ON ANGELESCUS POLYNOMIALS

by VIKRAMADITYA SINGH, *Department of Mathematics, University of Delhi.*

(Communicated by R. S. Varma, F.N.I.)

(*Received January 3 ; read October 9, 1953*)

INTRODUCTION

The Angelescus set of polynomials $[\Pi_n(x)]$ of degree n in x are defined as

$$\Pi_n(x) = e^x \left(\frac{d}{dx}\right)^n \{e^{-x} A_n(x)\} \quad .. \quad .. \quad .. \quad (1)$$

where

$$A_n(x) = (a_0, a_1, a_2, .. a_n \; x)^n \quad .. \quad .. \quad .. \quad .. \quad (2)$$

Shastri (1940) gave some recurrence relations for these polynomials and also their relations with well-known functions. In the present paper a Stieltjes integral representation of these polynomials has been given and it has been shown that the only orthogonal sub-set of this set is the set of confluent hypergeometric polynomials, the most well known among which is the Laguerre polynomial. This problem has also been treated in a more general form, by Meixner (1934) and Sheffer (1939). Meixner has used the generating function relation and Laplace transformation and Sheffer has used a property of zero type polynomials. I have, however, used the integral representation for these polynomials as given in the present paper. With the help of the integral representation some finite and infinite series involving Angelescus polynomials have also been summed.

I. SET OF ANGELESCUS POLYNOMIALS

We begin by proving the *lemma* :

The set of Angelescus polynomials is given by

$$\frac{\Pi_n(x)}{n\,!} = \int_{-\infty}^{\infty} L_n(x+t)\; d\beta(t), \quad .. \quad .. \quad .. \quad .. \quad (3)$$

where $L_n(x)$ is the Laguerre polynomial of degree n in x and $\beta(t)$ is such that for $n = 0, 1, 2, 3, \ldots.$ the moment constants $[\mu_n]$ given by

$$\mu_n = \int_{-\infty}^{\infty} t^n \; d\beta(t) \text{ exist and } \mu_0 \neq 0 . \quad .. \quad .. \quad .. \quad (4)$$

Proof: From equation (2) it is obvious that

$$\frac{d}{dx}\left\{\frac{A_n(x)}{n\,!}\right\} = \frac{A_{n-1}(x)}{(n-1)\,!},$$

which shows that the set of polynomials

$$\left[\frac{A_n(x)}{n!}\right]$$

belongs to the Appell set. It was proved by Sheffer (1945) that an Appell polynomial $P_n(x)$ can be represented as

$$P_n(x) = \int_0^\infty \frac{(x+t)^n}{n!}\, d\beta(t)$$

if

$$\mu_n = \int_0^\infty t^n d\beta(t) \qquad n = 0, 1, 2, \ldots .$$

exist and $\mu_0 \neq 0$.

Without any difficulty we can say that under the conditions on $\beta(t)$ as given in the lemma

$$P_n(x) = \int_{-\infty}^\infty \frac{(x+t)^n}{n!}\, d\beta(t).$$

Hence using (1) we have

$$\frac{\Pi_n(x)}{n!} = e^x \left(\frac{d}{dx}\right)^n \left\{e^{-x} \int_{-\infty}^\infty \frac{(x+t)^n}{n!}\, d\beta(t)\right\}$$

$$= e^x \int_{-\infty}^\infty \left(\frac{d}{dx}\right)^n \left\{e^{-x} \frac{(x+t)^n}{n!}\right\} d\beta(t).$$

There is no difficulty in differentiating under the integral sign, since the differentiated integral is uniformly convergent for all finite values of n and x. We get therefore

$$\frac{\Pi_n(x)}{n!} = e^x \int_{-\infty}^\infty e^t \left[\left\{\frac{d}{d(x+t)}\right\}^n \left\{e^{-(x+t)} \frac{(x+t)^n}{n!}\right\}\right] d\beta(t)$$

$$= e^x \int_{-\infty}^\infty e^t\, e^{-(x+t)} L_n(x+t)\, d\beta(t)$$

$$= \int_{-\infty}^\infty L_n(x+t)\, d\beta(t),$$

where we have used the well-known Rodrigue's formula for Laguerre polynomials. This establishes our lemma.

