THE IRREDUCIBLE REPRESENTATIONS OF THE LIE-ALGEBRA OF THE ORTHOGONAL GROUP WITH SPIN $\frac{3}{2}$

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1. Introduction

In recent years, the Lie-algebras of the orthogonal group, considered as associative algebras generated by a number of symbols satisfying a set of relations have been found to be of importance in the theory of elementary particles in Quantum Mechanics. The Clifford-Dirac algebra of 4 symbols which is the Lie-algebra of the orthogonal group in 5 dimensions was employed by Dirac in his theory of the electron whose spin is $\frac{1}{2}$. Explicit matrices of the representation of the Clifford-Dirac algebra with any number of symbols were given by Brauer and Weyl (1935). The Lie-algebra associated with an elementary particle of spin 1 was investigated by Kemmer (1939) and the matrices of the representations were obtained in an explicit form by D. E. Littlewood (1947).

The investigation of the Lie-algebras for higher spins is more complicated. It was, however, proved by Madhava Rao, Thiruvenkatachar and Venkatachaliengar (1946) that the algebra for the case of half-odd-integral spins is the direct product of the corresponding Clifford-Dirac algebra and another algebra called the ξ-algebra generated by the symbols $\xi_1, \xi_2, \ldots \xi_n$ satisfying the commutation rules

(i) $\{\xi_r, \{\xi_r, \xi_s\}\} = \xi_s,$

(ii) $[\xi_r, \{\xi_s, \xi_t\}] = 0 \, ; \, r \neq s \neq t \, ,$

where $\{a, b\}$ is the anticommutator $ab + ba$ and $[a, b]$ is the commutator $ab - ba$.

This direct product resolution simplifies the problem of determining the irreducible representations of the Lie-algebra considerably. The matrices of the irreducible representations of the Lie-algebra are then the Kronecker products of the matrices of the irreducible representations of the ξ-algebra and those of the corresponding Clifford-Dirac algebra. In the case of spin $\frac{3}{2}$, the symbols generating the original Lie-algebra satisfy a quartic and the corresponding symbols ξ_r satisfy the quadratic

$$\xi^2 = \tfrac{3}{4} - \xi.$$

In this paper, we take up the investigation of the ξ-algebra A_n generated by the n symbols $\xi_1, \xi_2, \ldots \xi_n$ with spin $\frac{3}{2}$. We show that the centre of the algebra is generated by a single element θ and obtain the minimal equation it satisfies. We set $\xi_{r-1} = \omega_{1r}$; $\{\omega_{1r}, \omega_{1s}\} = \omega_{rs}$ and show that the irreducible representations of the algebra A_n are given by

I (a) when n is even;

$$D_{nr}(\omega_{n,n+1}) = \tfrac{1}{2} E_{f_1} + \begin{vmatrix} \dfrac{(2r-n-6)}{2(n-2r+4)} & 1 \\ \dfrac{(n-2r+4)^2-1}{(n-2r+4)^2} & \dfrac{(2r-n-2)}{2(n-2r+4)} \end{vmatrix} \times E_{f_2} + \tfrac{1}{2} E_{f_3}$$

$$1 \leqslant r \leqslant \frac{n}{2}+1$$

(b) when n is odd, we have the same expression for $D_{nr}(\omega_{n,n+1})$ as I(a) with $1 \leqslant r \leqslant \dfrac{n+1}{2}$ and an additional representation.

$$D_{n,\frac{n+3}{2}}(\omega_{n,n+1}) = \tfrac{1}{2} E_{f_4} + -\tfrac{3}{2} E_{f_5}$$

where E_k is the unit matrix of order k.

II. $D_{nr}(\omega_{p,p+1}) = D_{n-1,r-1}(\omega_{p,p+1}) + D_{n-1,r}(\omega_{p,p+1})$

$$p = 1, 2, 3, \ldots (n-1).$$

By taking the anticommutators of the $\omega_{p,p+1}$ repeatedly, we obtain the matrices for ξ_r.

