RELATION BETWEEN MAXIMUM PRESSURE AND SHOT-START PRESSURE

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1. INTRODUCTION

The main problem of Internal Ballistics is the calculation of Maximum Pressure and Muzzle Velocity for given loading conditions in a gun.

The equations of Internal Ballistics in their general form are such that it is not possible to integrate them directly but can be treated by numerical integration or by the use of differential analyser. Mr. N. S. Venkatesan (1952) has recently given an explicit expression for the relation between the maximum pressure and shot-start pressure and has solved the equations of Internal Ballistics under the assumptions (Corner, 1950) $\theta = 0$, i.e., the propellant is tubular in shape and (Corner, 1951) $B = 0$, i.e., neglecting the covolume correction for gases.

The object of this paper is to extend the results of Mr. Venkatesan and obtain an explicit expression for the relation between maximum pressure and shot-start pressure for all values of θ. We shall assume that $B = 0$, i.e., the covolume of the gases equals the reciprocal of the density of the solid propellant. This is generally true except at high densities of loading.

2. FUNDAMENTAL EQUATIONS

The four fundamental equations of the Internal Ballistics, in non-dimensional parameters, are shown, in standard books, (Corner, 1950 and 1951), to be

$$z = \zeta\,(\xi - Bz) + \frac{\gamma - 1}{2M}\,\eta^2 \quad \ldots \quad (1)$$

$$M\zeta = \eta\,\frac{d\eta}{d\xi} \quad \ldots \quad (2)$$

$$\zeta = -\eta\,\frac{df}{d\xi} \quad \ldots \quad (3)$$

$$z = (1-f)\,(1+\theta f) \quad \ldots \quad (4)$$

where

$$\xi = 1 + \frac{x}{l}$$

$$\eta = \frac{AD}{F\beta C}\,v$$

$$\zeta = \frac{Al}{FC}\,p$$

$$M = \frac{A^2D^2}{F\beta^2 C m_1} \text{ (Central Ballistic Constant).}$$

For $B = 0$ the above equations reduce to

$$z = \zeta\xi + \frac{\gamma-1}{2M}\eta^2 \quad \text{(I)}$$

$$M\zeta = \eta\frac{d\eta}{d\xi} \quad \text{(II)}$$

$$\zeta = -\eta\frac{df}{d\xi} \quad \text{(III)}$$

$$z = (1-f)\,(1+\theta f) \quad \text{(IV)}$$

The initial conditions at the shot-start are,

$$\xi = 1\,;\ \eta = 0\,;\ \zeta = \zeta_0 \quad \text{and } z = z_0.$$

From (1) we see that $z_0 = \zeta_0$ so that z_0 itself is a measure of shot-start pressure.

From (II) and (III) we obtain that

$$M = -\frac{d\eta}{df}.$$

Integrating this we get,

$$\eta = M\,(f_0-f), \text{ where } f = f_0 \text{ when } \eta = 0. \quad (5)$$

Also from (IV), $z_0 = (1-f_0)\,(1+\theta f_0)$, so that we can express f_0 in terms of z_0 and f in terms of z, and hence from (5), z can be expressed in terms of η and z_0.

Equation (IV) can be written as:

$$\theta f^2-(\theta-1)f+(z-1) = 0$$

$$\therefore f = \frac{(\theta-1)\pm\sqrt{(\theta+1)^2-4\theta z}}{2\theta}$$

and hence

$$f_0 = \frac{(\theta-1)\pm\sqrt{(\theta+1)^2-4\theta z_0}}{2\theta}$$

Hence equation (5) becomes,

$$\eta = \pm\,\frac{M}{2\theta}\left[\sqrt{(\theta+1)^2-4\theta z_0} - \sqrt{(\theta+1)^2-4\theta z}\right] \quad (6)$$

Now $\eta = \frac{AD}{F\beta C}v$; A, D, F, β, C and v are all positive, therefore η is also positive.