II. Some series involving Angelescus Polynomials

Result 1. For all finite values of ω

$$\sum_{n=0}^{\infty} \frac{\Pi_n(x)}{n!} \frac{\omega^n}{n!} = e^{\omega} \int_{-\infty}^{\infty} J_0 \left\{2\sqrt{\omega(x+t)}\right\} d\beta(t) \cdot \quad \cdot\cdot \quad \cdot\cdot \quad (5)$$

Multiplying both the sides of (3) by $\frac{\omega^n}{n!}$ and interchanging the order of summation and integration on the right hand side we get

$$\sum_{n=0}^{\infty} \frac{\Pi_n(x)}{n!} \frac{\omega^n}{n!} = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} L_n(x+t) \frac{\omega^n}{n!} d\beta(t),$$

if the integral on the right exists.

Now, from Szëgo (1939; p. 98) we have

$$\sum_{n=0}^{\infty} L_n(x) \frac{\omega^n}{n!} = e^{\omega} J_0 \left(2\sqrt{x\omega}\right).$$

Hence

$$\sum_{n=0}^{\infty} \frac{\Pi_n(x)}{n!} \frac{\omega^n}{n!} = e^{\omega} \int_{-\infty}^{\infty} J_0 \left\{2\sqrt{\omega(x+t)}\right\} d\beta(t).$$

Result 2. If $I_0(x)$ is the Bessel function of purely imaginary argument

$$e^{\frac{\omega(x+y)}{1-\omega}} \sum_{n=0}^{\infty} \frac{\Pi_n(x)}{n!} \frac{\Pi_n(y)}{n!} \omega^n = \int_{-\infty}^{\infty} d\beta(t') \int_{-\infty}^{\infty} I_0 \left\{\frac{2\sqrt{\omega(x+t)(y+t')}}{1-\omega}\right\} e^{-\frac{\omega(t+t')}{1-\omega}} d\beta(t), \quad (6)$$

whenever the integral on the right exists.

Using (3) we get on interchanging the order of summation and integration, whenever it is possible to do so, that

$$\sum_{n=0}^{\infty} \frac{\Pi_n(x)}{n!} \frac{\Pi_n(y)}{n!} \omega^n = \int_{-\infty}^{\infty} d\beta(t') \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} L_n(x+t) L_n(y+t') \omega^n d\beta(t)$$

$$= \int_{-\infty}^{\infty} d\beta(t') \int_{-\infty}^{\infty} I_0 \left\{\frac{2\sqrt{\omega(x+t)(y+t')}}{1-\omega}\right\} \frac{e^{-\frac{\omega(x+y+t+t')}{1-\omega}}}{1-\omega} d\beta(t).$$

Wherein we have used the Mehler's formula for Laguerre polynomials Szëgo (1939; p. 98)

$$\sum_{n=0}^{\infty} L_n(x) L_n(y) \omega^n = \exp\left\{-(x+y)\frac{\omega}{1-\omega}\right\} I_0 \left\{\frac{2\sqrt{xy\omega}}{1-\omega}\right\} \frac{1}{1-\omega}.$$

Now multiplying both the sides by $e^{(x+y)\frac{\omega}{1-\omega}}$ we get the result.

