A THEORY OF RESISTANCE IN POTENTIAL FLOWS

(PARTS I–IV)

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PART I

D'ALEMBERT'S PARADOX AND ITS RESOLUTION

SEC. A

Introduction

1. *A Brief Survey of the Problem*

D'Alembert proved, as is well known, that when a perfect fluid flows past a solid body in potential motion the total pressure thrust of the fluid on the body (excluding buoyancy) vanishes. In actual experience it has always been found that a body moving through air or water does experience a resistance. The treatment of air and water as perfect fluids yielded, in Hydrostatics, results remarkably in accord with experience. The flow past solid bodies was actually observed to be very largely a potential flow. It was therefore expected that the theory of potential flows should account for at least a major part of the resistance experienced by solids. D'Alembert's results pointed to the contrary and were therefore regarded as paradoxical. They seemed to provide a proof of the inadequacy of the classical theory of fluid flows. (Durand, 1935; Goldstein, 1938; Prandtl-Tietjens, 1934; Birkhoff, 1950).

Helmholtz's attempts to obtain resistance in potential flows by means of the theory of free-stream-lines and wakes is well known. (Helmholtz, 1868; Kirchhoff, 1869.) The basic principles of his approach were founded on the observed fact that when a body moves through a fluid it is usually followed by a dead-water region relatively at rest with the body, called the wake, and the observed potential flow is outside the system formed by the body and the wake. If any resistance is experienced by the body in uniform motion it should be the total reaction of the fluid on the body-wake system upon which D'Alembert's paradox would again apply. To avoid this difficulty, known as Brillouin paradox (Villat, 1929), the wake had to be extended to infinity, which resulted in the outside potential flow being delimited by the so-called free-streamlines where the velocity was everywhere the same. These specifications were, however, sufficient to determine the flow uniquely (Riabouchinsky, 1930) and further application of this theory was made in the study of several problems (Leray, 1935). As for resistance, it was applied to the case of a flat plate moving at right angles to itself; a positive result was obtained but it was nearly about half the observed value. (Goldstein, 1938, Vol. I, p. 37.) The work of Levi-civita (1907) and Cisotti (1922) extended the applicability of the theory to the case of curved boundaries and some solutions for circular boundaries have been obtained. (Rosenhead, 1928; Brodetzky, 1926; Wenistein, 1933.)

But the essential objection to Helmholtz theory lies in the admission that the introduction of the body should modify the flow at infinity, i.e., the wake should extend to infinity—a contradiction to observed facts. Besides, when the body is moving the wake must have an infinite energy.

Another phenomenon that observation records, in addition to closed wakes, is the formation of vortices inside the wake. On the basis of this observation a theory of calculation of the *form drag* was constructed and gave much better results. It was, however, found that a symmetrical system of vortex filaments (in two-dimensional flow) could not be stable. The stable arrangement, known as the 'Karman vortex street' would introduce asymmetry and thereby disturb the steady character of the flow (Lamb, 1932; Milne Thomson, 1949; Durand, 1935; Goldstein, 1938.)

Meanwhile it was contended that a real fluid like air or water has an appreciable coefficient of viscosity and that D'Alembert's paradox was possibly a consequence of the neglect of viscosity. Attempts to introduce viscosity into the theory of fluid flows resulted in the modification of Euler's equations of motion into the Navier-Stokes equations. The order of the equation was raised and consequently additional boundary conditions were looked for. After a long and protracted controversy (Goldstein, 1938, Vol. II, Appendix) it was finally decided that in the case of viscous fluids there should be no tangential component of the velocity on the surface of a body; in other words 'there should be no slip'.

The intractability of the Navier-Stokes equations thwarted all attempts at exact solutions except in a few simple cases, and perhaps the Hagen-Poiseuille flow in capillary tubes is the strongest available evidence in favour of the no-slip hypothesis. Motions past spheres or cylinders were solved under highly restrictive conditions as in Stokes or Oseen's solutions. (Lamb, 1932.)

Prandtl (1904) effected a sort of compromise between observation and theory in what is known as the Boundary Layer Theory. The main arguments of the theory are as follows:—'Let us admit the fact that the flow of a real fluid past a solid body is mainly a potential flow except in a thin layer round the body and possibly the wake, where only viscosity comes into play. There should be no slip on the boundary but, instead of a sudden discontinuity into the main-stream potential flow, let us conceive of a thin layer of fluid where a quick transition from the no-velocity on the surface to the main-stream velocity takes place.' The theory explained the separation of the flow and the formation of vortices but its mathematical aspects were never regarded as too satisfactory (Krzywoblocki, 1953). It provides, however, the only basic method for the calculation of the skin-friction drag at present.

In practical aeronautics to-day, calculations are carried out in parts. The form-drag is calculated from the theory of vortices and the skin-friction-drag is calculated in an approximate manner by means of the Boundary Layer Theory. (Goldstein, 1938; Durand, 1935.)

Thus the problem of fluid-resistance cannot yet be said to be in all too happy a state.

2. *The physical background and the justification for a new outlook.*

Let us turn for a moment to the basic facts of experience and observations as recorded in the photographs taken at various laboratories. (Goldstein, 1938; Durand, 1935; Birkhoff, 1950.) These have been very ably summarised by Goldstein, 1938 (Vol. I, Chaps. I and II). They are as follows:—

(i) A real fluid like air or water has an appreciable amount of viscosity.

(ii) A resistance is experienced when a body moves steadily through a real fluid.

(iii) At start, the flow is practically wholly potential without wakes. (We shall henceforth characterise this as the *Primary Potential* flow.)

(iv) When a steady state is attained, the relative-flow picture (i.e. when the camera is fixed to the body) reveals usually a wake of finite dimensions and a potential flow outside the body-wake system. Inside the wake vortices are seen to exist. Separation takes place at some point wherefrom starts the boundary of the wake as in Fig. 1 (after

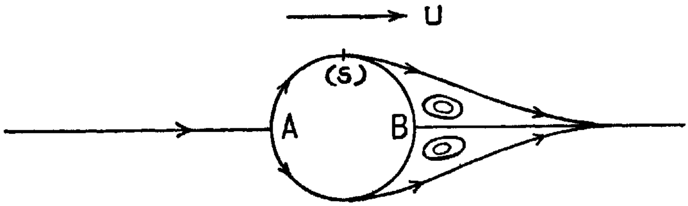

FIG. 1.

Goldstein, 1938, Vol. I, p. 64.) The separation point S, changes position and approaches the forward stagnation point A closer and closer as the velocity of the main flow is increased.

(v) Pressure measurements round circular cylinders and spheres even in flows with little wakes reveal a higher pressure in the foreparts than at the back. (See Goldstein, Vol. I, p. 24, Figs. 4, 5.)

(vi) The flow, a little away from the body, has been shown by computation to agree very closely with the theoretically calculated potential flow (Goldstein, 1938, Vol. II, p. 551).

(vii) By appropriate sucking at points of the body-surface it is possible to retard the separation of the flow and make it coincide very nearly with the primary potential flow, so that the non-zero surface-velocity of the potential flow shows itself almost immediately beside the surface.

Before proceeding further, it is necessary to remember that Jeffreys (1930) had given a theoretical proof that the only potential motion possible under the no-slip condition is a rigid body motion where the fluid and the solid move as a whole. It has been our convention, so far, that for $\mu \neq 0$ we must impose the no-slip condition on the boundary. This contradicts the possibility of potential flows when viscosity is taken into account.

Nature, however, reveals that in the case of real fluids where viscosity is not of a negligible order, potential motions with or without wakes do take place. Possibly Nature avoids Jeffrey's condition by forming an indefinitely thin layer of sticking fluid over which slipping takes place—as is generally envisaged in the Boundary Layer Theory. We shall see, as we proceed, that it is possible to resolve some of the long-standing paradoxes of fluid mechanics by discarding the 'hypothesis of no slip' on the surface in the case of real fluids.

3. *The New Approach and its Consequences*

With the above in view, we start with the assumptions—

(i) that in real fluids, in motion, $\mu \neq 0$.

(ii) that, despite (i), the fluid may have a tangential velocity on the body-surface.

We know that the general equations of motion for a viscous fluid are represented by the Navier-Stokes equations*. It is easy to see that all potential flows ($\nabla^2\psi = 0$) are solutions of these equations. This whole set of solutions stand barred when the no-slip hypothesis is brought in. Hence when the no-slip condition is no longer adhered to, potential flows of viscous fluids become possible. We shall develop our results on the basis of this possibility.

Consequences

As a consequence of the above we shall show without further assumptions or approximations that:—

1. A solid moving in a real fluid does experience a positive resistance, even when the flow has no circulation or vortices. This resolves D'Alembert's paradox.
2. The resolution of D'Alembert's paradox removes all objections to the possibility of *finite wakes* and hence resolves Brillouin's paradox.
3. Even in a primary potential flow, i.e., a flow without wakes, vortices, or circulation, the thrust in the foreparts of the obstacle does exceed the same in the hind parts.
4. That the form-drag and the skin-friction drag both exist in a definite way in every case of primary potential flow.
5. That it is possible to construct a general formula for the total drag on a cylinder in every case of flow, with or without wakes.

These constitute the major achievements of our approach.

Normal Thrust and Pressure

The usual concept of pressure in a fluid holds only when the fluid is assumed to be perfect or is at rest if viscous. With the introduction of viscosity and the fluid in motion we must replace the simple pressure concept by that of the normal and tangential tractions across a surface which may change with the orientation of the surface. The symbol 'p' used in the Navier-Stokes equations can no longer be interpreted as the normal thrust per unit area in every direction; it will merely represent the average normal stress across three orthogonal surfaces through a point of the fluid.

In the case of two-dimensions, 'p', therefore, no longer represents the normal action per unit length. That will, in general, be different from 'p' and we shall denote it by p_n.

In all pressure measurements with airfilled tubes like the Pitot tubes, the effective pressure on the gas is the normal thrust per unit area and hence must be a measure of p_n instead of p. Hence, all actual records of 'pressure' measurements will be interpreted to measure p_n and not p.

In section B, Part I, we shall work out a few results in support of our conclusions.

