldstein theorem (Goldstein, 1932)

$$\int_0^\infty e^{-kx} f(x)\,dx = 2a^\nu \int_0^\infty (x+k)^{-\nu} K_{2\nu}\{2\sqrt{a\,(x+k)}\}\, g(x)\,dx.$$

Hence, on replacing k by p, the theorem follows immediately.

2. (i) Taking (McLachlan and Humbert, 1950, p. 35)

$$g(x) = x^{\frac{1}{2}\lambda} I_\lambda(2\sqrt{ax})$$

$$\doteqdot a^{\frac{1}{2}\lambda} p^{-\lambda} e^{\frac{a}{p}}, \quad R(\lambda) > -1$$

we have

$$f(x) = a^{\frac{1}{2}\lambda}\, x^{2\nu-\lambda-2}$$
$$\doteqdot \Gamma(2\nu-\lambda-1)\, a^{\frac{1}{2}\lambda}\, p^{2+\lambda-2\nu}$$
$$= \phi(p), \quad R(2\nu-\lambda-1) > 0, \quad R(p) > 0.$$

Applying the theorem we get

$$p^{1+\lambda-2\nu} = \frac{2a^{\nu-\frac{1}{2}\lambda}}{\Gamma(2\nu-\lambda-1)} \int_0^\infty x^{\frac{1}{2}\lambda}\,(p+x)^{-\nu}\, I_\lambda(2\sqrt{ax}) K_{2\nu}\{2\sqrt{a(p+x)}\}\, dx \quad .. \quad (2)$$

valid, by A.C. (analytic continuation), for $R(\lambda) > -1$, $R(2\nu-\lambda-1) > 0$, $|\arg p| < \pi$, $|\arg a| < \pi$, $a \neq 0$, $p \neq 0$.

(ii) Starting with (Mitra, 1933)

$$\phi(p) = 2pK_\mu(\sqrt{2pa})\, I_\mu(\sqrt{2pa})$$
$$\doteqdot x^{-1} e^{-\frac{a}{x}} I_\mu\left(\frac{a}{x}\right) = f(x),$$

we have on term by term interpretation,

$$p^{2-2\nu} e^{\frac{a}{p}} f(p) = \frac{(\frac{1}{2}a)^\mu\, p^{1-\mu-2\nu}}{\Gamma(\mu+1)}\; {}_0F_1\left(\mu+1\,;\frac{a^2}{4p^2}\right)$$
$$\doteqdot \frac{(\frac{1}{2}a)^\mu\, x^{2\nu+\mu-1}}{\Gamma(\mu+1)\,\Gamma(2\nu+\mu)}\; {}_0F_3\left(\mu+1, \frac{2\nu+\mu}{2}, \frac{2\nu+\mu+1}{2}; \frac{a^2x^2}{16}\right)$$
$$= g(x), \quad R(2\nu+\mu) > 0.$$

Therefore, by the theorem, we have

$$K_\mu(\sqrt{2pa})\, I_\mu(\sqrt{2pa}) = \frac{2^{-\mu}\, a^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(2\nu+\mu)} \int_0^\infty (p+x)^{-\nu}\, x^{2\nu+\mu-1}$$
$$\times\; K_{2\nu}\{2\sqrt{a(p+x)}\}\; {}_0F_3\left(\mu+1, \frac{2\nu+\mu}{2}, \frac{2\nu+\mu+1}{2}; \frac{a^2x^2}{16}\right) dx \quad .. \quad .. \quad (3)$$

valid, by A.C., for $R(2\nu+\mu) > 0$, $|\arg p| < \pi$, $|\arg a| < \pi$, $p \neq 0$, $a \neq 0$.

Since

$$ber_\nu^2(\sqrt{2x}) + bei_\nu^2(\sqrt{2x}) = \frac{(\frac{1}{2}x)^\nu}{\Gamma^2(\nu+1)}\; {}_0F_3\left(\nu+1, \frac{\nu+2}{2}, \frac{\nu+1}{2}; \frac{x^2}{16}\right) \quad .. \quad (4)$$

we have on taking $\nu = \frac{1}{2}$ in (3),

$$K_\mu(\sqrt{2pa})\, I_\mu(\sqrt{2pa}) = \sqrt{a} \int_0^\infty (p+x)^{-\frac{1}{2}}\, K_1\{2\sqrt{a(p+x)}\}$$
$$\times \left\{ ber_\mu^2(\sqrt{2ax}) + bei_\mu^2(\sqrt{2ax}) \right\} dx \quad .. \quad (5)$$

where $R(\mu) > -1$, $|\arg p| < \pi$, $|\arg a| < \pi$, $p \neq 0$, $a \neq 0$.