RESULT 3.
$$\sum_{\nu=0}^{n} \frac{\Pi_\nu(x)}{\nu\,!} = -\frac{d}{dx}\frac{\Pi_{n+1}(x)}{(x+1)\,!}. \qquad \ldots \quad \ldots \quad \ldots \quad (7)$$

We get from (3) that

$$\sum_{\nu=0}^{n} \frac{\Pi_\nu(x)}{\nu\,!} = \int_{-\infty}^{\infty} \sum_{\nu=0}^{n} L_\nu(x+t)\, d\beta(t)$$

$$= \int_{-\infty}^{\infty} \frac{n+1}{x+t}\Big[L_n(x+t) - L_{n+1}(x+t)\Big] d\beta(t)$$

$$= -\int_{-\infty}^{\infty} \frac{d}{dx} L_{n+1}(x+t)\, d\beta(t)$$

$$= -\frac{d}{dx}\frac{\Pi_{n+1}(x)}{(n+1)\,!}.$$

Wherein we have used the well-known recurrence relations about Laguerre polynomials.

III. THE ORTHOGONAL SUB-SET

It is a well known result that the necessary and sufficient condition for a polynomial $P_n(x)$ of degree n in x, to be orthogonal is that it should satisfy a difference equation of the form

$$P_n(x) = (-a_n x + b_n)\, P_{n-1}(x) + C_n P_{n-2}(x)\,, \qquad \ldots \quad \ldots \quad (8)$$

where a_n, b_n, c_n are constants independent of x and c_n is negative.

From (3) it is clear that a particular sub-set of Angelescus (1938) set of polynomials can be obtained by properly choosing the function $\beta(t)$ and that for different $\beta(t)$ we will get different sets of polynomials all belonging to the Angelescus set. We will now apply the condition (8) to (3) in order to search for what functions $\beta(t)$ do the polynomials $\Pi_n(x)$ satisfy the condition of orthogonality and from there we will get the desired orthogonal sub-set.

Using the expansion of Laguerre polynomials as

$$\left.\begin{aligned} L_n(x+t) &= \sum_{\nu=0}^{n} (-)^\nu \binom{n}{\nu} \frac{(x+t)^\nu}{\nu\,!} \\ &= \sum_{\nu=0}^{n} (-)^\nu \binom{n}{\nu} \sum_{r=0}^{\nu} \binom{\nu}{r} \frac{1}{\nu!} x^r t^{\nu-r} \\ &= \sum_{r=0}^{n} (-x)^r \binom{n}{r} \sum_{\nu=0}^{n-r} (-t)^\nu \binom{n-r}{\nu} \frac{1}{(\nu+r)}\,, \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (9)$$

we get

$$\left.\begin{aligned}\frac{\Pi_n(x)}{n!} &= \sum_{r=0}^{n}(-x)^r\binom{n}{r}\sum_{\nu=0}^{n-r}\binom{n-r}{\nu}(-)^\nu\frac{\mu_r}{(\nu+r)!} \\ &= \mu_0\frac{(-x)^n}{n!}+(-x)^{n-1}n\left\{\frac{\mu_0}{(n-)!}-\frac{\mu_1}{n!}\right\} \\ &\quad+\sum_{r=0}^{n-2}(-x)^r\binom{n}{r}\sum_{\nu=0}^{n-r}(-)^\nu\binom{n-r}{\nu}\frac{\mu_\nu}{(\nu+r)!},\end{aligned}\right\} \quad \ldots \quad \ldots \quad (10)$$

wherein we have seperated the terms containing x^n and x^{n-1} from other terms.

In like manner we may write

$$\left.\begin{aligned}(-a_nx+b_n)\frac{\Pi_{n-1}(x)}{(n-1)!} &= \frac{a_n(-x)^n}{(n-1)!}\mu_0+(-x)^{n-1}\left\{\frac{b_n\mu_0}{(n-1)!}\right. \\ &\qquad\left.+(n-1)a_n\left(\frac{\mu_0}{(n-2)!}-\frac{\mu_1}{(n-1)!}\right)\right\} \\ &\quad+\sum_{r=0}^{n-2}(-x)^r\left\{\binom{n-1}{r}b_n\sum_{\nu=0}^{n-r-1}(-)^\nu\binom{n-r-1}{\nu}\frac{\mu_\nu}{(\nu+r)!}\right. \\ &\qquad\left.+\binom{n-1}{r-1}a_n\sum_{\nu=0}^{n-r}(-)^\nu\binom{n-r}{\nu}\frac{\mu_\nu}{(\nu+r)!}\right\}.\end{aligned}\right\} \quad (11)$$