Sec. B

The Resolution of D'Alembert's Paradox

4. *General Formulae.*

Let us consider a primary two-dimensional potential flow past any fixed cylinder. We assume that the flow takes place in the x-y plane and that ϕ_1, ψ_1

* In two dimensions it is

$$\frac{\partial}{\partial t}\nabla^2\psi+\psi_y\nabla^2\psi_x-\psi_x\nabla^2\psi_y=\nu\nabla^4\psi$$

where ψ represents the stream-function.

represent the velocity-potential and the stream-function respectively for the same, and that, ψ_1 vanishes on the surface S of the body. If

$$W_1 \equiv \phi_1 + i\psi_1 \qquad \ldots \qquad (4.1)$$

represents the complex potential, the whole of the x-y plane with the exception of the area occupied by the body is mapped into the W_1 plane with a slit on the ϕ_1-axis and any point in the x-y plane outside the body or on it can be represented by a pair of orthogonal curvilinear co-ordinates (ϕ_1, ψ_1). Let us assume that ϕ_1 increases from left to right and ψ_1 increases from below upwards.

If ds_1, ds_2 be the elements of length normal to $\phi_1 =$ constant and $\psi_1 =$ constant, respectively, at (ϕ_1, ψ_1) we have

$$\left.\begin{aligned} ds_1 &= \frac{1}{q_1}\, d\phi_1 \\ ds_2 &= \frac{1}{q_1}\, d\psi_1 \end{aligned}\right\} \qquad \ldots \qquad (4.2)$$

where q_1 represents the speed of flow at (ϕ_1, ψ_1).

Stresses.

In the general orthogonal system $\alpha =$ const., $\beta =$ const., $\gamma =$ const. in three dimensions, the stress components are given by

$$\left.\begin{aligned} p_{\alpha\alpha} &= -p+2\mu e_{\alpha\alpha} \\ p_{\beta\beta} &= -p+2\mu e_{\beta\beta} \\ p_{\alpha\beta} &= p_{\beta\alpha} = \mu e_{\alpha\beta} \\ &\text{etc.} \ldots \end{aligned}\right\} \qquad \ldots \qquad (4.3)$$

where

$$\left.\begin{aligned} e_{\alpha\alpha} &= \frac{1}{h_1}\frac{\partial u}{\partial \alpha} + \frac{v}{h_1 h_2}\frac{\partial h_1}{\partial \beta} + \frac{w}{h_3 h_1}\frac{\partial h_1}{\partial \gamma} \\ e_{\beta\beta} &= \frac{1}{h_2}\frac{\partial v}{\partial \beta} + \frac{w}{h_2 h_3}\frac{\partial h_2}{\partial \gamma} + \frac{u}{h_1 h_2}\frac{\partial h_2}{\partial \alpha} \\ e_{\alpha\beta} &= \frac{h_2}{h_1}\frac{\partial}{\partial \alpha}\left(\frac{v}{h_2}\right) + \left(\frac{h_1}{h_2}\right)\frac{\partial}{\partial \beta}\left(\frac{u}{h_1}\right) \\ &\text{etc} \ldots \end{aligned}\right\} \qquad \ldots \qquad (4.4a)$$

u, v, w representing the velocity components normal to the surfaces $\alpha =$ const., $\beta =$ const. and $\gamma =$ const. respectively and where

$$ds^2 = (h_1 d\alpha)^2 + (h_2 d\beta)^2 + (h_3 d\gamma)^2 \qquad \ldots \qquad (4.4b)$$

Hence for the two-dimensional ϕ_1, ψ_1 system where $\phi_1 \equiv \alpha$, $\psi_1 \equiv \beta$, $\gamma \equiv z$, $\left(\frac{\partial}{\partial z} = 0\right)$ we have

$$\left.\begin{aligned} h_1 &= h_2 = \frac{1}{q_1},\ h_3 = 1 \\ u &= q_1,\ v = 0,\ w = 0 \end{aligned}\right\} \qquad \ldots \qquad (4.5)$$

and consequently

$$\left.\begin{aligned} p_{\alpha\alpha} &\equiv p_{\phi_1\phi_1} = -p+\mu\frac{\partial q_1^2}{\partial \phi_1} \\ p_{\beta\beta} &\equiv p_{\psi_1\psi_1} = -p-\mu\frac{\partial q_1^2}{\partial \phi_1} \\ p_{\alpha\beta} &\equiv p_{\phi_1\psi_1} = \mu\frac{\partial q_1^2}{\partial \psi_1} \end{aligned}\right\} \quad \ldots \quad (4.6)$$

On the surface of the body $\psi_1 =$ const. and hence the components of the stress exerted by the fluid on the body are (Fig. 2).

(i) a normal stress $-p_{\psi_1\psi_1}$ per unit length towards the body;

(ii) a tangential stress $-p_{\psi_1\phi_1}$ per unit length in the positive sense of flow.

Denoting these two components of the surface action by p_n and p_t respectively, we have,

$$p_n = -p_{\psi_1\psi_1} = p+\mu\frac{\partial q_1^2}{\partial \phi_1} \quad \ldots \quad (4.7)$$

$$p_t = -p_{\psi_1\phi_1} = -\mu\frac{\partial q_1^2}{\partial \psi_1} \quad \ldots \quad (4.8)$$

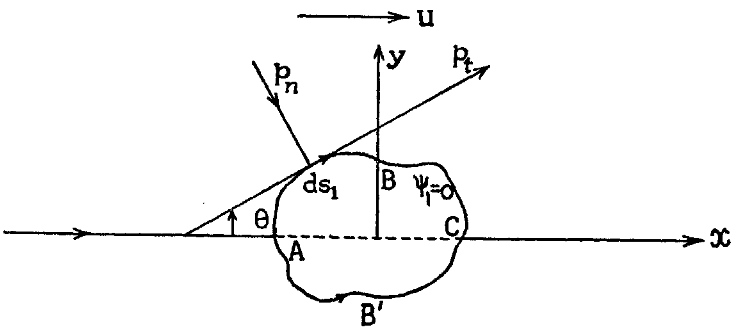

FIG. 2.

If now X and Y denote the x- and y-components of the total reaction we have (Fig. 2)

$$X = \int_{ABCB'A} (p_t \cos\theta + p_n \sin\theta)ds_1\,;\quad Y = \int_{ABCB'A} (p_t \sin\theta - p_n \cos\theta)ds_1.$$

Now,

$$\frac{dx}{ds_1} = \cos\theta,\ \frac{dy}{ds_1} = \sin\theta.$$

Hence substituting from (4.7) and (4.8) and remembering that

$$\int p\,dy = \int p\,dx = 0 \quad \ldots \quad (4.9)$$

by D'Alembert's paradox, we have

$$\left.\begin{aligned} X &= \mu \oint \left\{ \frac{\partial q_1^2}{\partial \phi_1} dy - \frac{\partial q_1^2}{\partial \psi_1} dx \right\} \\ Y &= -\mu \oint \left\{ \frac{\partial q_1^2}{\partial \psi_1} dy + \frac{\partial q_1^2}{\partial \phi_1} dx \right\} \end{aligned}\right] \quad \ldots \quad \ldots \quad \ldots \quad (4.10)$$

where the integrals are taken round the contour of the body in the clockwise sense in each case.

Modification of Blasius' Theorem.

Equations (4.10) provide the corresponding modification of the Blasius' theorem. In the general case where (4.9) is not assumed to hold, we have, in fact,

$$X - iY = -\tfrac{1}{2}\rho i \oint \left| \frac{dW_1}{dz} \right|^2 dz + 2\mu i \oint_{W_1 = \overline{W}_1} \frac{\partial q_1^2}{\partial \overline{W}_1} d\bar{z}, \quad \ldots \quad \ldots \quad (4.11)$$

where $\overline{W}_1$ and $\bar{z}$ represent the complex conjugates of W_1 and z.

Form-drag and Skin-friction-drag.

In every symmetrical case where the flow is parallel to the x-axis, $Y = 0$ and the total resistance X (per unit length of the cylinder) is contributed by both p_n and p_t. These two contributions are technically known as the *form-drag* and the *skin-friction-drag* respectively. Denoting them by X_N and X_t we have

$$X_N = \mu \oint \frac{\partial q_1^2}{\partial \phi_1} dy \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.12)$$

$$X_t = -\mu \oint \frac{\partial q_1^2}{\partial \psi_1} dx \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.13)$$

where (4.9) is assumed to hold. Of course,

$$X = X_N + X_t \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.14)$$

5. *Particular Cases.*

A. *Circular Cylinder* (radius $= a$)

$$W_1 = U\left(z + \frac{a^2}{z}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.1)$$

$$q_1^2 = U^2 \left\{ 1 - \frac{2a^2}{r^2} \cos 2\theta + \frac{a^4}{r^4} \right\} \quad \ldots \quad \ldots \quad \ldots \quad (5.1a)$$

On $r = a$,

$$q_1 \frac{\partial}{\partial \psi_1} = \frac{\partial}{\partial r}$$

$$q_1 \frac{\partial}{\partial \phi_1} = -\frac{1}{a}\frac{\partial}{\partial \theta} \text{ (as } \theta \text{ decreases with } s_1\text{)}$$

hence from (4.10), (4.12) and (4.13) we have,

$$\left.\begin{aligned} X &= 8\pi\mu U(\neq 0) \\ X_N &= 4\pi\mu U \\ X_t &= 4\pi\mu U \\ Y &= 0. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (5.2)$$

Besides, from (4.7) and (4.8)

$$p_n = p - \frac{4\mu U}{a}\cos\theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.2a)$$

$$p_t = \frac{4\mu U}{a}\sin\theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.2b)$$

where p is given by Bomoulli's equation, so that

$$p = \Pi - \tfrac{1}{2}\rho q_1^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.2c)$$

Π denoting the stagnation pressure (Fig. 3).

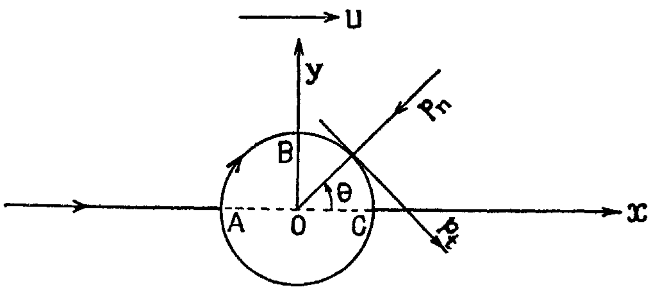

FIG. 3.