We prove also that the dimension of the algebra A_n is given by the simple expression $\dfrac{2}{n+2}\begin{pmatrix} 2n+1 \\ n \end{pmatrix}$. It follows, therefore that the dimension of the corresponding Lie-algebra of the orthogonal group is given by

$$\frac{2^{n+1}}{n+2}\begin{pmatrix} 2n+1 \\ n \end{pmatrix}.$$

2.   THE ξ-ALGEBRA

Let A_n be the ξ-algebra generated by the n symbols $\xi_1, \xi_2, \ldots \xi_n$, which satisfy the following relations :—

(1) (a). $$\xi^2 = \tfrac{3}{4} - \xi *$$

(1) (b). $$\{\xi_r, \{\xi_r, \xi_s\}\} = \xi_s; \ \{\xi_r, \xi_s\} = \xi_r\xi_s + \xi_s\xi_r.$$

(1) (c). $$[\xi_r, \{\xi_s, \xi_t\}] = 0 \quad [\xi_r, \xi_s] = \xi_r\xi_s - \xi_s\xi_r$$

$$r \neq s \neq t.$$

We set

$$\xi_{r-1} = \omega_{1r} = \omega_{r1}; \ [r = 2, 3, \ldots (n+1)]$$

and

$$\{\xi_{r-1}, \xi_{s-1}\} = \{\omega_{1r}, \omega_{1s}\} = \omega_{rs}.$$

It follows easily that $\{\xi_r, \xi_s\}^2 = \tfrac{3}{4} - \{\xi_r, \xi_s\}$.

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* In the paper by Madhava Rao and collaborators referred to, ξ is taken to satisfy $\xi^2 = \xi + \tfrac{3}{4}$. We have here replaced ξ by $-\xi$ for the slight simplification of some of the formulae.

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Hence the ω's satisfy the relations :

(1') (a). $$\omega_{pq}^2 = \tfrac{3}{4} - \omega_{pq} \;(p \neq q).$$

(1') (b). $$\{\omega_{1r}, \omega_{rs}\} = \omega_{1s}.$$

(1') (c). $$[\omega_{1r}, \omega_{st}] = 0\;;\;\omega_{pq} = \omega_{qb}.$$

$$r \neq s \neq t.$$

From 1' (a), it follows that in any representation ω_{pq} has the eigen-values $\tfrac{1}{2}$ or $-\tfrac{3}{2}$. Consider a representation in which ω_{1r} is diagonal, say $|\lambda_r|$ and let $\omega_{1s} = |a_{pq}|$.

We have $$\{\omega_{1r}, \omega_{1s}\} = |(\lambda_p + \lambda_q)\,a_{pq}|.$$

Since $$\{\omega_{1r}, \omega_{rs}\} = \omega_{1s}$$

$$(\lambda_p + \lambda_q)^2\,a_{pq} = a_{pq}$$

$$\therefore \quad 4\lambda_p^2\,a_{pp} = a_{pp},$$

i.e. either $$\lambda_p = \tfrac{1}{2} \text{ or } a_{pp} = 0.$$

Now $$\operatorname{spur}\omega_{1s} = \sum a_{pp}$$

and

$$\operatorname{spur}\{\omega_{1r}, \omega_{1s}\} = \sum 2\lambda_p\,a_{pp} = \sum a_{pp} = \operatorname{spur}\omega_{1s}.$$

Similarly, taking ω_{1s} diagonal, we obtain

$$\operatorname{spur}\omega_{1r} = \operatorname{spur}\{\omega_{1s}, \omega_{1r}\}.$$

We therefore have

(2) $$\operatorname{spur}\omega_{1r} = \operatorname{spur}\omega_{1s} = \operatorname{spur}\omega_{rs}.$$

It is clear that (2) is true for any spin.