Now, if $\frac{\Pi_n(x)}{n!}$ satisfies (8) we get on equating the coefficients of x^n and x^{n-1} in the right hand side of (10) and (11) that

$$\left.\begin{aligned}a_n &= \frac{1}{n}, \\ \text{and} \quad n\left\{\frac{\mu_0}{(n-1)!}-\frac{\mu_1}{n!}\right\} &= \left\{\frac{b_n\mu_0}{(n-1)!}+a_n(n-1)\left(\frac{\mu_0}{(n-2)!}-\frac{\mu_1}{(n-1)!}\right)\right\}\end{aligned}\right\} \quad (12)$$

$$b_n = \frac{2n-1+\alpha}{n}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

where

$$\alpha = -\frac{\mu_1}{\mu_0}; \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

and comparing the coefficients of the remaining powers of x on both sides of (8) for $r = 0, 1, 2, \ldots, n-2$, we have

$$C_n \binom{n-2}{r} \sum_{\nu=0}^{n-2-r} (-)^\nu \binom{n-r-2}{\nu} \frac{\mu_\nu}{(\nu+r)!} = \binom{n}{r} \sum_{\nu=0}^{n-r} (-)^\nu \binom{n-r}{r} \frac{\mu_\nu}{(\nu+r)!}$$

$$-\binom{n-1}{r} b_n \sum_{\nu=0}^{n-r-1} (-)^\nu \binom{n-r-1}{\nu} \frac{\mu_\nu}{(\nu+r)!}$$

$$+\binom{n-1}{r-1} a_n \sum_{\nu=0}^{n-r} (-)^\nu \binom{n-r}{\nu} \frac{\mu_\nu}{(\nu+r)!} \cdot$$

Putting the values of a_n and b_n in this we get after some rearrangement

$$C_n \binom{n-2}{r} \sum_{\nu=0}^{n-2-r} (-)^\nu \binom{n-r-2}{\nu} \frac{\mu_\nu}{(\nu+r)!} = \frac{1}{n^2} \binom{n}{r} \sum_{\nu=0}^{n-r} (-)^\nu \binom{n-r}{\nu}$$

$$\times \frac{\mu_\nu}{(\nu+r)!} \Big[-\alpha(n-r-\nu) - \{ n^2 - n(2r+2\nu+1) + r^2 + r\nu + r + \nu \} \Big] .$$

From the above equation it is quite clear that C_n should be a linear function of α in order that the term containing α in the right hand side may be balanced by a corresponding one on the left hand side. Let us therefore put

$$C_n = \frac{n-1}{n} (\alpha e_n + d_n) , \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

e_n and d_n depending only on n .
Transposing we get

$$\sum_{\nu=0}^{n-r} (-)^\nu \binom{n-r}{\nu} \frac{\mu_\nu}{(\nu+r)!} \Big[-\alpha(n-r-\nu) - \{ n^2 - n(2r+2\nu+1) + r^2 + r\nu + r + \nu \}$$

$$-(n-r-\nu)(n-r-\nu-1)(d_n + \alpha e_n) \Big] = 0$$

or,

$$\sum_{\nu=0}^{n-r} (-)^\nu \binom{n-r}{\nu} \frac{\mu_\nu}{(\nu+r)!} \Big[-\alpha(n-r-\nu) \{ 1 + e_n(n-r-\nu-1) \}$$

$$-\{ n^2 - n(2r+2\nu+1) + r^2 + r\nu + r + \nu + d_n(n-r-\nu)(n-\nu-r-1 \} \Big] = 0 \quad \ldots \quad (16)$$

It should be noted here that the above equation holds not only for $r = 0, 1, 2, \ldots, n-2$ but also for $r = n-1, n$. In the first case it gives the relation (14) and in the second case it shows that $\mu_0 \neq 0$.