Observations

(i) From (5.2) it is clear that a circular cylinder in a primary potential flow without wakes or vortices will actually experience a force towards the direction of the flow. There is a positive form-drag and a skin-friction drag and the two are equal.

(ii) (5.2*a*) shows obviously (Fig. 3) that the normal thrust at the foreparts of the cylinder where $\frac{\pi}{2} \leq \theta \leq \pi$ exceeds the same at corresponding points on the hind parts where $0 \leq \theta < \frac{\pi}{2}$. This verifies a well-known item of experience without bringing in wakes.

B. *Elliptic Cylinder: Flow parallel to the major axis*

In this case, the primary potential flow is represented by

$$W_1 = U\left[b\sqrt{\frac{a+b}{a-b}}\,e^{-\zeta} + cz\right] \quad \ldots \quad \ldots \quad \ldots \quad (5.3)$$

where

$$z = c\cosh\zeta \equiv c\cosh(\xi + i\eta). \quad \ldots \quad \ldots \quad \ldots \quad (5.4)$$

The surface of the body is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \dots \quad \dots \quad \dots \quad \dots \quad (5.4a)$$

where

$$\left.\begin{aligned} \xi &= \alpha \text{ (const.)} \\ a = c\cosh\alpha,\ b &= c\sinh\alpha,\ c^2 = a^2-b^2 \end{aligned}\right\} \quad \dots \quad \dots \quad (5.4b)$$

and

Now, on a $\xi=$ const.

$$\frac{\partial}{\partial n}=\frac{1}{h}\frac{\partial}{\partial\xi},\quad \frac{\partial}{\partial s}=-\frac{1}{h}\frac{\partial}{\partial\eta}$$

s being measured from left to right, so that

$$h^2=c^2(\sinh^2\xi+\sin^2\eta) \quad \dots \quad \dots \quad \dots \quad (5.4c)$$

Now for such a flow

$$q_1^2=\frac{U^2}{h^2}\left[h^2+ab\left(\frac{a+b}{a-b}\right)e^{-2\xi}-b(a+b)\cos 2\eta\right] \quad \dots \quad \dots \quad (5.5)$$

but, in general,

$$\left.\begin{aligned} q_1\frac{\partial}{\partial\psi_1}=\frac{\partial}{\partial n},\quad q_1\frac{\partial}{\partial\phi_1}&=\frac{\partial}{\partial s} \\ \psi_1 &= \text{const.} \end{aligned}\right\} \quad \dots \quad \dots \quad \dots \quad (5.6)$$

on

Hence,

$$X=-\mu\oint\frac{1}{q_1h}\left[\frac{\partial q_1^2}{\partial\xi}dx+\frac{\partial q_1^2}{\partial\eta}dy\right]$$

which gives on simplification (for $b\neq 0$)

$$X=4\pi\mu U\left(1+\frac{b}{a}\right) \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (5.7)$$

Also, from (4.12), (4.13), we have

$$\left.\begin{aligned} X_N &= 2\pi\mu U\left(1+\frac{b}{a}\right) \\ X_t &= 2\pi\mu U\left(1+\frac{b}{a}\right) \end{aligned}\right\} \quad \dots \quad \dots \quad \dots \quad (5.7a)$$

We notice that there is again a positive resistance and the form and skin-friction drags are equal in this case also.

In the limiting case when $b\to a$ either $\alpha\to\infty$ or $c\to 0$, but the ellipse becomes a circle and the result agrees with those for a circular cylinder for which the drags are independent of the radius (as is expected from dimensional considerations).

6. *Solid Sphere*

When a sphere moves through a fluid with uniform velocity the motion is three-dimensional but has an axis of symmetry. Calculations for the total resistance etc. can be carried out in a manner similar to the above as the primary potential motion for such a body is known; and the stress components can be calculated from (4.3), (4.4) without difficulty. Measuring θ from the back stagnation point in an anticlockwise sense we have the following results :—

$$\left.\begin{aligned} \phi_1 &= Ur\sin\theta+\tfrac{1}{2}U\frac{a^3}{r^2}\cos\theta \\ u &\equiv \frac{\partial\phi_1}{\partial r} = U\cos\theta\left(1-\frac{a^3}{r^3}\right) \\ v &\equiv \frac{1}{r}\frac{\partial\phi_1}{\partial\theta} = -U\sin\theta\left(1+\tfrac{1}{2}\frac{a^3}{r^3}\right) \end{aligned}\right\} \qquad (6.1)$$

$$q_1^2 = U^2\left[\cos^2\theta\left(1-\frac{a^3}{r^3}\right)^2+\sin^2\theta\left(1+\tfrac{1}{2}\frac{a^3}{r^3}\right)^2\right] \qquad (6.1a)$$

$$p_n = p-\frac{6\mu U}{a}\cos\theta \qquad (6.2a)$$

$$p_t = \frac{3\mu U}{a}\sin\theta \qquad (6.2b)$$

$$X_N = -\int_S p_n\cos\theta\,dS = 8\pi\mu Ua \qquad (6.3a)$$

$$X_t = \int_S p_t\sin\theta\,dS = 8\pi\mu Ua \qquad (6.3b)$$

$$X \equiv X_N+X_t = 16\pi\mu Ua \qquad (6.4)$$

Observations

(i) The remarks made on the results for a circular cylinder hold here also though the results are not quantitatively the same.

(ii) It should be remembered that Stokes' famous formula for the resistance on a sphere gives the value $6\pi\mu Ua$. Thus there is an agreement in order though not in value. There is, however, every reason why the two results should differ. Stokes' formula was derived on the assumption that the motion is very slow so that the inertia terms were neglected and, further, the no-slip condition on the boundary was also imposed. In our case, there is no approximation, but the existence of the potential flow is assumed (and consequently the no-slip condition discarded). A primary potential flow round a sphere is unlikely to be stable, as we shall see in Part II, but if such a flow holds at any time there is hardly any escape from the resistance being $16\pi\mu Ua$.

7. *Further possibilities*

(I) *Flows with circulation or vortices.*—It is obviously quite an easy matter to extend the calculations to those cases where the potential flow is attended with a circulation or vortices. We have merely to modify the complex potential W_1 but the general formulae derived in Art. 4 hold *mutatis mutandis*. The singularities of the field do not create any difficulty as integration on the body surface only is involved.

(II) *Potential flows in three-dimensions.*—For any three-dimensional system for which the primary potential flow is known the stress components on the body surface can always be calculated from (4.3), (4.4*a*) by a suitable choice of orthogonal co-ordinates or even in a Cartesian system. Hence it is always possible to calculate either the form-drag or the skin-friction drag for any body whatsoever, under any type of general motion, provided the primary potential flow round it is solved. I believe the procedure is direct enough to need no illustration.

Besides, it appears that the method opens up an extensive field of investigation specially to the advantage of aeronautical practice, where the flow round the aerofoil is supposed to be very nearly a potential flow. If the flow is not primary potential (with or without isolated vortices) it should be followed by a wake and we shall see in Part IV how the calculations can be extended to the cases of flows with wakes.

Abstract

This paper is divided into Sections A and B. Section A contains a critical survey of the existing theories of fluid resistance and pointing out some of the major inconsistencies explains that the age-old paradoxes of the Classical theory really arise out of a neglect of viscosity coupled with the hypothesis of no slip on the boundary. It further, points out that a change of outlook more in agreement with the facts of observation brings in a remarkable simplification into the whole problem and at the same time resolves effectively D'Alembert's paradox and Brillouin's paradox.

Section B gives a general formula—a modification of Blasins' formula—for the calculation of the resistance on any cylinder moving in a real incompressible fluid with uniform velocity and the results thereof, for the resistance of a circular cylinder, an elliptic cylinder and a sphere.

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Part II

Dissipation in Potential Motion

1. It is well known that the total rate of dissipation, i.e., the rate at which mechanical energy is disappearing, is given by (Lamb, 1932, pp. 379–381).

$$D = \mu \int (\xi^2+\eta^2+\zeta^2)\, dv - \mu \int \frac{\partial q^2}{\partial n}\, dS + 2\mu \int \begin{vmatrix} l, & m, & n \\ u, & v, & w \\ \xi, & \eta, & \zeta \end{vmatrix} dS . \quad .. \quad (1.1)$$

where

$D =$ the total rate of dissipation,

$u, v, w =$ the components of velocity,

$\xi, \eta, \zeta, =$ the components of vorticity,

$q =$ the speed of motion,

the first integral is taken over the entire range of the fluid, the second and third integrals are taken over all finite boundaries S (irrespective of slip or no slip on the surface) and l, m, n denote the direction-cosines of the normal drawn at a point of S into the fluid.

In the particular case where the flow is purely potential ξ, η, ζ vanish everywhere and the equation (1.1) reduces to

$$D = -\mu \int \frac{\partial q^2}{\partial n} \, . \, dS \, . \qquad \qquad (1.2)$$

$\frac{\partial}{\partial n}$ denoting the derivative taken along the normal drawn into the fluid.

If slipping is not permitted on the surface the dissipation given by (1.2) will vanish. But it is well known that D in general, can be written as (Lamb, *ibid.*, p. 381).

$$\mu \int \left[2\,(a^2+b^2+c^2)+f^2+g^2+h^2\right] dv \qquad \qquad (1.3)$$

where a, b, c, f, g, h denote the components of strain and the integration is taken over the whole volume of the fluid. Now the integrand in (1.3) is an essentially positive quantity and hence D can vanish only if

$$a = b = c = f = g = h = 0 \qquad \qquad (1.4)$$

that is, only if the motion reduces to a purely rigid-body motion. This is consistent with Jeffreys' result (Jeffreys, 1930). Hence in a potential flow, which is not merely one of rigid translation, *dissipation must exist* and can never be equal to zero. As potential flow of a real fluid past a solid obstacle is a fact, dissipation must be a necessary feature of such flows and, hence, the integral (1.2) never vanishes—in conformity with our rejection of the no-slip condition (Part I). Thus having allowed slip on the surface for the potential flow of a *real* fluid, we must admit that such a flow will always be attended with a dissipation given by (1.2).