We now proceed to obtain explicit matrices for the irreducible representations of the algebra A_n. From the theory of the orthogonal group (cf. F. D. Murnaghan : The theory of group representations) it follows that the dimension formula for the irreducible representations of the Lie-algebra (generated by n symbols) is

(3) (a) $$D_{\lambda_1 \lambda_2 \ldots \lambda_k} = \frac{2^k\,l_1'\,l_2' \ldots l_k'}{\underline{|1}\;\underline{|3} \ldots \underline{|2k-1}} \prod_{p<q}^{k} (l_p'^2 - l_q'^2)\;;\;n = 2k$$

and

(3) (b) $$D_{\lambda_1 \lambda_2 \ldots \lambda_k} = \frac{2^{k-1}}{\underline{|2}\;\underline{|4} \ldots \underline{|2k-2}} \prod_{p<q}^{k} (l_p^2 - l_q^2)\;;\;n = 2k-1,$$

where $\lambda_1, \lambda_2, \ldots \lambda_k$ are half-odd integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k$; $l_p = \lambda_p + (k-p)$; $l_p' = l_p + \tfrac{1}{2}$; and λ_1 is the spin. Denoting by $f_{\lambda_1 \lambda_2 \ldots \lambda_k}$, the dimension of the corresponding irreducible representation of the ξ-algebra, we have, since 2^k (or 2^{k-1}) is the dimension of the irreducible representation of the Clifford-Dirac algebra according as $n = 2k$ (or $2k-1$),

(3') (a) $$f_{\lambda_1 \lambda_2 \ldots \lambda_k} = \frac{D_{\lambda_1 \lambda_2 \ldots \lambda_k}}{2^k}\;;\;n = 2k,$$

(3') (b) $$f_{\lambda_1 \lambda_2 \dots \lambda_k} = \frac{D_{\lambda_1 \lambda_2 \dots \lambda_k}}{2^{k-1}} \; ; \; n = 2k-1 \; .$$

If spin $= \frac{3}{2}$, the λ's are all either $\frac{3}{2}$ or $\frac{1}{2}$.

Let $f_{nr} \equiv f_{\frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}}$ with $(r-1) \to \frac{3}{2}$'s and $(k-r+1) \to \frac{1}{2}$'s, r taking the values $1, 2, \dots (k+1)$ where $n = 2k$ or $2k-1$; there will be $(k+1)$ irreducible representations of the algebra A_n.

We obtain after some simplification that the formulae 3'(a) and 3'(b) both reduce to

(4) $$f_{nr} \equiv f_{\frac{3}{2} \dots \frac{3}{2}, \frac{1}{2} \dots \frac{1}{2}} = \frac{n-2r+4}{n+2} \binom{n+2}{r-1}$$

we define $f_{nr} = 0$ for $r \geqslant \frac{n+4}{2}$, since $f_{nr} = 0$ for $n = 2r-4$ and negative for $n < 2r-4$. i.e. r takes the values

$$1, 2, 3, \dots \frac{n}{2}+1 \text{ if } n \text{ is even}$$

and

$$1, 2, 3, \dots \frac{n+1}{2}, \frac{n+3}{2} \text{ if } n \text{ is odd.}$$

We denote by D_{nr}, the irreducible representation (of the algebra A_n) whose dimension is f_{nr} and by $D_{nr}(\omega_{pq})$ the matrix of representation for ω_{pq} in the representation D_{nr}.

It follows from the theory of the orthogonal group that the algebra A_n branches over the algebra A_{n-1} according to the law

(5) $$D_{nr}(\omega_{pq}) = D_{n-1, r-1}(\omega_{pq}) + D_{n-1, r}(\omega_{pq})$$

[It is verified easily that $f_{nr} = f_{n-1, r-1} + f_{n-1, r}$].