The above equations on being solved will give us the values of μ_ν. Now, from (4) we see that μ_ν is always independent of n and we will utilize this fact to show that

$$d_n = -1 \text{ and } e_n = \frac{a}{n-1} \cdot \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

Putting $r=n-2$ in (16) we get

$$\mu_0(n-1)\left[-\alpha(1+e_n)-(d_n+1)\right]-\mu_1\{-\alpha+(n-1)\}+\mu_2=0$$

$$\therefore \quad \mu_2=-\mu_1 a-\mu_1 e_n(n-1)+\mu_0(n-1)(d_n+1),\ldots$$

which shows that in order that μ_2 be independent of n, e and d_n must satisfy (17).

Therefore we can write

$$\mu_2=-\mu_1\alpha-a\mu_1=-(\alpha-1)\mu_1+(a+1)\alpha\mu_0 .$$

With these values of e_n and d_n (16) reduces to

$$\sum_{\nu=0}^{n-r}(-)^\nu\binom{n-r}{\nu}\frac{\mu_\nu}{(\nu+r)!}\left[-\left\{1+\frac{a}{n-1}(n-r-\nu-1)\right\}\alpha(n-r-\nu)+\nu(\nu+r)\right]=0$$

$$r=0,1,2,\ldots\ldots\ldots,n. \quad \ldots \quad \ldots \quad (18)$$

We will now show that the system of equations (18) is equivalent to the system

$$\mu_n=-(a-n+1)\mu_{n-1}+\alpha(a+1)(n-1)\mu_{n-2}, \quad \ldots \quad \ldots \quad (19)$$

for $\quad n=1,2,3,\ldots.$

if we take $\quad \mu_{-1}=0.$

Putting $\quad n-r=p$ (18) reduces to

$$\sum_{\nu=0}^{p}(-)^\nu\binom{p}{\nu}\frac{\mu_\nu}{(\nu+n-p)!}\left[-\left\{\left(1-\frac{p-r-1}{n-1}\right)+\frac{a+1}{n+1}(p-\nu-1)\right\}\alpha(p-\nu)\cdot\right.$$

$$\left.+\nu(\nu+n-p)\right]=0$$

or,

$$\sum_{\nu=0}^{p}(-)^\nu\binom{p}{\nu}\frac{\mu_\nu}{(\nu+n-p)!}\left[-\{(n+\nu-p)+(a+1)(p-\nu-1)\}\right.$$

$$\left.\alpha(p-\nu)+\nu(\nu+n-p)\{(\nu+n-p-1)-(\nu-p)\}\right]=0$$

or,

$$\sum_{\nu=0}^{p}\frac{(-)\binom{p}{\nu}\mu_\nu}{(\nu+n-p)!}\left[-\alpha(a+1)(p-\nu)(p-\nu-1)+\nu(\nu+n-p)(\nu+n-p-1)\right.$$

$$\left.+(\alpha-\nu)(p-\nu)(n+\nu-p)\right]=0$$

or,

$$\sum_{\nu=0}^{p-2}\frac{(-)^\nu p!\,(a+1)\alpha\mu_\nu}{(p-\nu-2)!\,\nu!\,(\nu+n-p)!}+\sum_{\nu=0}^{p-2}\frac{(-)^\nu\mu_{\nu+2}\,p!}{(p-\nu-2)!\,(\nu+1)!\,(\nu+n-p)!}-\frac{\mu_1 p}{(n-p-1)!}$$

$$+\sum_{\nu=0}^{p-2}\frac{(-)^\nu\mu_{\nu+1}\,p!\,(\alpha-\nu-1)}{(p-\nu-2)!\,(\nu+1)!\,(\nu+n-p)!}-\frac{\mu_0\alpha p}{(n-p-1)!}=0$$

$$\sum_{\nu=0}^{p-2}\frac{(-)^\nu p!}{(p-\nu-2)!\,(\nu+1)!\,(\nu+n-p)!}\left[\mu_{\nu+2}-\alpha(\nu+1)(a+1)\mu_\nu+(\alpha-\nu-1)\mu_{\nu+1}\right]=0$$

$$\ldots \quad (20)$$

From (20) it is quite evident that if (19) is satisfied (18) will automatically hold.