It must be noted that (1.1) or for the matter of that (1.2), has been actually arrived at by converting the volume integral (1.3) into the surface integrals in parts. Hence any singularities in the flow will modify (1.1) and hence also (1.2) and the corresponding corrections must be introduced.

2. *Dissipation in primary potential flows past cylindrical boundaries.*

Let us suppose that we are considering the potential flow of a real fluid past a solid cylinder and that the motion is defined by the complex potential $W_1 = \phi_1 + i\psi_1$ as in Art. 4, Part I. As the surface of the body in a potential flow must be a streamline, and for such a surface (cf. Eqns. (4.2), I)

$$q_1 \frac{\partial}{\partial \psi_1} = \frac{\partial}{\partial n}$$

$$\frac{1}{q_1} d\phi_1 = ds,$$

we can write (1.2) in the form

$$D = -\mu \oint \frac{\partial q_1^2}{\partial \psi_1} . d\phi_1 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

where q_1 represents the speed and the integral is taken round the entire contour of the body.

The quantity D given by (1.2) or (2.1) can be easily evaluated for a sphere, a circular cylinder or an elliptic cylinder as the stream-functions for such flows are known.

We have the following results:

I. Circular cylinder (radius a)

$$D = 8\pi\mu U^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

II. Elliptic cylinder (axes $2a$, $2b$)

$$D = 2\pi\mu U^2 \left(1+\frac{b}{a}\right)^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

III. Sphere (radius a)

$$D = 12\pi\mu a U^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

where U represents the velocity of the fluid at infinity parallel to the axis of x.

3. *Maintenance of Steady Relative Motion.*

When a body moves through an otherwise still fluid with uniform speed U, the fluid in the neighbourhood of the body is thrown into motion and at any instant the body is followed by a field of disturbed fluid surrounding it, the disturbance dying away at an infinite distance from the body. Now, for a real fluid ($\mu \neq 0$) the motion will entail a dissipation of energy. The picture relative to the body cannot become steady unless the rate of supply of energy to the fluid just balances the rate of dissipation and in that case the rate of change of kinetic energy with the motion will vanish. This result can be established in the following general manner :

Let $\Gamma_{\nu x}$, $\Gamma_{\nu y}$, $\Gamma_{\nu z}$ be the components of the stress per unit area at any point of a surface bounding the fluid exerted by the surface on the fluid. Then the total work done on the fluid per unit time is, in general, given by

$$\int_{S+\Sigma} (\Gamma_{\nu x} . u + \Gamma_{\nu y} . v + \Gamma_{\nu z} . w) \, dS \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

the integral being taken over the finite body surface S as well as the surfaces Σ at infinity. Following the principle that the fluid at infinity remains undisturbed when the body moves through it, the integral over Σ vanishes and hence (3.1) reduces to

$$\int_{S} (\Gamma_{\nu x} . u + \Gamma_{\nu y} . v + \Gamma_{\nu z} . w) \, dS \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

the integral being now taken over the body-surface only.

Now, (3.2) can be written as

$$\sum_{u,\,v,\,w} \int u \left(\frac{\partial \Gamma_{xx}}{\partial x} + \frac{\partial \Gamma_{xy}}{\partial y} + \frac{\partial \Gamma_{xz}}{\partial z} \right) dv + \int \Phi \, dv$$

where Φ represents the rate of dissipation per unit volume. Hence, substituting in the first term from the Navier-Stokes equations we have

$$\int_S (\Gamma_{\nu x} \,.\, u + \Gamma_{\nu y} \,.\, v + \Gamma_{\nu z} \,.\, w) \, dS = \frac{1}{2} \rho \frac{D}{Dt} \int q^2 \, dv + D$$

where D is given by (1.3) and $\frac{1}{2}\rho q^2$ represents the kinetic energy per unit volume. For a steady relative picture we must have

$$\frac{D}{Dt} \int \frac{1}{2} \rho q^2 \,.\, dv = 0$$

and hence

$$\int (\Gamma_{\nu x} \,.\, u + \Gamma_{\nu y} \,.\, v + \Gamma_{\nu z} \,.\, w) \, dS = D. \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

Now, if R be the resistance experienced by a body moving with uniform velocity U, the rate of work done on the fluid is $R.U$ and hence, finally, we must have

$$R.U = D. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

Equation (3.4) must be regarded as very fundamental for the maintenance of a steady relative picture and we shall impose this as a necessary condition for the preservation of a steady flow.

In Part I, Art. 5, we have calculated the total resistance R in a few symmetrical cases of primary potential flow and represented the same by X. Also equations (2.2), (2.3), (2.4) give us the corresponding values of D. The results are shown in the following table :—

	Body	RU	D
1.	Circular cylinder	$8\pi\mu U^2$	$8\pi\mu U^2$
2.	Elliptic cylinder	$4\pi\mu U^2 \left(1+\frac{b}{a}\right)$	$2\pi\mu U^2 \left(1+\frac{b}{a}\right)^2$
3.	Sphere	$16\pi\mu a U^2$	$12\pi\mu a U^2$

From the above table it becomes clear that whereas equation (3.4) is obeyed for a circular cylinder, it is not so for either an elliptic cylinder or a sphere. Hence, on the assumption of the primary potential motion, the flow round a circular cylinder only can be maintained steadily. That round a sphere or an elliptic cylinder cannot be so maintained and if started at any time it will change soon. The additional energy supply to the fluid by the work of the body-forces in the two latter cases will, it appears, tend to create a wake and enlarge it until the two rates just balance and equation (3.4) is satisfied for the altered motion. Our next enquiry therefore will be to look for such a situation. With a view to that we shall develop the theory of closed wakes bounded by potential flows in Part III of this work.

Abstract

This paper contains a discussion of the dissipation in potential flow of a real incompressible fluid and establishes the condition for the maintenance of a steady flow. Calculations show that the classical potential flow round a circular cylinder can be maintained steadily but that round an elliptic cylinder or a sphere cannot be so maintained. This result is interpreted to indicate the formation of wakes.

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Part III

Closed Wakes in Two Dimensions.

1. *Introduction*

The Classical wake or the Helmholtz wake being supposed to extend to infinity the position was never regarded as satisfactory as was observed in Part I. Closed wakes are a reality that could hardly be denied. For a long time there seemed to have developed a sort of resignation to the situation for lack of new light. Recently, however, the question seems to have been reopened and several attempts have already been made to introduce and discuss the question of closed wakes. (Klose, 1941; Kolscher, 1940; Southwell and Vasey, 1946; Manarini, 1948; Allen, 1949; Lighthill, 1949; Kármán, 1949).

So long, however, as D'Alembert's paradox remained unresolved objections against closed wakes could not be overridden—so far, at least, as motion with uniform velocity is concerned. In Part I of our present theory we have resolved D'Alembert's paradox and so the chief argument *against* closed wakes has already been removed. We shall therefore proceed with the analysis of closed wakes in potential flows.

2. *The Requirements.*

Let us consider a two-dimensional primary potential flow past a fixed cylinder and let W_1 represent the complex potential for the flow. The character of the primary potential flow is that the stream-line $\psi_1 = 0$ where

$$W_1 = \phi_1 + i\psi_1 \qquad (2.1)$$

enclasps the body surface completely, meeting at the front stagnation point A and leaving it at the back stagnation point B as shown by the thick line Fig. 1. The

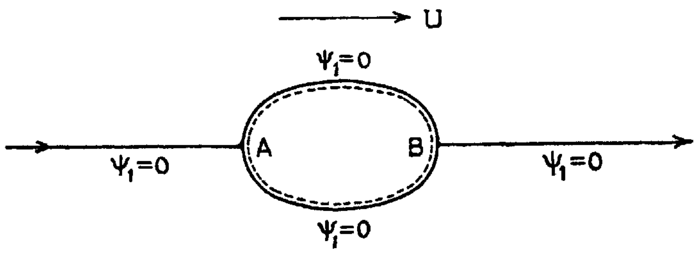

Fig. 1

dotted line, inside, indicates the body surface. In the case of motion with a wake we would look for a flow where one stream-line should usually meet the body surface at the front stagnation point A, separate and run along the surface of the body for some distance (AC, AC'), leave the body surface at two points C, C' on the two sides and meet again at a distance from the body, say at D. The whole area enclosed in the loop thereby formed, will then contain the body and the wake (Fig. 2).

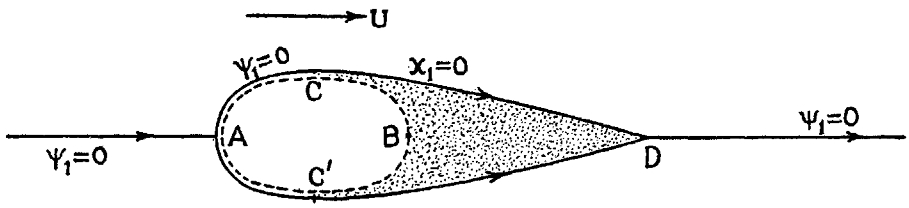

FIG. 2

These requirements will be readily satisfied if the stream-function ψ is of the form

$$\psi = \psi_1 \chi_1 . \qquad (2.2)$$

For, then, $\psi = 0$ will be satisfied by $\psi_1 = 0$ as well as $\chi_1 = 0$ so that when χ_1 is suitably chosen the wake boundary CD or $C'D$ will be represented by

$$\chi_1 = 0 \qquad (2.3)$$

The points C and D can then be easily obtained as the intersections of

$$\left.\begin{array}{l} \psi_1 = 0 \\ \chi_1 = 0 \end{array}\right\} \qquad (2.4)$$

with

Now, by our *assumption* the motion outside is to be a purely potential motion and if there be any vortices they may lie inside the loop, i.e., in the wake. For a steady motion outside, it is enough that the wake boundary remains fixed.

Thus the requirements for the altered, i.e., the wake-flow are:—

(1) It should be a potential flow without singularities outside the body-wake system.

(2) It should reduce itself to the primary potential flow at large distances from the body (observational result mentioned in Part I).