We next show that if S_{nr} is the spur of any ω_{pq} in the irreducible representation D_{nr}, then

(6) $$S_{nr} = f_{nr} \frac{n^2 + (5-4r)n + 4(r-1)(r-3)}{2n(n+1)}$$

$$= (n-2r+4) \frac{(n-1)(n-2) \dots (n-r+4)}{2 \lfloor r-1} \times$$

$$\times \{n^2 + (5-4r)n + 4(r-1)(r-3)\}$$

Proof :

We assume the result to be true for the algebra A_{n-1} and prove it for A_n. From (5) it follows that

$$S_{nr} = S_{n-1, r-1} + S_{n-1, r}.$$

Now $S_{n-1, r-1} + S_{n-1, r} =$

$$= \frac{(n-2r+5)}{2 \lfloor r-2} (n-2)(n-3) \dots (n-r+4) \{n^2 + (7-4r)n + 4(r-2)(r-3)\}$$

$$+ \frac{(n-2r+3)}{2 \lfloor r-1} (n-2)(n-3) \dots (n-r+3) \{n^2 + (3-4r)n + 4(r-1)(r-2)\}$$

$$= \frac{(n-2) \dots (n-r+4)}{2 \lfloor r-1} \left[\begin{array}{l} (n-2r+5)(r-1) \{n^2 + (7-4r)n + 4(r-2)(r-3)\} + \\ (n-2r+3)(n-r+3) \{n^2 + (3-4r)n + 4(r-1)(r-2)\} \end{array} \right]$$

i.e. $S_{n-1,\,r-1} + S_{n-1,\,r} =$

$$= \frac{(n-2)\ldots(\,n-r+4)}{2\,\lfloor r-1\rfloor}\left[(n-2r+4)(n-1)\,\{n^2+(5-4r)n+4(r-1)(r-3)\}\right]$$

on simplification.

$= S_{nr}.$

Now $D_{n1}(\omega_{pq}) = \frac{1}{2}$ for all n so that $S_{n1} = \frac{1}{2}$ and for the algebra A_2, one can see easily that

$$D_{22}(\omega_{12}) = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{3}{2} \end{vmatrix} \text{ or } S_{22} = -1.$$

That is, the formula is true for S_{n1} and S_{22} and hence by induction it is universally true.

3. THE IRREDUCIBLE REPRESENTATIONS OF A_n

We consider the algebra A_n as generated by the n symmetric symbols ω_{12} $\omega_{23}, \ldots \omega_{p,\,p+1}, \ldots \omega_{n,\,n+1}$ and define

$$\omega_{rs} = \{\omega_{1r}, \omega_{1s}\}\,;\ r \neq s \text{ with the relations (1')}.$$

From (1') (c) it follows that

$$[\omega_{pq}, \omega_{st}] = 0\,;\quad p, q \neq s, t.$$

Therefore, $\omega_{n,\,n+1}$ commutes with $\omega_{12}, \omega_{23}, \ldots \omega_{n-2,\,n-1}$, i.e. with the algebra A_{n-2}. We also have the branching law

$$D_{nr}(\omega_{p,\,p+1}) = D_{n-1,\,r-1}(\omega_{p,\,p+1}) + D_{n-1,\,r}(\omega_{p,\,p+1})\,;\ p = 1, 2, 3, \ldots (n-1).$$

We now show that—

(7) (a). When n is even

$$D_{nr}(\omega_{n,\,n+1}) = \tfrac{1}{2}\,E_{f_1} + \begin{vmatrix} \dfrac{2r-n-6}{2(n-2r+4)} & 1 \\[2mm] \dfrac{(n-2r+4)^2-1}{(n-2r+4)^2} & \dfrac{(2r-n-2)}{2(n-2r+4)} \end{vmatrix} \times E_{f_2} + \tfrac{1}{2}\,E_{f_3}$$

$$1 \leqslant r \leqslant \frac{n}{2} + 1.$$

(7) (b). When n is odd, we have the same expression for $D_{nr}(\omega_{n,\,n+1})$ as (7) (a) with