(iii) Next taking (McLachlan and Humbert, 1950, p. 51)

$$f(x) = x^{-k} e^{-\frac{a}{2x}} W_{k,m}\left(\frac{a}{x}\right)$$

$$\doteqdot 2\sqrt{a}\, p^{k+\frac{1}{2}} K_{2m}(2\sqrt{ap}) = \phi(p), \quad R(p) > 0,$$

we have

$$p^{2-2\nu} e^{\frac{a}{p}} f(p) = p^{2-2\nu-k} e^{\frac{a}{2p}} W_{k,m}\left(\frac{a}{p}\right)$$

$$\doteqdot x^{k+2\nu-2} \sum_{m,-m} \left\{\frac{\Gamma(-2m)(ax)^{m+\frac{1}{2}}}{\Gamma(\frac{1}{2}-m-k)\Gamma(k+2\nu+m-\frac{1}{2})} \times {}_1F_2(\tfrac{1}{2}+m-k\,;\, 1+2m,\, k+2\nu+m-\tfrac{1}{2}\,;\, ax)\right\}$$

$$= g(x), \quad R(k+2\nu\pm m-\tfrac{1}{2}) > 0 \text{ and } 2m \text{ not an integer.}$$

The theorem then gives

$$K_{2m}(2\sqrt{ap}) = a^{\nu-\frac{1}{2}} p^{\frac{1}{2}-k} \int_0^\infty (p+x)^{-\nu} x^{k+2\nu-2} K_{2\nu}\{2\sqrt{a(p+x)}\} \times \left\{\sum_{m,-m} \frac{\Gamma(-2m)(ax)^{m+\frac{1}{2}}}{\Gamma(\frac{1}{2}-m-k)\Gamma(k+2\nu+m-\frac{1}{2})} \times {}_1F_2(\tfrac{1}{2}+m-k\,;\, 1+2m,\, 2\nu+k+m-\tfrac{1}{2}\,;\, ax)\right\} dx \quad \ldots \quad (6)$$

valid, by A.C., for $R(k+2\nu\pm m-\frac{1}{2}) > 0$, $|\arg p| < \pi$, $|\arg a| < \pi$, $a \neq 0$, $p \neq 0$.

A few special cases of (6) are worth mentioning:

(*a*) When $k = m+\frac{1}{2}$, we get

$$K_{2m}(2\sqrt{ap}) = \frac{a^{\nu+m} p^{-m}}{\Gamma(2m+2\nu)} \int_0^\infty (p+x)^{-\nu} x^{2m+2\nu-1} K_{2\nu}\{2\sqrt{a(p+x)}\}\, dx \quad \ldots \quad (7)$$

$$R(m+\nu) > 0, \quad |\arg p| < \pi, \ |\arg a| < \pi, \ p \neq 0, \ a \neq 0.$$

(*b*) If we take $\nu = \frac{1}{2}-k$, and use the relation

$$2K_\nu(2\sqrt{z}) = \sum_{\nu,-\nu} \Gamma(-\nu)\, z^{\frac{1}{2}\nu}\, {}_0F_1(1+\nu\,;z) \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

we have

$$K_{2m}(2\sqrt{ap}) = \frac{2(ap)^{\frac{1}{2}-k}}{\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)} \int_0^\infty (p+x)^{k-\frac{1}{2}} x^{-k-\frac{1}{2}} \times K_{2k-1}\{2\sqrt{a(p+x)}\}\, K_{2m}(2\sqrt{ax})dx \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

$$R(\tfrac{1}{2}-k\pm m) > 0, \ |\arg p| < \pi, \ |\arg a| < \pi, \ p \neq 0, \ a \neq 0.$$