(3) Its stream-function should be of the form (2.2).

(4) The equations (2.4) should (for a closed wake) have at least, one real solution within the range AB and one real finite solution on the right of B.

3. *The Complex Potential in Symmetrical Flows*

We shall designate the new wake-flow the 'secondary' flow and denote its complex potential by W_2. Further, we shall confine ourselves only to the consideration of flows symmetrical about the axis of x.

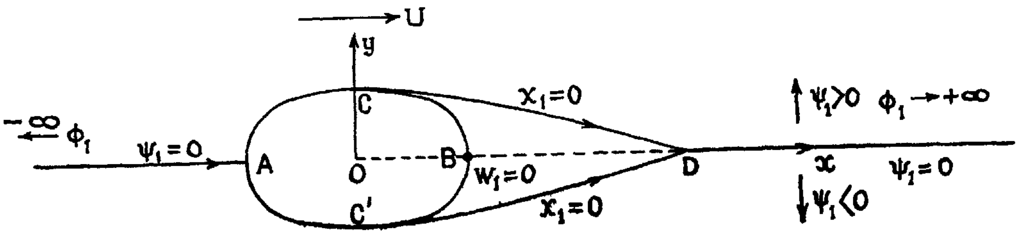

FIG. 3

As before, we assume that the primary potential flow is represented by (2.1) and that the flow has symmetry about the x-axis in the real, i.e., the flow plane. Thus ψ_1 vanishes on the axis of X as well as on the body-surface and ϕ_1 may vary from $-\infty$ on the extreme left to $+\infty$ on the extreme right, ψ_1 being positive above and negative below the x-axis. If, now, the points in the flow plane be expressed in curvilinear co-ordinates (ϕ_1, ψ_1) this symmetry will be disturbed unless the origin of W_1 (i.e., the point $\phi_1 = 0$, $\psi_1 = 0$) is situate on the axis of x. The most convenient point for such a choice for $W_1 = 0$ is, however, the rear stagnation point B, for the primary flow. We shall therefore suppose, in all our subsequent analysis, that

$$\phi_1 = \psi_1 = 0 \quad \text{at } B. \qquad (3.1)$$

Now, to satisfy requirement (1) it is enough to have W_2 as a function of W_1. To satisfy (2) this function of W_1 must $\to W_1$ as $|W_1| \to \infty$. Hence we assume, generally,

$$W_2 = W_1 + \sum_0^\infty \frac{\alpha_r}{W_1^r} \qquad (3.2)$$

the quantities α_r being constants or at least independent of co-ordinates (x, y).

Now, W_1 being an analytic function, W_2 is also so, outside $W_1 = 0$. But $W_1 = 0$ at the rear stagnation point B. So, W_2 given by (3.2) will be analytic everywhere outside a small area surrounding B. As under condition (4) B will be safely enclosed inside the wake, W_2 will be analytic everywhere outside the loop. In fact, W_2, given by (3.2), can be made to satisfy all necessary requirements for a *finite wake.*

Let us put

$$W_2 = \phi_2 + i\psi_2 \qquad (3.3)$$

and by (2.1)

$$W_1 = \phi_1 + i\psi_1 \equiv \rho e^{i\sigma}, \text{ say}; \qquad (3.4)$$

so that

$$\phi_1 = \rho \cos \sigma, \quad \psi_1 = \rho \sin \sigma. \qquad (3.4a)$$

To cover all points in the W_1 plane we shall take

$$\left.\begin{array}{l} \rho > 0, \quad W_1 \neq 0 \\ \rho = 0, \quad W_1 = 0 \end{array}\right\} \qquad (3.5)$$

so that σ may vary between 0 and 2π. Besides, on

$$\left.\begin{array}{l} \psi_1 = 0 \\ \quad \sigma = 0, \quad \phi_1 > 0 \\ \quad \sigma = \pi, \quad \phi_1 < 0 \end{array}\right\} \qquad (3.5a)$$

Further, owing to the assumed symmetry it will be enough to consider the upper half of the W_1-plane (including the origin) and for this domain

$$\left.\begin{array}{l} 0 \leqslant \sigma \leqslant \pi, \quad \rho > 0 \\ \rho = 0, \quad W_1 = 0 \end{array}\right\} \qquad (3.5b)$$

is sufficient for our analysis, conditions being exactly the same on the other side of the ϕ_1-axis.

4. *The Wake Boundary*

With (3.3), (3.4), (3.4*a*) we have from (3.2)

$$\phi_2 = \alpha_0 + \rho \cos\sigma \left\{ 1 + \sum_1^\infty \frac{\alpha_r \cdot \cos r\sigma}{\rho^{r+1} \cdot \cos\sigma} \right\} \quad \ldots \quad \ldots \quad (4.1)$$

$$\psi_2 = \rho \sin\sigma \left\{ 1 - \sum_1^\infty \frac{\alpha_r \cdot \sin r\sigma}{\rho^{r+1} \cdot \sin\sigma} \right\} \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

$$= \psi_1 \left\{ 1 - \sum_1^\infty \frac{\alpha_r}{\rho^{r+1}} \cdot \frac{\sin r\sigma}{\sin\sigma} \right\} \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.2a)$$

Comparing with (2.2), where ψ_2 takes the place of ψ, we have

$$\chi_1 \equiv \left\{ 1 - \sum_1^\infty \frac{\alpha_r \cdot \sin r\sigma}{\rho^{r+1} \cdot \sin\sigma} \right\} \quad \ldots \quad \ldots \quad \ldots \quad (4.3)$$

Hence, the equation of the wake-boundary is given, in general, by

$$1 - \sum_1^\infty \frac{\alpha_r \cdot \sin r\sigma}{\rho^{r+1} \cdot \sin\sigma} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4.4)$$

where, of course, ρ and σ are as defined by (3.4*a*). The intersection of the wake-boundary with $\psi_1 = 0$ is therefore given by the roots of the equation

$$\underset{\psi_1 \to 0}{\text{Lt.}} \left\{ 1 - \sum_1^\infty \frac{\alpha_r \cdot \sin r\sigma}{\rho^{r+1} \cdot \sin\sigma} \right\} \equiv \underset{\substack{\sigma \to 0 \\ \rho \to \phi_1}}{\text{Lt.}} \left\{ 1 - \sum_1^\infty \frac{\alpha_r \cdot \sin r\sigma}{\rho^{r+1} \cdot \sin\sigma} \right\} = 0,$$

that is, by the real roots of

$$1 - \sum_1^\infty \frac{r\alpha_r}{\phi_1^{r+1}} = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.5)$$

in general.

It is obvious that when a finite number of terms are considered the sum of the roots must vanish in every case. Besides, it is important to note that every real intersection of $\chi_1 = 0$ with $\psi_1 = 0$ and hence every real root of (4.5) *must be a stagnation point*. This can be shown quite easily.

For by (2.2)

$$\psi_2 = \psi_1 \chi_1 \cdot$$

Therefore

$$\nabla \psi_2 = \psi_1 \nabla \chi_1 + \chi_1 \nabla \psi_1 \cdot \quad \ldots \quad \ldots \quad \ldots \quad (4.6)$$

But, at a point of intersection, $\psi_1 = 0$ and $\chi_1 = 0$, hence

$$\nabla \psi_2 = 0 \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.7)$$

which proves the result.

5. *Points of Separation and Reunion of the Flow*

As the rear stagnation point has been chosen as the origin $W_1 = 0$, the front stagnation point A will be given by $\psi_1 = 0$ and a negative value for ϕ_1, for ϕ_1 is supposed to increase in the direction of the flow. Let this value be ϕ_{1A}. Then, for a real separation of the flow from the body surface equation (4.5) must have at least one negative root lying in the range

$$\phi_{1A} \leqslant \phi_1 \leqslant 0. \qquad (5.1)$$

For two or more coincident negative roots within the range the separation will take place tangentially. In the absence of any real positive root of (4.5) the flow does not unite. For only one real positive root the union or confluence takes place at an angle. For two or more coincident positive roots the confluence takes place in the form of a *cusp*.

Thus it is clear that the coefficients α_r determine the pattern of the wake. Given any specified pattern it is possible to choose the coefficients suitably. In general, however, there doesn't seem to exist any definite physical principle which may specify the constants uniquely. It is possible that flows for all values of α_r are permitted but only some of them are stable. This is a mere surmise and does not presume to be a proof. Experimental photographs show the existence of different types of wake-boundaries for the same obstacle when subjected to flows with different Reynolds numbers, i.e., for different velocities of the undisturbed flow. All these patterns of the wake boundary must obviously belong to the same topological group so that one can be transformed into another by a continuous deformation—and physical intuition suggests that the sole parameter of this deformation must be the speed U or to be exact the corresponding Reynolds number.

Speaking of experimental results, it may be observed that photographs (Goldstein, 1938, Vol. I, plates 7, 8) actually reveal the existence of what looks like stagnation areas near the points of separation and reunion.

6. *Angle of Separation and Angle of Reunion or Confluence*

It is possible to construct a general result which will give the angle at which the separation from the body-surface or at which reunion of the flow on the x-axis takes place.