$1 \leqslant r \leqslant \dfrac{n+1}{2}$ and an additional representation

(8) $$D_{n,\,\frac{n+3}{2}}(\omega_{n,\,n+1}) = \tfrac{1}{2}\,E_{f_4} + -\tfrac{3}{2}\,E_{f_5},$$

where $E_f, E_{f_2}, E_{f_3}, E_{f_4}, E_{f_5}$ are unit matrices of orders

$$f_1 = f_{n-2,\,r-2}$$
$$f_2 = f_{n-2,\,r-1}$$
$$f_3 = f_{n-2,\,r}$$

$$f_4 = f_{n-2,\,\frac{n-1}{2}}$$

$$f_5 = f'_{n-2,\,\frac{n+1}{2}}\ \text{respectively}$$

with $E_k = 0$ for $k \leqslant 0$ and $E_1 = 1$.

Proof:

We first of all determine $D_{n,\,\frac{n+3}{2}}(\omega_{n,\,n+1})$ when n is odd. We have, by the branching law, that $D_{n,\,\frac{n+3}{2}}$ is the same as $D_{n-1,\,\frac{n+1}{2}}$ over the algebra A_{n-1}. Taking the branching again over A_{n-2}, we have

$$D_{n,\,\frac{n+3}{2}}(\omega_{p,\,p+1}) = D_{n-2,\,\frac{n-1}{2}}(\omega_{p,\,p+1}) + D_{n-2,\,\frac{n+1}{2}}(\omega_{p,\,p+1})\ \text{over } A_{n-2};$$

$$(1 \leqslant p \leqslant n-2).$$

Since $\omega_{n,\,n+1}$ commutes with the algebra A_{n-2}, we have by the Schur lemma,

$$D_{n,\,\frac{n+3}{2}}(\omega_{n,\,n+1}) = \lambda_1 E_{f_4} + \lambda_2 E_{f_5}.$$

Writing $n = 2m+1$, we have

$$f_4 = f_{2m-1,\,m} = \frac{3(m+3)(m+4)\ldots.(2m)}{\underline{|m-1}}$$

$$f_5 = f_{2m-1,\,m+1} = \frac{(m+2)(m+3)\ldots.(2m)}{\underline{|m}}.$$

Hence taking the spur of $\omega_{n,\,n+1}$, we have

(9) $$\lambda_1 f_4 + \lambda_2 f_5 = S_{n,\,\frac{n+3}{2}} = S_{2m+1,\,m+2}$$

$$= \frac{-3(m+3)(m+4)\ldots.(2m)}{\underline{|m}},$$

i.e. $$\lambda_1 3m + \lambda_2(m+2) = -3$$

or $$\lambda_1 = \tfrac{1}{2}\ \text{and}\ \lambda_2 = -\tfrac{3}{2}.$$

Thus, we have

(8) $$D_{n,\,\frac{n+3}{2}}(\omega_{n,\,n+1}) = \tfrac{1}{2} E_{f_4} + -\tfrac{3}{2}E_{f_5}.$$

To prove (7), we observe, first of all by the branching law, that

$$D_{nr}(\omega_{p,\,p+1}) = D_{n-1,\,r-1}(\omega_{p,\,p+1}) + D_{n-1,\,r}(\omega_{p,\,p+1})\ \text{over } A_{n-1}$$

$$= D_{n-2,\,r-2}(\omega_{p,\,p+1}) + D_{n-2,\,r-1}(\omega_{p,\,p+1})$$

$$+ D_{n-2,\,r-1}(\omega_{p,\,p+1}) + D_{n-2,\,r}(\omega_{p,\,p+1})\ \text{over } A_{n-2}$$

$$= D_{n-2,\,r-2}(\omega_{p,\,p+1}) + E_2 \times D_{n-2,\,r-1}(\omega_{p,\,p+1})$$

$$+ D_{n-2,\,r}(\omega_{p,\,p+1})\ \text{over } A_{n-2}.$$

Since $\omega_{n,\,n+1}$ commutes with the algebra A_{n-2}, we have by the Schur lemma,