To prove the converse we use the method of induction. We have already seen that (19) holds for $n = 2$ and it can easily be verified that it holds for $n = 3$ also. Let us now suppose that it holds for $n = 1, 2, \ldots, m$. Then putting $n = m+1$ in (20) we get

$$\sum_{\nu=0}^{m-1} \frac{(-)^\nu p!\, \{\mu_{\nu+2}-(\nu+1)(a+1)\mu_\nu+(\alpha-\nu-1)\mu_{\nu+1}\}}{(p-\nu-2)!\,(\nu+1)!\,(\nu+n-p)!} = 0\,.$$

Whence

$$\mu_{m+1}+(\alpha-m)\mu_m-m(a+1)\mu_{m-1} = 0\,,$$

because by hypothesis the other terms reduce to zero. But this equation is obtained by putting $n = m+1$ in (19). We thus see that if (19) holds for $n = m$ it also holds for $n = m+1$. Hence it holds for all positive values of n.

IV. The Function $\beta(t)$

We have now to determine the function $\beta(t)$ when μ_n satisfies the difference equation (19). It has to be noted that $\beta(t)$ is independent of n. Substituting from (4) in (19) we have for $n = 1, 2, \ldots.$

$$\int_{-\infty}^{\infty} t^n\, d\beta(t) = -(\alpha-n+1)\int_{-\infty}^{\infty} t^{n-1}\, d\beta(t)+\alpha(a+1)(n-1)\int_{-\infty}^{\infty} t^{n-2}\, d\beta(t)$$

$$= -(\alpha+1)\int_{-\infty}^{\infty} t^{n-1} d\beta(t) - \int_{-\infty}^{\infty} t^n d^2\beta(t) - \Big[t^n\, d\beta(t)\Big]_{-\infty}^{\infty}$$

$$+\alpha(a+1)\Big[t^{n-1} d\beta(t)\Big]_{-\infty}^{\infty} - \alpha(a+1)\int_{-\infty}^{\infty} t^{n-1} d^2\beta(t)\,.$$

Where we have assumed $\beta(t)$ to be absolutely continuous. Taking

$$\Big[t^{n-1}\, d\beta(t)\Big]_{-\infty}^{\infty} = 0 \quad n = 1, 2, 3, \ldots, \qquad \ldots \quad \ldots \quad (21)$$

and we get

$$\{t+(a+1)\alpha\}\, \frac{d^2\beta(t)}{dt^2} + (t+\alpha+1)\, \frac{d\beta(t)}{dt} = T(t) \qquad \ldots \quad \ldots \quad (22)$$

such that

$$\int_{-\infty}^{\infty} x^i T(x)\, dx = 0\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

Integrating (22) we get

$$\frac{d\beta(t)}{dt} = e^{-t}(t+a\alpha+\alpha)^{a\alpha-1}\left[A+\int^t \frac{T(u)e^u du}{(u+a\alpha+\alpha)^{a\alpha}}\right].$$

This implies the existence of the integrals of the form

$$\int^{t} \frac{T(u)e^{u}}{(u+a\alpha+\alpha)^{a\alpha}}\,du\,,$$

so that we find from (21) that the limits of integration would be $(-a\alpha-\alpha, \infty)$, $a\alpha<1$.