At any point let q_2, θ_2 denote the speed and direction of the secondary flow in the real plane (z-plane); let q', θ' denote the same in the W_1-plane and let q_1, θ_1 denote the speed and direction of the primary flow in the actual flow plane (i.e., z-plane). Then,

$$\left.\begin{aligned} \frac{dW_1}{dz} &= u_1 - iv_1 = q_1 e^{-i\theta_1} \\ \frac{dW_2}{dz} &= u_2 - iv_2 = q_2 e^{-i\theta_2} \\ \frac{dW_2}{dW_1} &= u' - iv' = q' e^{-i\theta'} \end{aligned}\right\} \qquad (6.1)$$

and

where the suffix 1 refers to the primary flow, 2 to the secondary flow, both in the z-plane and the dash (′) refers to the secondary flow in the W_1-plane. But

$$\frac{dW_2}{dz} = \frac{dW_2}{dW_1} \cdot \frac{dW_1}{dz}. \qquad (6.2)$$

Hence,

$$q_2 e^{-i\theta_2} = q' q_1 e^{-i(\theta'+\theta_1)}$$

which shows

$$q_2 = q' q_1 \qquad \text{(6.3)}$$

$$\theta_2 = \theta' + \theta_1 \qquad \text{(6.4)}$$

But $\theta_2 - \theta_1$ is the angle between the directions of the primary and the secondary flow in the real (i.e., z) plane. Hence by (6.4) the angle through which the velocity at any point is turned in passing from the primary to the secondary flow is given by

$$\theta' = \theta_2 - \theta_1 \qquad \text{(6.5)}$$

so that the angle θ of separation or confluence which means the angle of turning of the flow at either point is given by

$$\tan\theta = \tan\theta' = \frac{v'}{u'} \qquad \text{(6.6)}$$

At a stagnation point $v' = u' = 0$, because $q = 0$ but $q_1 \neq 0$. Hence for such points we have

$$\tan\theta = \text{Lt.}\frac{v'}{u'}. \qquad \text{(6.7)}$$

But

$$u' = \frac{\partial\psi_2}{\partial\psi_1},\ v' = -\frac{\partial\psi_2}{\partial\phi_1}$$

so that, for separation or confluence

$$\tan\theta = -\underset{\substack{\psi_1\to 0\\ \chi_1\to 0}}{\text{Lt.}}\left[\frac{\dfrac{\partial\psi_2}{\partial\phi_1}}{\dfrac{\partial\psi_2}{\partial\psi_1}}\right] \qquad \text{(6.8)}$$

$$\left.\begin{array}{ll}\text{For a point of separation,} & \chi_1 \to 0 \ \text{and}\ \sigma\to\pi\\ \text{for a point of reunion,} & \chi_1\to 0 \ \text{and}\ \sigma\to 0\end{array}\right\} \qquad \text{(6.8}a\text{)}$$

We shall distinguish between these two angles by the suffixes s and m, so that

$$\left.\begin{array}{l}\theta_s = \text{angle of separation}\\ \theta_m = \text{angle of reunion}\end{array}\right\} \qquad \text{(6.8}b\text{)}$$

With (2.2) and (6.8b), equation (6.8) can be written as

$$\tan\theta_s = -\underset{\sigma\to\pi}{\text{Lt.}}\left[\frac{\dfrac{\partial\chi_1}{\partial\phi_1}}{\dfrac{\partial\chi_1}{\partial\psi_1}}\right] \qquad \text{(6.9}a\text{)}$$

$$\tan\theta_m = -\underset{\sigma\to 0}{\text{Lt.}}\left[\frac{\dfrac{\partial\chi_1}{\partial\phi_1}}{\dfrac{\partial\chi_1}{\partial\psi_1}}\right] \qquad \text{(6.9}b\text{)}$$

Alternatively, it is easy to see from the principles of conformal transformations that θ_m or θ_s are the angles at which the curve $\chi_1 = 0$ in the W_1-plane meets the ϕ_1 axis. Hence, if $\chi_1 = 0$ is expressed in polar co-ordinates (ρ, σ) in the W_1-plane,

$$\left.\begin{aligned} \tan \theta_m &= \underset{\sigma \to 0}{\text{Lt.}} \left(\rho \frac{d\sigma}{d\rho}\right) \\ \text{and} \qquad \tan \theta_s &= \underset{\sigma \to \pi}{\text{Lt.}} \left(\rho \frac{d\sigma}{d\rho}\right) \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (6.10)$$

Referring to the general equation (4.4) for $\chi_1 = 0$ it becomes obvious that the last two formulae are most convenient for the calculation of θ_m and θ_s.

7. *Special Types of Wake-flows*

We shall now describe a few simple types of closed wakes, as particular cases of the general results established above.

Type I. $[\alpha_r \neq 0,\ r \leqslant 1;\ \alpha_r = 0,\ r > 1.]$

In this case

$$W_2 = \alpha_0 + W_1 + \frac{\lambda^2}{W_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.1)$$

(where λ^2 has been put for α_1)

$$\phi_2 = \alpha_0 + \phi_1 \left(1 + \frac{\lambda^2}{\rho^2}\right) \quad \ldots \quad \ldots \quad \ldots \quad (7.2)$$

(α_0 being supposed real),

$$\psi_2 = \psi_1 \left(1 - \frac{\lambda^2}{\rho^2}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.3)$$

where

$$\rho^2 \equiv \phi_1^2 + \psi_1^2; \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.4)$$

$$\chi_1 = \left(1 - \frac{\lambda^2}{\rho^2}\right), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.5)$$

so that the equation to the wake boundary in the W_1-plane is

$$\rho = \lambda \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.5a)$$

This circle will necessarily have real intersections with $\psi_1 = 0$, so that the corresponding wake in the z-plane must be closed. The intersections are given by

$$\phi_1 = \pm \lambda \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.6)$$

and by (6.10), or obviously

$$\theta_m = \theta_s = \frac{\pi}{2} \quad \text{(signs being omitted)} \quad \ldots \quad \ldots \quad (7.7)$$

Thus in this case the flow separates from the body surface normally and meets on the x-axis again normally.

(Photographic plates actually record cases where the reunion appears to be normal. For example, Durand (1935), Vol. III, plate III, fig. 30.)

Type II. $[\alpha_r \neq 0,\ r < 2,\ \alpha_r = 0,\ r > 2]$

In this case,

$$W_2 = \alpha_0 + W_1 + \frac{\alpha_1}{W_1} + \frac{\alpha_2}{W_1^2} \qquad (7.8)$$

$$\chi_1 = 1 - \frac{\alpha_1}{\rho^2} - \frac{2\alpha_2 \cos\sigma}{\rho^3} \qquad (7.9)$$

The wake boundary $\chi_1 = 0$ is given by

$$\rho^3 - \rho\alpha_1 - 2\alpha_2 \cos\sigma = 0 \qquad (7.10)$$

The points of separation and reunion are given by the roots of

$$\phi_1^3 - \alpha_1\phi_1 - 2\alpha_2 = 0. \qquad (7.11)$$

For a closed wake, there must be at least two real roots one positive and one negative. Hence all the roots must be real. For a physically interpretable case (only one separation point and one confluence point) two of the roots must be equal. If $\lambda_1, \lambda_2, \lambda_3$ be the three roots we must have

$$\lambda_1 + \lambda_2 + \lambda_3 = 0 \qquad (7.11a)$$

Besides, say,

$$\lambda_1 = \lambda_2 \qquad (7.11b)$$

then (7.11*a*) becomes

$$2\lambda_1 + \lambda_3 = 0 \qquad (7.12)$$

Comparing with

$$(\phi_1 - \lambda_1)^2 (\phi_1 - \lambda_3) = 0$$

we have, besides (7.12),

$$\left.\begin{aligned} \lambda_1^2 + 2\lambda_1\lambda_3 &= -\alpha_1 \\ \lambda_1^2\lambda_3 &= 2\alpha_2 \end{aligned}\right\} \qquad (7.12a)$$

Hence eliminating λ_1, λ_3 from (7.12) and (7.12*a*) we have

$$\alpha_1^3 = 27\,\alpha_2^2, \qquad (7.13)$$

(The result could be obtained from the discriminant for a cubic) so that it is necessary that

$$\alpha_1 > 0 \qquad (7.14)$$

$$\alpha_2 = \pm \left(\frac{\alpha_1}{3}\right)^{\frac{3}{2}} \qquad (7.15)$$

Thus *two cases* are possible, viz.,

$$\left.\begin{aligned} \lambda_3 < 0,\ \lambda_1 > 0,\ \alpha_2 < 0 \\ \lambda_3 > 0,\ \lambda_1 < 0,\ \alpha_2 > 0 \end{aligned}\right\} \qquad (7.16)$$

Hence

(i) When the last coefficient α_2 is negative there is one negative root and two equal positive roots. The flow leaves the surface at an angle and meets back in a cusp.

(ii) When the last coefficient is positive, there are two negative equal roots and one positive root. In this case separation takes place tangentially and the confluence takes place at an angle.

Both the above cases are possible. But in either, equations (7.14) and (7.15) must be obeyed. Thus out of the two constants α_1 and α_2 only one becomes independent.

Putting

$$\alpha_1 = 3\lambda^2 \qquad .. \quad .. \quad .. \quad .. \quad (7.17a)$$

we have from (7.15)

$$\alpha_2 = \mp \lambda^3, \qquad .. \quad .. \quad .. \quad .. \quad (7.17b)$$

the upper and the lower sign corresponding to the cases (i) and (ii) respectively. Now, (7.11) can be put in the form

$$\phi_1^3 - 3\lambda^2\phi_1 \pm 2\lambda^3 = 0 \qquad .. \quad .. \quad .. \quad (7.18)$$

so that the roots in the two cases are

$$\left.\begin{array}{l} -2\lambda, \ \lambda, \ \lambda \\ -\lambda, -\lambda, \ 2\lambda \end{array}\right\} \qquad .. \quad .. \quad .. \quad .. \quad (7.18a)$$

respectively.
Correspondingly,

$$W_2 = \alpha_0 + W_1 + \frac{3\lambda^2}{W_1} \mp \frac{\lambda^3}{W_1^2} \qquad .. \quad .. \quad .. \quad .. \quad (7.19)$$

$$\phi_2 = \alpha_0 + \rho \cos\sigma \left\{ 1 + \frac{3\lambda^2}{\rho^2} \mp \frac{\lambda^3}{\rho^3} \cdot \frac{\cos 2\sigma}{\cos\sigma} \right\} \qquad .. \quad .. \quad (7.19a)$$

$$\psi_2 = \rho \sin\sigma \left\{ 1 - \frac{3\lambda^2}{\rho^2} \pm \frac{2\lambda^3}{\rho^3} \cos\sigma \right\} \qquad .. \quad .. \quad .. \quad (7.19b)$$

$$\chi_1 = 1 - \frac{3\lambda^2}{\rho^2} \pm \frac{2\lambda^3 \cos\sigma}{\rho^3} \qquad .. \quad .. \quad .. \quad .. \quad .. \quad (7.19c)$$

and for case (i) $\theta_s = \frac{\pi}{2}, \ \theta_m = 0$

for case (ii) $\theta_s = 0, \ \theta_m = -\frac{\pi}{2}$ .. .. (7.20)

Thus in case (i) the separation is orthogonal and union is in cusp and in case (ii) the separation is tangential but union is orthogonal.