(*c*) Finally, we take $\nu = 2m+\frac{3}{4}$ and $k = -m$ and use the relation

$$2\Gamma(\mu+1)I_\mu(z)K_\mu(z) = \Gamma(\mu)\, {}_1F_2(\tfrac{1}{2}\,;\, 1+\mu,\, 1-\mu\,;\, z^2) + (\tfrac{1}{2}z)^{2\mu}\, \Gamma(-\mu)\, {}_1F_2(\tfrac{1}{2}+\mu\,;\, 1+\mu,\, 1+2\mu\,;\, z^2) \quad (10)$$

we then get

$$K_{2m}(2\sqrt{ap})=\frac{2a^{m+\frac{3}{2}}\,p^{m+\frac{1}{2}}}{\Gamma(2m+\frac{1}{2})}\int_0^\infty (p+x)^{-2m-\frac{3}{2}}\,x^{2m}$$

$$\times\, K_{4m+\frac{3}{2}}\,\{2\sqrt{a(p+x)}\}\, I_{2m}\,(\sqrt{ax})\, K_{2m}\,(\sqrt{ax})\,dx \qquad .. \quad (11)$$

$$R(m)> -\tfrac{1}{4},\ |\arg p\,|<\pi,\ |\arg a\,|<\pi,\ p\neq 0,\ a\neq 0.$$

(iv) Now taking (Meijer, 1936, p. 18)

$$f(x) = x^{-3\mu-\frac{1}{2}}\,e^{-\frac{a}{2x}}\,W_{\mu,\,\mu}\left(\frac{a}{x}\right)$$

$$\doteqdot \frac{2}{\sqrt{\pi}}\,a^{\frac{1}{2}-\mu}\,p^{2\mu+1}\,K^2_{2\mu}\,(\sqrt{ap})$$

$$= \phi(p),\quad R(p)>0,\quad R(a)>0,$$

we have

$$p^{2-2\nu}\,e^{\frac{a}{p}}\,f(p) = p^{\frac{3}{2}-2\nu-3\mu}\,e^{\frac{a}{2p}}\,W_{\mu,\,\mu}\left(\frac{a}{p}\right)$$

$$\doteqdot x^{3\mu+2\nu-\frac{3}{2}}\left\{\frac{\Gamma(-2\mu)(ax)^{\mu+\frac{1}{2}}}{\Gamma(\frac{1}{2}-2\mu)\Gamma(4\mu+2\nu)}\,{}_1F_2(\tfrac{1}{2}\,;\,1+2\mu,\,4\mu+2\nu\,;\,ax)\right.$$

$$\left.+\frac{\Gamma(2\mu)(ax)^{-\mu+\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(2\mu+2\nu)}\,{}_1F_2(\tfrac{1}{2}-2\mu\,;\,1-2\mu,\,2\mu+2\nu\,;\,ax)\right\}$$

$$= g(x),\quad R(\mu+\nu)>0,\quad R(2\mu+\nu)>0.$$

Applying the theorem we have

$$K^2_{2\mu}(\sqrt{ap}) = \sqrt{\pi}\,p^{-2\mu}\,a^{\nu+\mu}\int_0^\infty (p+x)^{-\nu}\,x^{3\mu+2\nu-1}\,K_{2\nu}\,\{2\sqrt{a(p+x)}\}$$

$$\times\left\{\frac{\Gamma(-2\mu)(ax)^{\mu}}{\Gamma(\frac{1}{2}-2\mu)\Gamma(4\mu+2\nu)}\,{}_1F_2(\tfrac{1}{2}\,;\,1+2\mu,\,4\mu+2\nu\,;\,ax)\right.$$

$$\left.+\frac{\Gamma(2\mu)(ax)^{-\mu}}{\Gamma(\frac{1}{2})\Gamma(2\mu+2\nu)}\,{}_1F_2(\tfrac{1}{2}-2\mu\,;\,1-2\mu,\,2\mu+2\nu\,;\,ax)\right\}dx \quad (12)$$

valid, by A.C., for $R(\mu+\nu)>0$, $R(2\mu+\nu)>0$, $|\arg p\,|<\pi$, $|\arg a\,|<\pi$, $p\neq 0$, $a\neq 0$.