$$D_{nr}(\omega_{n,\,n+1}) = a_{11}\,E_{f_1} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \times E_{f_2} + a_{44}\,E_{f_3},$$

where
$$f_1 = f_{n-2,\,r-2}; \; f_2 = f_{n-2,\,r-1}; \; f_3 = f_{n-2,\,r}.$$

Since
$$\overset{2}{\omega}_{n,\,n+1} = \tfrac{3}{4} - \omega_{n,\,n+1},$$

$$a_{11},\, a_{44} = \tfrac{1}{2} \text{ or } -\tfrac{3}{2}; \; a_{22} + a_{33} = -1$$

and
$$a_{22}^2 + a_{23}\,a_{32} = \tfrac{3}{4} - a_{22}.$$

Taking the spur of $\omega_{n,\,n+1}$, we have

$$a_{11} f_1 + (a_{22} + a_{33}) f_2 + a_{44} f_3 = S_{nr},$$

i.e.

(10)
$$\frac{(n-r+4)\,(n-r+5)\ldots(n-1)}{\underline{r-3}} \left\{ a_{11}\,(n-2r+6) - \frac{(n-2r+4)\,(n-r+3)}{r-2} \right.$$
$$\left. + a_{44}\,\frac{(n-2r+2)\,(n-r+2)\,(n-r+3)}{(r-1)\,(r-2)} \right\}$$
$$= (n-2r+4)\,\frac{(n-r+4)\,(n-r+5)\ldots(n-1)}{2\,\underline{r-1}} \times$$
$$\times\; \{ n^2 + (5-4r)\,n + 4\,(r-1)\,(r-3) \}.$$

It is easily seen that (10) will be consistent for the value $\frac{1}{2}$ only for a_{11} and a_{44}.

We now assume the result (7) for $n = m$ and prove it for $(m+1)$.

We have just proved that

$$D_{m+1,\,r}\,(\omega_{m+1,\,m+2}) = \tfrac{1}{2} E_{f_1'} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \times E_{f_2'} + \tfrac{1}{2} E_{f_3'},$$

where $\;f_1' = f_{m-1,\,r-2}; \; f_2' = f_{m-1,\,r-1}; \; f_3' = f_{m-1,\,r},$ and $a_{22} + a_{33} = -1$.

Hence
$$\text{spur}\,(\omega_{m+1,\,m+2}) = \tfrac{1}{2} f_1' - f_2' + \tfrac{1}{2} f_3' = S_{m+1,\,r}.$$

Now
$$D_{m+1,\,r}(\omega_{m,\,m+1}) = D_{m,\,r-1}\,(\omega_{m,\,m+1}) + D_{m,\,r}\,(\omega_{m,\,m+1}) \text{ by the branching law}$$

$$= \tfrac{1}{2} E_{f_1''} + \begin{vmatrix} \dfrac{(2r-m-8)}{2(m-2r+6)} & 1 \\[2mm] \dfrac{(m-2r+6)^2-1}{(m-2r+1)^2} & \dfrac{(2r-m-4)}{2(m-2r-6)} \end{vmatrix} \times E_{f_2''} + \tfrac{1}{2} E_{f_3''}$$

$$+\; \tfrac{1}{2} E_{f_1} + \begin{vmatrix} \dfrac{(2r-m-6)}{2(m-2r+4)} & 1 \\[2mm] \dfrac{(m-2r+4)^2-1}{(m-2r+4)^2} & \dfrac{(2r-m-2)}{2(m-2r+4)} \end{vmatrix} \times E_{f_2} + \tfrac{1}{2} E_{f_3},$$

where
$$f_1' = f_{m-2, r-3}; \quad f_2' = f_{m-2, r-2}; \quad f_3' = f_{m-2, r-1}$$
$$f_1 = f_{m-2, r-2}; \quad f_2 = f_{m-2, r-1}; \quad f_3 = f_{m-2, r}.$$