Type III. *Separation tangential, Union in a cusp.*

Retaining only four terms in the expansion (3.2) we have

$$W_2 = \alpha_0 + W_1 + \frac{\alpha_1}{W_1} + \frac{\alpha_2}{W_1^2} + \frac{\alpha_3}{W_1^3} \qquad .. \quad .. \quad .. \quad (7.21)$$

Then,

$$\left.\begin{array}{l} \phi_2 = \alpha_0 + \rho \cos\sigma \left\{ 1 + \frac{\alpha_1}{\rho^2} + \frac{\alpha_2 \cos 2\sigma}{\rho^3 \cos\sigma} + \frac{\alpha_3}{\rho^4} \cdot \frac{\cos 3\sigma}{\cos\sigma} \right\} \\ \psi_2 = \rho \sin\sigma \left\{ 1 - \frac{\alpha_1}{\rho^2} - \frac{\alpha_2 . 2\cos\sigma}{\rho^3} - \frac{\alpha_3}{\rho^4}(3 - 4\sin^2\sigma) \right\} \end{array}\right] \qquad .. \quad (7.21a)$$

so that

$$\chi_1 = 1 - \frac{\alpha_1}{\rho^2} - \frac{\alpha_2 . 2\cos\sigma}{\rho^3} - \frac{\alpha_3}{\rho^4}(3 - 4\sin^2\sigma) \qquad .. \quad .. \quad .. \quad (7.22)$$

The points of intersection are given by

$$\phi_1^4 - \alpha_1\phi_1^2 - 2\alpha_2\phi_1 - 3\alpha_3 = 0 \qquad .. \quad .. \quad .. \quad (7.22a)$$

We shall consider only that case where the separation and the union are both tangential, i.e., where $\theta_m = \theta_s = 0$. For this it is necessary that (7.22*a*) should have two distinct pairs of equal roots and since the sum of the roots always vanishes two of these must be negative and two positive. Applying the necessary conditions from the theory of biquadratics, we have

$$\left.\begin{aligned} \alpha_1 &> 0 \\ \alpha_2 &= 0 \\ -3\alpha_3 &= \frac{\alpha_1^2}{4} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.23)$$

Hence putting

$$\alpha_1 = 2\lambda^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.24)$$

we have from (7.22*a*)

$$\phi_1^4 - 2\lambda^2\phi_1^2 + \lambda^4 \equiv (\phi_1^2 - \lambda^2)^2 = 0$$

so that the points of intersection of the wake-boundary with $\psi_1 = 0$ are given by

$$\phi_1 = -\lambda, -\lambda, \lambda, \lambda .$$

furnishing the necessary pairs of coincident roots. It should be noted that the points of separation and reunion are at equal distances from the back stagnation point in the W_1-plane. In this case, as expected, $\theta_s = \theta_m = 0$ and,

$$\phi_2 = \alpha_0 + \rho \cos\sigma \left\{ 1 + \frac{2\lambda^2}{\rho^2} - \frac{1}{3}\frac{\lambda^4}{\rho^4}(4\cos^2\sigma - 3) \right\} \quad \ldots \quad \ldots \quad (7.25a)$$

$$\psi_2 = \rho \sin\sigma \left\{ 1 - \frac{2\lambda^2}{\rho^2} + \frac{1}{3}\frac{\lambda^4}{\rho^4}(3 - 4\sin^2\sigma) \right\} \quad \ldots \quad \ldots \quad (7.25b)$$

$$\chi_1 = 1 - \frac{2\lambda^2}{\rho^2} + \frac{1}{3}\frac{\lambda^4}{\rho^4}(3 - 4\sin^2\sigma) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.25c)$$

α_0, of course, being arbitrary.

8. *Application to Circular and Elliptic Cylinders*

To calculate the wake-flow past any cylinder we have merely to substitute the appropriate expression for W_1 as a function of z, so adjusted that the origin for W_1 is transferred to the back stagnation point. Thus, for

(I) *the Circular Cylinder*

$$W_1 = -2aU + Uz\left(1 + \frac{a^2}{z}\right) \quad \ldots \quad \ldots \quad \ldots \quad (8.1)$$

which makes

$$\phi_1 = -2aU + Ux\left(1 + \frac{a^2}{r^2}\right) \quad \ldots \quad \ldots \quad \ldots \quad (8.1a)$$

$$\psi_1 = Uy\left(1 - \frac{a^2}{r^2}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8.1b)$$

so that ϕ_1, ψ_1 vanish at the back stagnation point $(a, 0)$.
Further,

$$\phi_{1A} = \left(\phi_1\right)_{\substack{x=-a \\ y=0}} = -4aU \quad \ldots \quad \ldots \quad \ldots \quad (8.2)$$

so that for flows of the different types enumerated before, we must have the follow limitations on the parameter λ in order that the separation may remain betw the stagnation points A and B for the primary flow.

Type I	$0 \leqslant \lambda \leqslant 4aU$	.. .. (8
Type II (i)	$0 \leqslant 2\lambda \leqslant 4aU$	.. .. (8
Type II (ii)	$0 \leqslant \lambda \leqslant 4aU$	.. .. (8
Type III	$0 \leqslant \lambda \leqslant 4aU$	.. .. (8

(II) *For an Elliptic Cylinder*

$$\left.\begin{aligned} W_1 &= -U(a+b)+U\left[b\sqrt{\frac{a+b}{a-b}}e^{-\zeta}+\sqrt{a^2-b^2}\cosh\zeta\right] \\ z &= c\cosh\zeta \equiv c\cosh(\xi+i\eta) \end{aligned}\right\} \quad .. \quad ($$

so that

$$\phi_1 = -U(a+b)+U\left[b\sqrt{\frac{a+b}{a-b}}e^{-\xi}+\sqrt{a^2-b^2}\cosh\xi\right]\cos\eta \quad .. \quad (8$$

$$\psi_1 = -U\left[b\sqrt{\frac{a+b}{a-b}}e^{-\xi}-\sqrt{a^2-b^2}\sinh\xi\right]\sin\eta \quad .. \quad .. \quad (8$$

and

$$\phi_{1A} = -2U(a+b) \,. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (8$$

In the subsequent part (Part IV) of this paper we shall discuss the questior resistance and dissipation in wake flows.

Abstract

The resolution of D'Alembert's paradox removes the objection to closed wakes—a diffic met by Helmholtz by the free-streamline theory. In this part, therefore, the author deve the Theory of closed wakes in potential flow. A general formula for the wake flow in dimensions past any cylinder is established and by an analysis of the wake-boundary sev types of closed wakes are demonstrated.

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PART IV

RESISTANCE AND DISSIPATION IN MOTIONS WITH CLOSED WAKES

1. *Introduction*

In Part I of this paper we have resolved D'Alembert's paradox and calculated the resistance in the primary potential flow of a real incompressible fluid on obstacles of different shapes. In Part II we have seen that the energy supplied to the fluid by the work of the body forces exceeds, in some cases, the amount of mechanical energy necessary to maintain a steady flow, proving thereby, that the primary potential flows concerned could not be maintained steadily in such cases without modification. We interpreted this result as leading to the formation of wakes behind those obstacles. But wakes must necessarily be closed. With the resolution of D'Alembert's paradox objections to the existence of closed wakes were removed and in Part III we formulated the theory of potential flows with closed wakes—in contrast to the Helmholtz theory of infinite wakes. In the few simple cases discussed in Art. 7, Part III, it could be noted that the parameter λ (which really determines the point of separation of the flow and the end of the wake, in every case considered) was left arbitrary. By the application of the Principle of Maintenance enunciated in Art. 3, Part II, we shall be in a position to determine this parameter.

With this in view we shall first indicate how the Resistance and the Dissipation in potential flows with closed wakes can be obtained. The expressions for both will involve the unknown parameter λ and the equation expressing the condition for maintenance will be the one determining λ. The result in every case will conceivably depend on the primary flow chosen, i.e., on the nature of the cylindrical obstacle selected for study. With the writing down of the equation of maintenance, however, the theoretical part of the present investigations is brought to its completion.

2. *Resistance in motion with closed wakes*

We have indicated previously in the introduction to Part I that we conceive of the wake as a mass of dead-water following the body and relatively at rest with it in the sense that the velocity of the centre of mass of the particles constituting the wake is the same as that of the body and hence U, by assumption, although the elements of the fluid may be executing vortex motions inside the region bounded by the body-surface and the wake-boundary between the points of separation and confluence as shown in Fig. 1, Part I. Thus the total linear momentum of the fluid in the wake remains invariable when the motion becomes steady. Consequently the resultant of the stresses acting on the wake alone across its bounding surfaces must vanish. If now, a force P is necessary to drive the body steadily through the fluid, since the body also is supposed to move with the uniform velocity U, Newton's Second Law applied to the system of the body and the wake taken as a whole gives P as merely balancing the total reaction of the outside fluid on the body-wake system exerted across the boundary of the combined system, i.e., across the boundary of the loop inside $\psi_2 = 0$, mentioned in Art. 2, Part III (see Fig. 3, Part III). The above argument, however, applies equally well to any three-dimensional problem and hence we can say generally that—

'When a body moves steadily through a fluid and a closed wake of finite dimensions forms behind it, the total resistance experienced by the body can be calculated merely as the resultant effect of the surface stresses exerted by the rest of the fluid across the boundary of the body-wake system.'

This result is independent of the nature of the fluid motion outside the body-wake system and also independent of the internal motions executed by the fluid particles inside the wake so long as the wake maintains its boundary and its total linear momentum invariable.

In our present discussion the flow outside the body-wake system is potential by assumption. Hence by a reference to Art. 4, Part I, we notice that the total resistance can be calculated from formula (4.10), Part I (by merely replacing ϕ_1, ψ_1 by ϕ_2, ψ_2 and integrating over the boundary of the body-wake system $ACDC'A$ (Fig. 3, Part III) instead of that of the body surface alone. Thus we have, (for a two-dimensional flow only)—

$$X = \mu \oint \left\{ \frac{\partial q^2}{\partial \phi_2} dy - \frac{\partial q^2}{\partial \psi_2} dx \right\} \quad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

$$Y = -\mu \oint \left\{ \frac{\partial q^2}{\partial \phi_2} dx + \frac{\partial q^2}{\partial \psi_2} dy \right\} \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

q representing the speed in the secondary flow W_2 ($= \phi_2 + i\psi_2$) and the integrals being taken round the body-wake contour $ACDC'A$ (Fig. 3, Part III), in a clockwise sense.