In particular, taking $2\mu+\nu = \frac{1}{4}$ and using (8), we get

$$K^2_{2\mu}(\sqrt{ap}) = \frac{2p^{-2\mu}\,a^{\frac{1}{2}-\mu}}{\Gamma(\frac{1}{2}-2\mu)}\int_0^\infty (p+x)^{2\mu-\frac{1}{4}}\,x^{-\mu-\frac{1}{2}}\,K_{2\mu}\,(2\sqrt{ax})$$

$$\times\, K_{4\mu-\frac{1}{2}}\,\{2\sqrt{a(p+x)}\}\,dx \qquad .. \qquad .. \quad (13)$$

$$R(\mu)<\tfrac{1}{4},\ |\arg p\,|<\pi,\ |\arg a\,|<\pi,\ p\neq 0,\ a\neq 0.$$

On the other hand, if we take $\nu = \frac{1}{2}-3\mu$ and use (10), we have

$$K^2_{2\mu}\,(\sqrt{ap}) = \frac{2\sqrt{\pi}\,p^{-2\mu}a^{\frac{1}{2}-\mu}}{\Gamma(\frac{1}{2}-2\mu)}\int_0^\infty (p+x)^{3\mu-\frac{1}{2}}\,x^{-2\mu}$$

$$\times K_{6\mu-1}\{2\sqrt{a(p+x)}\}I_{-2\mu}(\sqrt{ax})K_{2\mu}(\sqrt{ax})dx \quad .. \quad (14)$$

$$R(\mu)<\tfrac{1}{4},\ |\arg p\,|<\pi,\ |\arg a\,|<\pi,\ p\neq 0,\ a\neq 0.$$

(v) Next taking (Meijer, 1936, p. 19)

$$f(x) = x^{-1}e^{-\frac{a}{2x}}W_{\frac{1}{2},\mu}\left(\frac{a}{x}\right)$$

$$\doteqdot 2p^{3/2}\sqrt{\frac{a}{\pi}}K_{\mu+\frac{1}{2}}(\sqrt{ap})K_{\mu-\frac{1}{2}}(\sqrt{ap})$$

$$= \phi(p),\ R(p) > 0,\ R(a) > 0,$$

we have

$$p^{2-2\nu}e^{\frac{a}{p}}f(p) = p^{1-2\nu}e^{\frac{a}{2p}}W_{\frac{1}{2},\mu}\left(\frac{a}{p}\right)$$

$$\doteqdot x^{2\nu-1}\sum_{\mu,\,-\mu}\left\{\frac{\Gamma(-2\mu)(ax)^{\mu+\frac{1}{2}}}{\Gamma(-\mu)\Gamma(\frac{1}{2}+\mu+2\nu)}\,{}_1F_2(\mu\,;\ 1+2\mu,\ 2\nu+\mu+\tfrac{1}{2}\,;\ ax)\right\}$$

$$= g(x),\ R(2\nu\pm\mu+\tfrac{1}{2}) > 0.$$

Hence, by the theorem we get

$$K_{\mu+\frac{1}{2}}(\sqrt{ap})K_{\mu-\frac{1}{2}}(\sqrt{ap}) = a^{\nu}\sqrt{\frac{\pi}{p}}\int_0^{\infty}(p+x)^{-\nu}K_{2\nu}\left\{2\sqrt{a(p+x)}\right\}x^{2\nu-\frac{1}{2}}$$

$$\times\left\{\sum_{\mu,\,-\mu}\frac{\Gamma(-2\mu)(ax)^{\mu}}{\Gamma(-\mu)\Gamma(2\nu+\mu+\frac{1}{2})}\,{}_1F_2(\mu\,;\ 1+2\mu,\ 2\nu+\mu+\tfrac{1}{2}\,;\ ax)\right\}dx \quad .. \quad (15)$$

valid, by A.C., for

$$R(2\nu\pm\mu+\tfrac{1}{2}) > 0,\ |\arg p| < \pi,\ |\arg a| < \pi,\ p\neq 0,\ a\neq 0.$$