As t increases z increases; therefore $z > z_0$ and also the maximum value of z is unity. Since η is always positive, we have to choose the positive sign in (6). Substituting the value of ζ from (II) in equation (1) we have,

$$z = \frac{\eta}{M}\frac{d\eta}{d\xi}\xi + \frac{\gamma-1}{2M}\eta^2$$

or

$$\frac{d\xi}{\xi} = \frac{\eta\,d\eta}{Mz-\frac{\gamma-1}{2}\eta^2} \quad (7)$$

Equation (6) can be written as :

$$\frac{2\theta\eta}{M} = \left[a - \sqrt{(\theta+1)^2 - 4\theta z} \right]$$

where $$a = \sqrt{(\theta+1)^2 - 4\theta z_0}$$

Squaring and making some simplification, we have,

$$Mz = \frac{1}{M}\left[M^2 z_0 + aM\eta - \theta\eta^2 \right],$$ for all values of θ, negative or positive.

Substitution of this value of Mz in equation (7) gives,

$$\frac{d\xi}{\xi} = \frac{2M\eta\, d\eta}{2M^2 z_0 + 2aM\eta - (\overline{\gamma-1}\, M + 2\theta)\eta^2} \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

Hence $$\frac{d\xi}{\xi} = \frac{2M\eta\, d\eta}{G + 2F\eta - E\eta^2},$$

where

$$G = 2M^2 z_0,\ F = aM,\ E = (\overline{\gamma-1}\, M + 2\theta)$$

Integration of this equation gives,

$$\frac{1}{M}\log\xi = -\frac{1}{E}\log(G + 2F\eta - E\eta^2) - \frac{F}{E^2K}\log\frac{\eta - \frac{F}{E} - K}{\eta - \frac{F}{E} + K} + \text{constant}$$

where $K = \sqrt{\left(\frac{G}{E} + \frac{F^2}{E^2}\right)}$

Initially, when $\xi = 1$, $\eta = 0$, $\zeta = \zeta_0$ and $z = z_0$ and hence the value of the constant is,

$$\log G^{\frac{1}{E}} \left[\frac{-\frac{F}{E} - K}{-\frac{F}{E} + K} \right]^{\frac{F}{E^2K}}$$

Therefore

$$\frac{1}{M}\log\xi = -\frac{1}{E}\log(G + 2F\eta - E\eta^2) - \frac{F}{E^2K}\log\frac{\eta - \frac{F}{E} - K}{\eta - \frac{F}{E} + K} + \log G^{\frac{1}{E}} \left[\frac{-\frac{F}{E} - K}{-\frac{F}{E} + K} \right]^{\frac{F}{E^2K}}$$

$$\text{or } \xi^{\frac{E}{M}}(G + 2F\eta - E\eta^2) = G\left[\frac{K + \eta - \frac{F}{E}}{K - \eta + \frac{F}{E}} \right]^{\frac{F}{EK}} \left[\frac{K + \frac{F}{E}}{K - \frac{F}{E}} \right]^{\frac{F}{EK}}$$

Substituting the values of G, F and E we have,

$$\xi^{\frac{(\overline{\gamma-1}M+2\theta)}{M}}\left[2M^2z_0+2aM\eta-(\overline{\gamma-1}M+2\theta)\eta^2\right]$$

$$=2M^2z_0\left[\frac{K+\eta-\dfrac{aM}{\overline{\gamma-1}M+2\theta}}{K-\eta+\dfrac{aM}{\overline{\gamma-1}M+2\theta}}\right]^{\frac{aM}{(\overline{\gamma-1}M+2\theta)K}}\cdot\left[\frac{K+\dfrac{aM}{\overline{\gamma-1}M+2\theta}}{K-\dfrac{aM}{\overline{\gamma-1}M+2\theta}}\right]^{\frac{aM}{(\overline{\gamma-1}M+2\theta)K}} \quad \ldots \quad \text{(V)}$$

where

$$K=\sqrt{\left\{\frac{2M^2z_0}{(\overline{\gamma-1}M+2\theta)}+\frac{a^2M^2}{(\overline{\gamma-1}M+2\theta)^2}\right\}}$$

3. Maximum Pressure

Equation (8) can be written as,

$$\frac{d\eta}{d\xi}=\frac{2M^2z_0+2aM\eta-(\overline{\gamma-1}M+2\theta)\eta^2}{2M\eta\xi}$$

Using this quantity in (II) we have,

$$\zeta=\frac{[2M^2z_0+2aM\eta-(\overline{\gamma-1}M+2\theta)\eta^2]}{2M^2\xi} \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