Let us now define

$$T_1(t) = \frac{d\beta(t)}{dt} + ce^{-t}\,(t+a\alpha+\alpha)^{a\alpha-1}, \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

and determine c such that

$$\int_{-a\alpha-\alpha}^{\infty} T_1(t)\,dt = \int_{-a\alpha-\alpha}^{\infty} \left[\frac{d\beta(t)}{dt} + c\,e^{-t}\,(t+a\alpha+\alpha)^{a\alpha-1}\right] dt = 0\,. \ldots \quad (25)$$

Again, we have, when i is an integer $\geqslant 1$

$$\int_{-a\alpha-\alpha}^{\infty} \frac{T_1(x)}{e^{-x}(x+a\alpha+\alpha)^{a\alpha-1}}\,\frac{d}{dx}\left\{(x+a\alpha+\alpha)^{a\alpha+i-1}\,e^{-x}\right\} dx$$

$$= \left[(x+a\alpha+\alpha)^{i}\left\{\frac{d}{dx}\,\beta(x) + ce^{-x}\,(x+a\alpha+\alpha)^{a\alpha-1}\right\}\right]_{-a\alpha-\alpha}^{\infty}$$

$$+ \int_{-a\alpha-\alpha}^{\infty} (x+a\alpha+\alpha)^{i-1}\,T(x)\,dx$$

$$= 0\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

The integral vanishing by virtue of (23) and the integrating term vanishing by virtue of (21).

But we find that the left hand side of (26) is

$$\int_{-a\alpha-\alpha}^{\infty} T_1(x)\left\{(a\alpha+i-1)\,(x+a\alpha+\alpha)^{i-1} - (x+a\alpha+\alpha)^{i}\right] dx = 0\,.$$

Now putting $i = 1$ we find with the help of (25) that

$$\int_{-a\alpha-\alpha}^{\infty} T_1(x)\,(x+a\alpha+\alpha)\,dx = 0\,,$$

i.e.

$$\int_{-a\alpha-\alpha}^{\infty} xT_1(x)\,dx = 0\,.$$

Similarly putting $i = 2, 3, 4,$ in succession we find that

$$\int_{-a\alpha-\alpha}^{\infty} x^{i}T_1(x)\,dx = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

i.e. $T_1(x)$ is a function whose moments of all order vanish. Hence for all practical purposes we can neglect the function $T_1(t)$ and write

$$\frac{d\beta(t)}{dt} = A\, e^{-t}\,(t+a\alpha+\alpha)^{a\alpha-1} \qquad \infty \geqslant t \geqslant -a\alpha-\alpha$$

$$= B\, e^{-t}\, t^{a\alpha-1} \qquad \infty \geqslant t \geqslant 0$$

With this value of $\frac{d\beta(t)}{dt}$ we will find the generating function for Angelescus Polynomials to be of the form

$$B\,\frac{\Gamma(a\alpha)}{(1-t)^{-a\alpha}}$$

which is of the same form as that of Laguerre polynomials.

Acknowledgements

My thanks are due to Dr. R. S. Varma, Senior Scientist, Defence Science Organization, for suggesting me the problem and for guidance in the preparation of the paper. Thanks are also due to Dr. Ram Behari, Head of the Mathematics Department, Delhi University, for kindly providing me all facilities to carry on research work and for his encouragement.

References

Angelescus, C. R. (1938). *Acad. Sci. Roum.*, **2**, 199–201. (vide also *Zbl. f. Math.*, **1**, 1938, (8), 356.)

Meixner, J, (1934). *Jour. Lond. Math. Soc.*, **9**, 6–13.

Shastri, N. A. (1940). Angelescus Polynomial. *Proc. Ind. Acad. Sci.*, **11**, 312–317.

Sheffer, I. M. (1939). Some properties of polynomial sets of type zero. *Duke. Math. Journ.*, **5**, 590.

Sheffer, I. M. (1945). Note on Appell Polynomials. *Bull. Amer. Math. Soc.*, **51**, 739–744.

Singh, V. Appell set of polynomials (in press)

Szego, G. (1939). Orthogonal Polynomials, Ch. V.

Issued June 25, 1954.