(vi) Lastly, taking (Watson, 1944, p. 439)

$$f(x) = \frac{1}{x}\,e^{-\frac{\alpha+\beta}{x}}K_{\mu}\left(\frac{2\sqrt{\alpha\beta}}{x}\right)$$

$$\doteqdot 2pK_{\mu}(2\sqrt{\alpha p})K_{\mu}(2\sqrt{\beta p})$$

$$= \phi(p),\ R(\alpha) > 0,\ R(\beta) > 0,\ R(p) > 0$$

we have, on taking $a = (\sqrt{\alpha}+\sqrt{\beta})^2$ in the theorem

$$p^{2-2\nu}e^{\frac{(\sqrt{\alpha}+\sqrt{\beta})^2}{p}}f(p) = p^{1-2\nu}e^{\frac{2\sqrt{\alpha\beta}}{p}}K_{\mu}\left(\frac{2\sqrt{\alpha\beta}}{p}\right)$$

$$= \tfrac{1}{2}p^{1-2\nu}e^{\frac{2\sqrt{\alpha\beta}}{p}}\sum_{\mu,\,-\mu}\left\{\Gamma(-\mu)\left(\frac{\sqrt{\alpha\beta}}{p}\right)^{\mu}{}_0F_1\left(1+\mu\,;\ \frac{\alpha\beta}{p^2}\right)\right\}$$

$$= \tfrac{1}{2}p^{1-2\nu}\sum_{\mu,\,-\mu}\left\{\Gamma(-\mu)\left(\frac{\sqrt{\alpha\beta}}{p}\right)^{\mu}{}_1F_1\left(\mu+\tfrac{1}{2}\,;\ 2\mu+1\,;\ \frac{4\sqrt{\alpha\beta}}{p}\right)\right\}$$

by virtue of the relation

$$ {}_1F_1(\alpha\,;\ 2\alpha\,;\ 2z) = e^{z}\,{}_0F_1(\alpha+\tfrac{1}{2}\,;\ \tfrac{1}{4}z^2). \quad .. \quad .. \quad .. \quad (16)$$

On finding the original function, we have

$$g(x) = \tfrac{1}{2}x^{2\nu-1} \sum_{\mu,\,-\mu} \left\{ \frac{\Gamma(-\mu)}{\Gamma(2\nu+\mu)} (\sqrt{\alpha\beta}\, x)^{\mu} {}_1F_2(\mu+\tfrac{1}{2}\,;\; 2\mu+1,\, 2\nu+\mu\,;\; 4\sqrt{\alpha\beta}\, x) \right\}$$

$$R(2\nu\pm\mu) > 0.$$

Thus the theorem gives

$$2K_\mu(2\sqrt{\alpha p})K_\mu(2\sqrt{\beta p}) = (\sqrt{\alpha}+\sqrt{\beta})^{2\nu} \int_0^\infty (p+x)^{-\nu} K_{2\nu}\{2(\sqrt{\alpha}+\sqrt{\beta})\sqrt{p+x}\}$$

$$\times x^{2\nu-1} \left\{ \sum_{\mu,\,-\mu} \frac{\Gamma(-\mu)}{\Gamma(2\nu+\mu)} (\sqrt{\alpha\beta}\, x)^{\mu} {}_1F_2(\mu+\tfrac{1}{2}\,;\, 2\mu+1,\, 2\nu+\mu\,;\, 4\sqrt{\alpha\beta}\, x) \right\} dx \quad .. \quad (17)$$

valid, by A.C., for

$R(2\nu\pm\mu) > 0$, $|\arg p| < \pi$, $|\arg \alpha| < \pi$, $|\arg \beta| < \pi$, $p \neq 0$, $\alpha \neq 0$, $\beta \neq 0$.

In particular, when $\nu = \frac{1}{4}$, this result simplifies with the help of the relations

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z} \quad .. \quad .. \quad .. \quad .. \quad (18)$$

$$\Gamma(\tfrac{1}{2}+z)\Gamma(\tfrac{1}{2}-z) = \pi \sec \pi z \quad .. \quad .. \quad .. \quad (19)$$

and

$$\sqrt{\pi}\, \Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}) \quad .. \quad .. \quad .. \quad (20)$$

to

$$K_\mu(2\sqrt{\alpha p})K_\mu(2\sqrt{\beta p}) = \cos(\mu\pi) \int_0^\infty (px+x^2)^{-\frac{1}{2}} e^{-2(\sqrt{\alpha}+\sqrt{\beta})\sqrt{p+x}}$$

$$\times K_{2\mu}(4\sqrt{\sqrt{\alpha\beta}\, x})dx \quad .. \quad .. \quad .. \quad (21)$$

$|R(\mu)| < \frac{1}{2}$, $|\arg p| < \pi$, $|\arg \alpha| < \pi$, $|\arg \beta| < \pi$, $p \neq 0$, $\alpha \neq 0$, $\beta \neq 0$.