Inserting the value of ξ from (V) here we get,

$$\zeta=\frac{[2M^2z_0+2aM\eta-(\overline{\gamma-1}M+2\theta)\eta^2]^{\frac{\gamma M+2\theta}{\overline{\gamma-1}M+2\theta}}}{2M^2\left[2M^2z_0\left(\dfrac{1+K_1\eta}{1-K_2\eta}\right)^{\frac{aM}{(\overline{\gamma-1}M+2\theta)K}}\right]^{\frac{M}{\overline{\gamma-1}M+2\theta}}} \quad \ldots \quad \ldots \quad \text{(VI)}$$

where

$$\frac{1}{K_1}=K-\frac{aM}{\overline{\gamma-1}M+2\theta}$$

and

$$\frac{1}{K_2}=K+\frac{aM}{\overline{\gamma-1}M+2\theta}$$

Now maximum pressure is obtained by eliminating η between (VI) and $\dfrac{d\zeta}{d\eta}=0$. Differentiating (9) with respect to η we get,

$$\frac{d\zeta}{d\eta}=\frac{-[2M^2z_0+2aM\eta-(\overline{\gamma-1}M+2\theta)\eta^2]}{2M^2}\frac{1}{\xi^2}\frac{d\xi}{d\eta}+\frac{[2aM-2(\overline{\gamma-1}M+2\theta)\eta]}{2M^2\xi}$$

Substituting the value of $\dfrac{d\xi}{d\eta}$ in $\dfrac{d\zeta}{d\eta}=0$ and simplifying, we get,

$$\eta=\frac{aM}{(\gamma M+2\theta)}=\eta_1 \text{ (say)}$$

Hence maximum pressure is given by,

$$\zeta_1' = \frac{2M^2z_0+2aM\eta_1-(\overline{\gamma-1}M+2\theta)\eta_1^2}{2M^2\xi_1} \quad \ldots \quad \text{(VII)}$$

where ξ_1 is given by (V) by putting $\eta = \eta_1$.

The fact that z_0 is small can be utilised in expanding ζ_1' in powers of z_0 giving an explicit expression for the relation between maximun pressure and shot-start pressure. We get after some simplification :

$$\zeta_1' = \left[\frac{(\theta+1)^2(\overline{\gamma+1}M+2\theta)}{2(\gamma M+2\theta)^2\, b_0^{\frac{M}{\gamma M+2\theta}}}\right]^{\frac{\gamma M+2\theta}{\overline{\gamma-1}M+2\theta}} \cdot Z^{\left[-\frac{Mz_0}{(\theta+1)^2}+z_0^2\frac{\{\frac{3}{2}M^2(\gamma-1)^2+2M\theta(\gamma-1)-2\theta^2\}M}{(\theta+1)^4(\overline{\gamma-1}M+2\theta)}\right]}$$

$$\times\Bigg[1+z_0\left[\frac{M}{(\theta+1)^2}\log b_0-\frac{Ma_0'}{\overline{\gamma-1}M+2\theta}+M\frac{\gamma M+2\theta}{\overline{\gamma-1}M+2\theta}\cdot\frac{2\gamma^2M+4\theta(\gamma-1)}{(\theta+1)^2(\overline{\gamma+1}M+2\theta)}\right]$$

$$+\frac{z_0^2}{2}\Bigg[\frac{M^2}{(\theta+1)^4}(\log b_0)^2-\frac{M\{3M^2(\gamma-1)^2+4M\theta(\gamma-1)-4\theta^2\}\log b_0}{(\theta+1)^4(\overline{\gamma-1}M+2\theta)}$$

$$-\frac{2M^2a_0'\log b_0}{(\overline{\gamma-1}M+2\theta)(\theta+1)^2}$$

$$+\frac{2M}{(\overline{\gamma-1}M+2\theta)}\left\{\left(\frac{a_0'^2}{2}-a_1'\right)+\frac{a_0'(\overline{\gamma-1}M+2\theta)}{(\theta+1)^2}+\frac{a_0'^2M}{2(\overline{\gamma-1}M+2\theta)}\right\}$$

$$+\left\{\frac{M}{(\theta+1)^2}\log b_0-\frac{Ma_0'}{\overline{\gamma-1}M+2\theta}\right\}2M\cdot\frac{\gamma M+2\theta}{\overline{\gamma-1}M+2\theta}\cdot\frac{2\gamma^2M+4\theta(\gamma-1)}{(\theta+1)^2(\overline{\gamma+1}M+2\theta)}$$