For motions with symmetry about the x-axis, Y vanishes as before and then (2.1) can be written as

$$X = 2\mu \int_A^D \left\{ \frac{\partial q^2}{\partial \phi_2} dy - \frac{\partial q^2}{\partial \psi_2} dx \right\} \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

the path of integration being the half-contour ACD.

An alternative form.

It would sometimes be more convenient to put (2.1) in a more readily adaptable form. For this we see that, in general, q^2 may be regarded as a function of the complex conjugate variables W_2 and $\overline{W}_2$ where

$$\left. \begin{aligned} W_2 &= \phi_2 + i\psi_2 \\ \overline{W}_2 &= \phi_2 - i\psi_2 \end{aligned} \right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

and as

$$\left. \begin{aligned} \frac{\partial}{\partial \phi_2} &= \frac{\partial}{\partial W_2} + \frac{\partial}{\partial \overline{W}_2} \\ \frac{\partial}{\partial \psi_2} &= i\left(\frac{\partial}{\partial W_2} - \frac{\partial}{\partial \overline{W}_2} \right) \end{aligned} \right\} \quad \ldots \quad \ldots \quad \ldots \quad (2.5)$$

we have

$$\left. \begin{aligned} \frac{\partial}{\partial \phi_2} + i\frac{\partial}{\partial \psi_2} &= 2\frac{\partial}{\partial \overline{W}_2} \\ dx - idy &= d\bar{z} \end{aligned} \right\} \quad \ldots \quad \ldots \quad \ldots \quad (2.6)$$

also

With these (2.1) becomes

$$X = -2\mu I \oint \frac{\partial q^2}{\partial \overline{W}_2} \cdot d\bar{z} \quad \ldots \quad \ldots \quad \ldots \quad (2.7)$$

where I stands for the imaginary part and, of course,

$$q^2 = \left| \frac{dW_2}{dz} \right|^2 \quad \ldots \quad (2.8)$$

the integration being taken round the whole contour in a clockwise sense. The equation of the path of integration ($W_2 = \overline{W}_2$) is given by

$$W_1 = \overline{W}_1, \; z \neq \bar{z} \quad \ldots \quad (2.9)$$

for the parts AC, AC' (Fig. 3, Part III) and

$$W_2 = \overline{W}_2, \; W_1 \neq \overline{W}_1 \quad \ldots \quad (2.10)$$

for the remaining parts CD, $C'D$.

The formula (2.7) becomes most convenient where the inverse transformation

$$z \equiv z(W_2)$$

for the flow is easily obtained. For in that case the whole path of integration can merely be taken as

$$W_2 = \overline{W}_2 \quad \ldots \quad (2.10a)$$

and Cauchy's theorem can be directly applied for the evaluation of X. For such a procedure we shall obviously be carrying out the integration in the W_2-plane instead of the real (i.e., z) plane.

In the types of flow discussed previously, however, the inverse transformation merely adds to the complication instead of simplifying it. It is therefore useful to investigate the possibilities of integration in the real or z-plane. For this we see that

$$q^2 = \frac{dW_2}{dz} \cdot \frac{d\overline{W}_2}{d\bar{z}} \quad \ldots \quad (2.11)$$

$$\therefore \quad \frac{\partial q^2}{\partial \overline{W}_2} = \frac{dW_2}{dz} \cdot \frac{d^2\overline{W}_2}{d\bar{z}^2} \cdot \frac{d\bar{z}}{d\overline{W}_2}$$

$$= \frac{\dfrac{dW_2}{dz}}{\dfrac{d\overline{W}_2}{d\bar{z}}} \cdot \frac{d^2\overline{W}_2}{d\bar{z}^2}, \quad \ldots \quad (2.12)$$

and for the path of integration (2.10a)

$$\frac{dW_2}{dz} = \frac{d\overline{W}_2}{d\bar{z}} \cdot \frac{d\bar{z}}{dz} \quad \ldots \quad (2.13)$$

so that we have

$$\frac{\partial q^2}{\partial \overline{W}_2} = \frac{d^2\overline{W}_2}{d\bar{z}^2} \cdot \frac{d\bar{z}}{dz} \quad \ldots \quad (2.14)$$

and so (2.7) gives

$$X = -2\mu I \oint \frac{d^2\overline{W}_2}{d\bar{z}^2} \cdot \frac{d\bar{z}}{dz} \cdot d\bar{z}. \quad \ldots \quad (2.15)$$

The disadvantage of this form is that Cauchy's formula for integration cannot be directly applied. For, the equation of the two parts of the path CAC' and CDC'

in the z-plane are given by different equations as shown in (2.9) and (2.10); so working with (2.15) we have to split up the integral into two parts and assuming symmetry we have

$$X = -2\mu I \left[2 \int_{A}^{C} \frac{d^2\overline{W}_2}{d\bar{z}^2} \cdot \frac{d\bar{z}}{dz} \cdot d\bar{z} + 2 \int_{C}^{D} \frac{d^2\overline{W}_2}{d\bar{z}^2} \cdot \frac{d\bar{z}}{dz} \cdot d\bar{z} \right] \quad \ldots \quad (2.16)$$
$$\begin{pmatrix} w_1 = \overline{w}_1 \\ z \neq \bar{z} \end{pmatrix} \qquad \begin{pmatrix} w_2 = \overline{w}_2 \\ w_1 \neq \overline{w}_1 \end{pmatrix}$$

The correctness of these formulae can be verified from the fact that these give back our previous results of Part I when W_2 is replaced by W_1 and C and D coincide.

In the more general case, where no symmetry is assumed the modified Blasiu's formula for the wake-flow can be at once written down in the same way in which (2.7) was obtained. We shall thereby obtain:

$$X - iY = -\frac{1}{2}\rho i \oint \left| \frac{dW_2}{dz} \right|^2 dz + 2\mu i \oint \frac{\partial q^2}{\partial \overline{W}_2} d\bar{z} \,. \quad \ldots \quad \ldots \quad (2.17)$$

with q^2 given by (2.8) and the path of integration being the entire contour of the body-wake system in a clockwise sense.

3. *The Rate of Dissipation and the Maintenance of the Flow*

The rate of dissipation in the potential flow outside the body-wake system must for the same reasons as in Part II be totally compensated for by the work of the body-wake system on the fluid outside, for a steady motion. Besides, the dissipation in the potential flow outside can be obtained from equ. (2.1), Part II, in the manner indicated in the preceding article. We thus have in this case,

$$D = -\mu \oint \frac{\partial q^2}{\partial \psi_2} \cdot d\phi_2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

where q^2 is given by (2.8), as before.

With (2.4) and (2.5). this can be written as

$$D = -\mu i \oint \left\{ \frac{\partial q^2}{\partial W_2} - \frac{\partial q^2}{\partial \overline{W}_2} \right\} dW_2 \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

provided (2.10a) is taken into account as defining the path of integration.

Alternatively, with (2.14) and a similar relation we can write (3.2) in the form

$$D = \mu i \oint \left\{ \frac{d^2 W_2}{dz^2} \cdot \frac{dz}{d\bar{z}} - \frac{d^2\overline{W}_2}{d\bar{z}^2} \cdot \frac{d\bar{z}}{dz} \right\} \frac{dW_2}{dz} \cdot dz \quad \ldots \quad \ldots \quad (3.3)$$

for integration in the z-plane, the integral being taken round the whole body wake contour in an ***anti-clockwise*** sense.

Maintenance of the Flow

From what has been said above and from the discussion in Art. 3, Part II, it is clear that the motion relative to the moving body-wake will remain steady if

$$X \,.\, U = D \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

in the case of flows with symmetry, X and D being now given by (2.1) and (3.1) respectively. In other words, for a steady symmetrical flow past a cylinder we must have

$$\mu \oint \left\{ \frac{\partial q^2}{\partial \phi_2} dy - \frac{\partial q^2}{\partial \psi_2} dx \right\} = - \frac{\mu}{U} \oint \frac{\partial q^2}{\partial \psi_2} d\phi_2 \qquad \ldots \qquad \ldots \qquad (3.5)$$

The importance of equation (3.5), as mentioned in the introduction to this part, lies in the fact that it provides us with the tool for the determination of the arbitrary parameter λ which characterises the different types of flow discussed in Art. 7, Part III. Having chosen a particular cylinder for study, the first thing we have to do is to determine W_1, the primary potential flow round it—by the means made available by well-known classical theories. Having obtained W_1 we next choose a particular type of the closed wake, like the cases shown in Art. 7, Part III, for study. The character of the wake boundary being sufficiently restricted to leave only one parameter open, we can apply equation (3.5) to test if the chosen type of wake is possible for the cylinder under consideration. For, a real value of the parameter satisfying (3.5) we can safely say that the type of flow chosen may be exhibited when the particular cylinder is in motion in a real fluid steadily. Besides, the value of the parameter thus obtained will fix up the position of the wake boundary in relation to the body and for the types of closed wakes detailed in Part III such a value of the parameter will fix up the position of the separation point as well as the end of the wake represented by the 'confluence' or the 'union' as we have called it.

Conclusion

This brings to a completion the theoretical formulations of our present investigations.

The investigation of possible closed-wake flows round a circular or an elliptic cylinder is a matter of detail and we prefer to postpone it for a future communicacation on the subject.

Finally, I must thank the authorities of the National Institute of Sciences of India who by selecting me a Senior Research Fellow of the Institute gave me leisure and opportunity to complete this theory the rudiments of which had been hanging around my mind for some time past. Lastly I take this opportunity to express my most sincere gratefulness to Prof. N. R. Sen, in whose laboratory this work was pursued and to whose constant encouragement and untiring watchfulness I owe the major part of my success in completing this theory.

Abstract

This part obtains a general formula for resistance in Potential flows with wakes past any cylinder and also the condition for the maintenace of such a flow.

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