3. It is possible to transform the theorem into another form which involves Whittaker function. For example, if we use the integral (Goldstein, 1932)

$$2K_{2\nu}(2\sqrt{x}) = x^{\frac{1}{2}-l} \int_0^\infty e^{-\frac{x}{u}-\frac{1}{2}u} u^{l-2} W_{l,\nu}(u)\, du \quad .. \quad .. \quad (22)$$

in (1), taking $a = 1$, we have

$$\phi(p) = p \int_0^\infty (p+x)^{\frac{1}{2}-l-\nu} g(x) dx \int_0^\infty e^{-\frac{p+x}{u}-\frac{1}{2}u} u^{l-2} W_{l,\nu}(u) du$$

$$= p \int_0^\infty e^{-\frac{p}{u}-\frac{1}{2}u} u^{l-2} W_{l,\nu}(u) du \int_0^\infty (p+x)^{\frac{1}{2}-\nu-l} e^{-\frac{x}{u}} g(x)\, dx$$

on reversing the order of integration, a process which can easily be justified if, for small x, $g(x) = o(x^{\mu})$ where $R(\mu) > -1$, and $g(x) = o(x^{\lambda})$ for large x.

The result may then be stated in the following form: If

$$f(x) \doteqdot \phi(p) \text{ and } p^{2-2\nu} e^{\frac{1}{p}} f(p) \doteqdot g(x)$$

then

$$\phi(p) = p \int_0^\infty e^{-\frac{p}{u}-\frac{1}{2}u} u^{l-2} W_{l,\nu}(u)\, F(p, u)\, du \quad \ldots \quad \ldots \quad (23)$$

where

$$F(p, u) = \int_0^\infty (p+x)^{\frac{1}{2}-\nu-l} e^{-\frac{x}{u}} g(x)\, dx$$

provided that the integrals are convergent.

As an illustration, we take (McLachlan and Humbert, 1950, p. 16)

$$f(x) = e^{-\frac{1}{x}} x^{-m-1}$$
$$\doteqdot 2p^{\frac{1}{2}m+1} K_m(2\sqrt{p}) = \phi(p)$$

we then have

$$p^{2-2\nu} e^{\frac{1}{p}} f(p) = p^{-m-2\nu+1}$$
$$\doteqdot \frac{x^{2\nu+m-1}}{\Gamma(2\nu+m)} = g(x), \quad R(2\nu+m) > 0$$

and

$$F(p, u) = \frac{1}{\Gamma(2\nu+m)} \int_0^\infty (p+x)^{\frac{1}{2}-\nu-l} x^{2\nu+m-1} e^{-\frac{x}{u}}\, dx$$
$$= e^{\frac{p}{2u}} (pu)^{\frac{1}{2}(\nu+m-l-\frac{1}{2})} u W_{-\frac{1}{2}(3\nu+l+m-\frac{3}{2}),\, \frac{1}{2}(\nu+m-l+\frac{1}{2})} \left(\frac{p}{u}\right)$$
$$R(2\nu+m) > 0.$$

on using the integral (Whittaker and Watson, 1935, p. 340)

$$\int_0^\infty t^{\alpha}(t+z)^{\beta} e^{-kt}\, dt = \Gamma(\alpha+1) e^{\frac{1}{2}zk} z^{\frac{1}{2}(\alpha+\beta)}$$
$$\times k^{-\frac{1}{2}(\alpha+\beta+2)} W_{\frac{1}{2}(\beta-\alpha),\, \frac{1}{2}(\alpha+\beta+1)}(kz), \quad R(\alpha+1) > 0.$$

On applying (23), we get

$$2K_m(2\sqrt{p}) = p^{\frac{1}{2}(\nu-l-\frac{1}{2})} \int_0^\infty e^{-\frac{1}{2}\left(u+\frac{p}{u}\right)} u^{\frac{1}{2}(\nu+m+l-\frac{5}{2})}$$
$$\times W_{l,\nu}(u) W_{-\frac{1}{2}(3\nu+l+m-\frac{3}{2}),\, \frac{1}{2}(\nu+m-l+\frac{1}{2})} \left(\frac{p}{u}\right) du \qquad (24)$$

The restriction $R(2\nu+m) > 0$ may now be removed by A.C. When $m = -2\nu$, this yields (22).

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