$$+M^2\left(\frac{\gamma M+2\theta}{\overline{\gamma-1}M+2\theta}\right)\left(\frac{M}{\overline{\gamma-1}M+2\theta}\right)\cdot\left\{\frac{2\gamma^2M+4\theta(\gamma-1)}{(\theta+1)^2(\overline{\gamma+1}M+2\theta)}\right\}^2\Bigg]$$

$$+\ldots\ldots\ldots\ldots\Bigg] \quad \ldots \quad \text{(VIII)}$$

where

$$a_0' = \frac{a_0}{b_0} \text{ and } a_1' = \frac{a_1}{b_0}$$

Here the constants a_0, a_1 and b_0 stand for the following expressions :—

$$a_0 = \frac{2M^2(\gamma^2+2\gamma-1)-8\theta^2}{(\overline{\gamma+1}M+2\theta)^2}$$

$$a_1 = -\frac{\{(\gamma^2+6\gamma+1)M^2+4\theta(\gamma+3)M+4\theta^2\}(\overline{\gamma-1}M+2\theta)^2}{2(\theta+1)^2(\overline{\gamma+1}M+2\theta)^3}$$

and $$b_0 = \frac{2(\theta+1)^2}{(\overline{\gamma+1}M+2\theta)}.$$

We can tabulate, in double entry table forms, the values of ζ_1' against z_0 and M, for different values of θ. This is done in the table given below for $\theta = 1$.

TABLE FOR ζ_1'

$\gamma = 1{\cdot}24$ $\theta = 1$

Z_0 \ M	1	2	3	4	5
0·00	·5531	·3829	·2929	·2372	·1992
0·01	·5625	·3952	·3063	·2513	·2137
0·02	·5700	·4050	·3165	·2625	·2252
0·03	·5771	·4141	·3271	·2728	·2357
0·04	·5836	·4225	·3364	·2826	·2457
0·05	·5898	·4306	·3454	·2920	·2554

The table shows that maximum pressure increases as the shot-start pressure increases but decreases as M increases.

4. Covolume correction for maximum pressure

We can apply the covolume correction for maximum pressure for different values of M in the following manner:

$$\zeta_1 = \zeta_1'/[1-B\{\zeta_1'+(1+\theta)^2\gamma\lambda_1/M\}] \quad .. \quad \text{[Corner, 1951]}$$

where ζ_1 = maximum value of ζ

ζ_1' = maximum value of ζ for $B = 0$

$$\lambda_1 = \frac{\dfrac{M}{\theta+\frac{1}{2}(\gamma-1)M}}{\left[\dfrac{M}{\theta+\frac{1}{2}(\gamma-1)M}+1\right]\left[\dfrac{M}{\theta+\frac{1}{2}(\gamma-1)M}+2\right]}$$

and

$$B = \left(b-\frac{1}{\delta}\right)\frac{C}{Al}$$

Now we apply this correction to our table for a particular case for which,

Shot-weight $m = 20$ lbs.
Shot travel $x_3 = 36{\cdot}959$ in.
Chamber capacity $k_0 = 82$ cub. in.
Total capacity $k_3 = 491$ cub. in.
Form coefficient $\theta = 1$.
Ballistic size $D = {\cdot}016$.

Propellant data are:

$F = 2010$ inch-tons/lb.,
$\beta = 1{\cdot}08$,
$\gamma = 1{\cdot}24$,
$\frac{1}{\delta} = 17{\cdot}3$,
$b = 25{\cdot}5$ c.in./lb.

For the above data

$$A = \frac{K_3 - K_0}{x_3} = 11 \cdot 0663$$

We know that B is a function of C, and also M is a function of C in this particular case,

$$M = \frac{A^2 D^2}{F\beta^2 C m_1} \text{ where } m_1 = 1 \cdot 06m + \tfrac{1}{3} \cdot C$$

For this particular case,

$$M = \frac{k}{C(1 \cdot 06m + \frac{1}{3}C)}, \text{ where } k = \frac{A^2 D^2}{F\beta^2}$$

or

$$C^2 + 3 \cdot 18\, mC - \frac{3k}{M} = 0$$

This gives,

$$C = \frac{-3 \cdot 18m + \sqrt{(3 \cdot 18m)^2 + \frac{12k}{M}}}{2}$$

In this present case we get the values of B for different values of M as given below, and graphs have been drawn in this particular case, showing relation between M and C; and also between B and C. (See graphs 1 and 2.)