It follows easily from $(1')(b)$ and $(1')(c)$ that $\{\omega_{rs}, \omega_{st}\} = \omega_{rt}$ and hence spur $\{\omega_{m, m+1}, \omega_{m+1, m+2}\}$ is also $S_{m+1, r}$; we thus have

(11)
$$\tfrac{1}{2} f_1' - \frac{(m-2r+8)}{2(m-2r+6)} f_2' - \frac{(m-2r+4)}{(m-2r+6)} a_{22} f_2'$$

$$+ a_{22} f_3' + a_{33} f_1 - \frac{(m-2r+6)}{(m-2r+4)} a_{33} f_2$$

$$- \frac{(m-2r+2)}{2(m-2r+4)} f_2 + \tfrac{1}{2} f_3 = S_{m+1, r}$$

[Observe that $f_1' - f_1'' = f_2''; \ f_2' - f_2'' = f_3''; \ f_2' - f_1 = f_2; \ f_3' - f_2 = f_3$].

We also have
(12)
$$a_{22} + a_{33} = -1.$$

On solving for a_{22} and a_{33} from (11) and (12), we obtain,

$$a_{22} = \frac{2r-m-7}{2(m-2r+5)}$$

$$a_{33} = \frac{2r-m-3}{2(m-2r+5)}$$

from $a_{22}^2 + a_{32} a_{23} = \tfrac{3}{4} - a_{22}$, we have

$$a_{32} a_{23} = \frac{(m-2r+5)^2 - 1}{(m-2r+5)^2} \ .$$

If $a_{22} = \tfrac{1}{2}$ or $-\tfrac{3}{2}$, $r = \frac{m+6}{2}$ or $\frac{m+4}{2}$ respectively and this is clearly not possible. Hence $a_{22} \neq \tfrac{1}{2}$ or $-\tfrac{3}{2}$ or $a_{32} a_{23} \neq 0$.

We now effect a similarity transformation of the matrices $D_{m+1, r} (\omega_{m+1, m+2})$ and $D_{m+1, r} (\omega_{m, m+1})$ by the matrix $\frac{1}{a_{23}} E_{f_{m, r-1}} + E_{f_{m, r}}$. This leaves $D_{m+1, r} (\omega_{m, m+1})$ unaltered while in $D_{m+1, r} (\omega_{m+1, m+2})$ it changes a_{23} to 1 and a_{32} to $a_{32} a_{23}$. We have thus proved the result for $n = m+1$ if it is true for $n = m$. Before completing the proof by induction, we observe that the foregoing does not cover the case $r = \frac{n}{2} + 1$ when n is even; for then

$$D_{n, \frac{n+2}{2}} (\omega_{n-1, n}) = D_{n-1, \frac{n}{2}} (\omega_{n-1, n}) + D_{n-1, \frac{n+2}{2}} (\omega_{n-1, n})$$

$$= D_{n-1, \frac{n}{2}} (\omega_{n-1, n}) + \tfrac{1}{2} E_{f_{n-3, \frac{n-2}{2}}} + -\tfrac{3}{2} E_{f_{n-3, \frac{n}{2}}}$$

We therefore treat this case separately : i.e. writing $n = 2p$, we show that

$$D_{2p, p+1} (\omega_{2p, 2p+1}) = \tfrac{1}{2} E_{f_{2p-2, p-1}} + \begin{vmatrix} -1 & 1 \\ \tfrac{3}{4} & 0 \end{vmatrix} \times E_{f_{2p-2, p}}.$$

We assume the result for $n = 2p$ and prove it for $2p+2$. The preceding result shows that (7) is true for $n = 2p+1$.