M	1	2	3	4	5	6	7	8	9	10
C	·5411	·2717	·1814	·1362	·1090	·0908	·0779	·0682	·0606	·0546
B	·0611	·0288	·0189	·0140	·0112	·0093	·0079	·0069	·0061	·0055

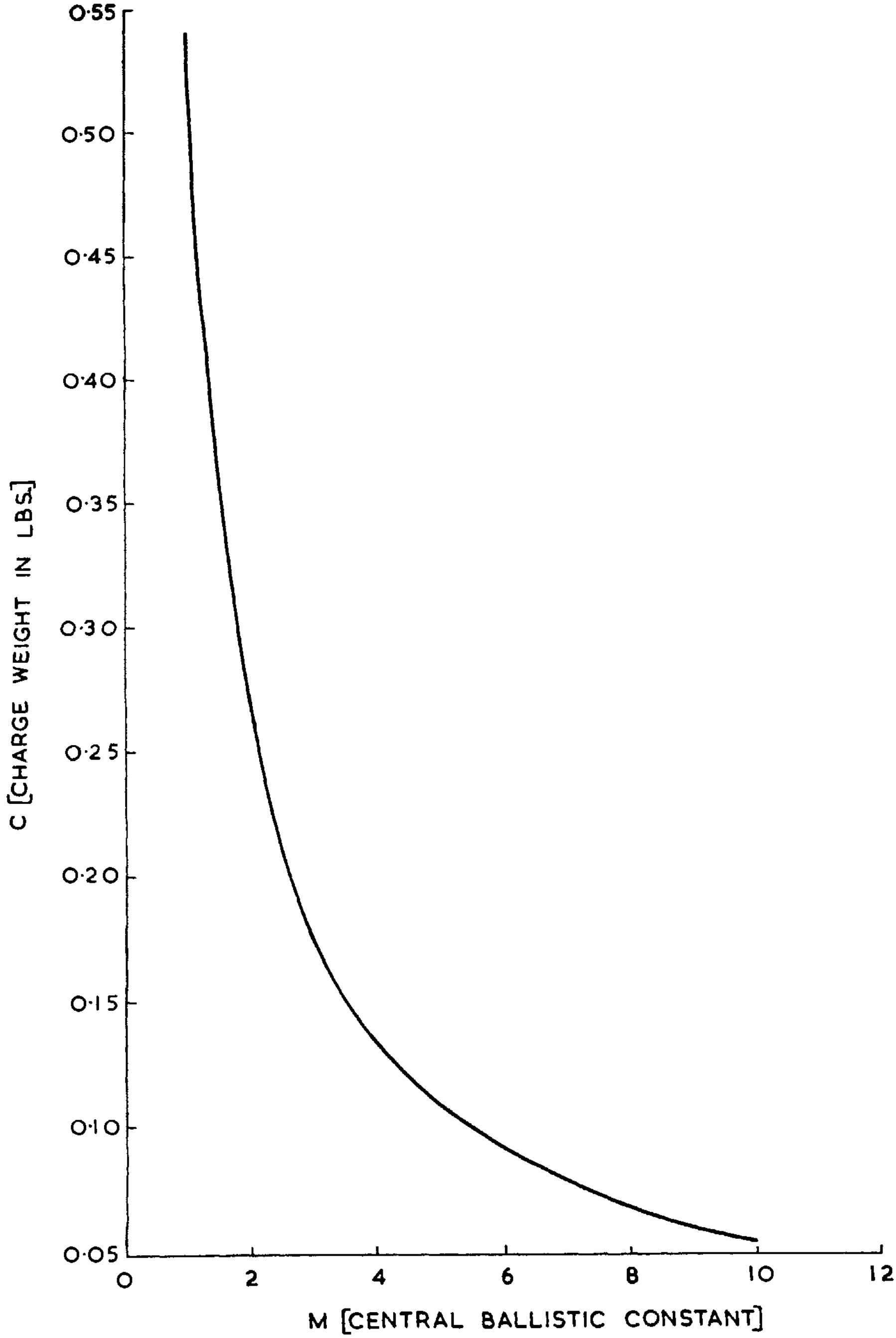
Applying the covolume correction the modified table becomes as follows:

TABLE FOR ζ_1

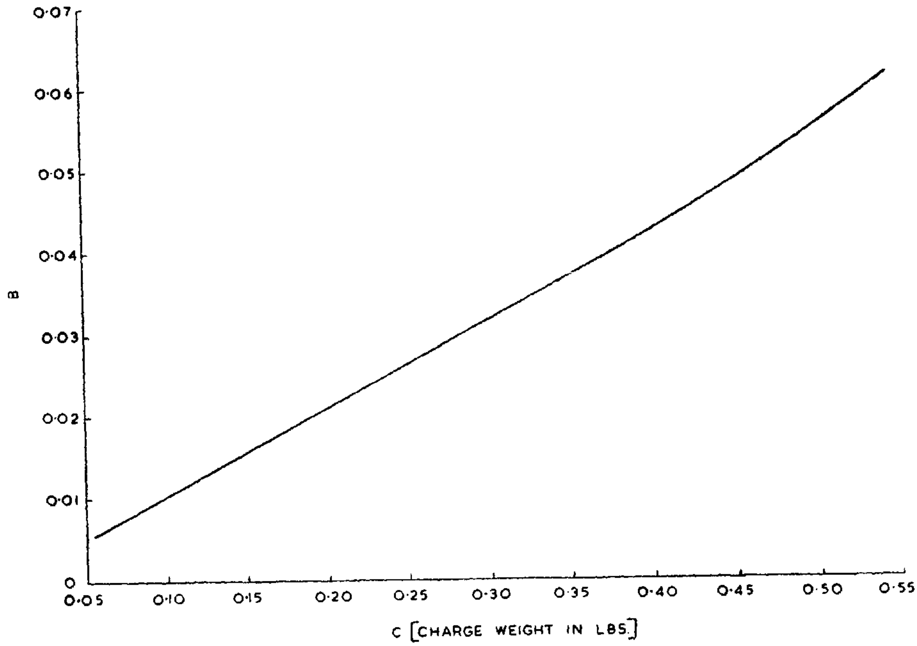
(Applying covolume correction)

$\gamma = 1 \cdot 24$ $\theta = 1$

Z_0 \ M	1	2	3	4	5
0·00	·5801	·3884	·2949	·2382	·1997
0·01	·5904	·4011	·3085	·2524	·2143
0·02	·5986	·4111	·3188	·2637	·2259
0·03	·6063	·4204	·3296	·2741	·2364
0·04	·6133	·4291	·3390	·2839	·2465
0·05	·6201	·4374	·3482	·2934	·2562



GRAPH I. RELATION BETWEEN CHARGE WEIGHT AND CENTRAL BALLISTIC CONSTANT (IN THE PARTICULAR CASE)



GRAPH 2. RELATION BETWEEN CHARGE WEIGHT AND B $\left[B=(b-\frac{1}{\delta})\frac{C}{Al}\right]$
(IN THE PARTICULAR CASE)

5. It is to be noted that there is a limitation to the value of M. Equation (6) can be written for the maximum pressure as,

$$\eta_1 = \frac{M}{2\theta}\left[\sqrt{(\theta+1)^2-4\theta z_0}-\sqrt{(\theta+1)^2-4\theta z_1}\right]$$

From this we get,

$$Mz_1 = \frac{1}{M}\left[M^2z_0+aM\eta_1-\theta\eta_1^2\right]$$

Since $z_1 \leqslant 1$ we must have,

$$\frac{1}{M^2}\left[M^2z_0+aM\eta_1-\theta\eta_1^2\right]\leqslant 1.$$

or

$$\frac{1}{M^2}\left[M^2z_0+\frac{a^2M^2}{(\gamma M+2\theta)}-\frac{\theta a^2M^2}{(\gamma M+2\theta)^2}\right]\leqslant 1$$

or

$$(\gamma M+\theta)a^2 \leqslant (1-z_0)(\gamma M+2\theta)^2$$

Substituting the value of a we get,

$$(\gamma M+\theta)\{(\theta+1)^2-4\theta z_0\} \leqslant (1-z_0)(\gamma M+2\theta)^2$$

Simplifying we get,

$$\left[\gamma M-\frac{(\theta+1)^2}{2(1-z_0)}\right]^2-\frac{\theta(\theta-1)^2}{(1-z_0)}-\frac{(\theta-1)^4}{4(1-z_0)^2}\geqslant 0$$

$$\therefore M \geqslant \left\{\sqrt{\frac{\theta(\theta-1)^2}{(1-z_0)}+\frac{(\theta-1)^4}{4(1-z_0)^2}}+\frac{(\theta-1)^2}{2(1-z_0)}\right\}\Big/\gamma$$

This gives a lower limit to the value of M. For all values of M less than this value, the maximum pressure occurs at 'all burnt' position.

In particular when $\theta = 0$ the inequality for M is,

$$M \geqslant \frac{1}{\gamma(1-z_0)}$$

And when $\theta = 1$, the lower limit for M is given by,

$$M > 0$$

Also in particular when $\theta = 0$ the results (V), (VI) and (VIII) reduce to the following results due to N. S. Venkatesan.

$$\xi^{\gamma-1}[2Mz_0+2\eta-(\gamma-1)\eta^2] = 2Mz_0\left[\frac{K+\frac{1}{\gamma-1}}{K-\frac{1}{\gamma-1}}\right]^{\frac{1}{K(\gamma-1)}} \cdot \left[\frac{K+\eta-\frac{1}{\gamma-1}}{K-\eta+\frac{1}{\gamma-1}}\right]^{\frac{1}{K(\gamma-1)}}$$

where $$K = \sqrt{\left\{\frac{1}{\gamma-1}\left(2Mz_0+\frac{1}{\gamma-1}\right)\right\}}$$

$$\zeta = \frac{[2Mz_0+2\eta-(\gamma-1)\eta^2]^{\frac{\gamma}{\gamma-1}}}{2M\left[2Mz_0\left(\frac{1+K_1\eta}{1-K_2\eta}\right)^{\frac{1}{K(\gamma-1)}}\right]^{\frac{1}{\gamma-1}}},$$

where $$\frac{1}{K_1} = K-\frac{1}{\gamma-1}$$

and $$\frac{1}{K_2} = K+\frac{1}{\gamma-1}$$

Maximum pressure for this case is given by,

$$\zeta_1' = \left[\frac{\gamma+1}{2M\gamma^2 b_0^{\frac{1}{\gamma}}}\right]^{\frac{\gamma}{\gamma-1}} \cdot Z_0^{\left[-MZ_0+\frac{3}{2}M^2Z_0^2(\gamma-1)\right]}$$

$$\times\Bigg[1+z_0\left\{M\log b_0-\frac{a_0'}{\gamma-1}+\frac{2M\gamma^3}{(\gamma-1)(\gamma+1)}\right\}$$

$$+\frac{z_0^2}{2}\Bigg\{M^2(\log b_0)^2-3M^2(\gamma-1)\log b_0-\frac{2Ma_0'\log b_0}{\gamma-1}$$

$$+2a_0'M+\frac{a_0'^2\gamma}{(\gamma-1)^2}-\frac{2a_1'}{(\gamma-1)}+\left(M\log b_0-\frac{a_0'}{\gamma-1}\right)\frac{4M\gamma^3}{(\gamma-1)(\gamma+1)}$$

$$+\frac{4M^2\gamma^6}{(\gamma-1)^2(\gamma+1)^2}\Bigg\}$$

$$+\cdots\cdots\cdots\cdots\Bigg]$$

where
$$a_0 = \frac{2(\gamma^2+2\gamma-1)}{(\gamma+1)^2}, \qquad a_0' = \frac{a_0}{b_0}$$

$$a_1 = -\frac{(\gamma^2+6\gamma+1)(\gamma-1)^2 M}{2(\gamma+1)^3}, \qquad a_1' = \frac{a_1}{b_0}$$

$$b_0 = \frac{2}{M(\gamma+1)}$$

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SUMMARY

Neglecting the covolume correction and taking any shape of the propellant, i.e., any value of θ, the equations of the internal ballistics of a conventional gun are integrated and an explicit relation between maximum pressure and shot-start pressure is derived. A double entry table is worked out to illustrate the variation of maximum pressure with shot-start pressure and the central ballistic constant; and further this table has been modified by applying the covolume correction for maximum pressure.

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