Now

$$D_{2p+2,\,p+2}\,(\omega_{k,\,k+1}) = D_{2p+1,\,p+1}\,(\omega_{k,\,k+1}) + D_{2p+1,\,p+2}\,(\omega_{k,\,k+1}) \text{ over } A_{n-1}$$

$$= D_{2p,\,p}\,(\omega_{k,\,k+1}) + D_{2p,\,p+1}\,(\omega_{k,\,k+1}) + D_{2p,\,p+1}\,(\omega_{k,\,k+1}) \text{ over } A_{n-2}.$$

Hence from the Schur lemma,

$$D_{2p+2,\,p+2}\,(\omega_{2p+2,\,2p+3}) = a_{11}\, E_{f_{2p,\,p}} + \begin{vmatrix} a_{22} & 1 \\ a_{32} & a_{33} \end{vmatrix} \times E_{f_{2p,\,p+1}}$$

where a_{23} is taken to be 1 as before.

From $\omega_{2p+2,\,2p+3}^{2} = \frac{3}{4} - \omega_{2p+2,\,2p+3}$, we have again $a_{11} = \frac{1}{2}$ or $-\frac{3}{2}$ and $a_{22}+a_{33} = -1$. Taking the spur of $\omega_{2p+2,\,2p+3}$, we have

$$a_{11}\frac{4(p+4)(p+5)\ldots(2p+1)}{\underline{|\,p-1}} - \frac{2(p+3)(p+4)\ldots(2p+1)}{\underline{|\,p}} = S_{2p+2,\,p+2}$$

$$= -6\frac{(p+1)(p+4)(p+5)\ldots(2p+1)}{\underline{|\,p+1}}$$

from which $a_{11} = \frac{1}{2}$ only. Therefore,

$$D_{2p+2,\,p+2}\,(\omega_{2p+2,\,2p+3}) = \frac{1}{2}\, E_{f_{2p,\,p}} + \begin{vmatrix} a_{22} & 1 \\ a_{32} & a_{23} \end{vmatrix} \times E_{f_{2p,\,p+1}}$$

and

$$D_{2p+2,\,p+2}\,(\omega_{2p+1,\,2p+2}) = D_{2p+1,\,p+1}\,(\omega_{2p+1,\,2p+2}) + D_{2p+1,\,p+2}\,(\omega_{2p+1,\,2p+2}),$$

i.e.

$$D_{2p+2,\,p+2}\,(\omega_{2p+1,\,2p+2}) =$$

$$\frac{1}{2}\, E_{f_{2p-1,\,p-1}} + \begin{vmatrix} -\frac{5}{6} & 1 \\ \frac{8}{9} & -\frac{1}{6} \end{vmatrix} \times E_{f_{2p-1,\,p}} + \frac{1}{2}\, E_{f_{2p-1,\,p+1}}$$

$$+ \frac{1}{2}\, E_{f_{2p-1,\,p}} + -\frac{3}{2}\, E_{f_{2p-1,\,p+1}}.$$

Since spur of $\{\omega_{2p+1,\,2p+2},\ \omega_{2p+2,\,2p+3}\}$ is also $S_{2p+2,\,p+2}$ we have

$$\frac{1}{2}f_{2p-1,\,p-1} - \frac{5}{6}f_{2p-1,\,p} - \frac{1}{3}f_{2p-1,\,p}\,a_{22} + f_{2p-1,\,p+1}\,a_{22} + f_{2p-1,\,p}\,a_{33} - 3f_{2p-1,\,p+1}$$

$$= S_{2p+2,\,p+2} = \frac{1}{2}f_{2p,\,p} - f_{2p,\,p+1}$$

and we also have $a_{22}+a_{33} = -1$. On solving for a_{22}, a_{33}, we obtain

$$a_{22} = -1,\ a_{33} = 0\,;\ a_{32} = \frac{3}{4}.$$

We have thus shown that in all cases (7) is true for $n = m+1$ if it is true for $n = m$. Now for the algebra A_2, one can show easily that

$$D_{22}(\omega_{12}) = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{3}{2} \end{vmatrix}\,;\ D_{22}(\omega_{23}) = \begin{vmatrix} -1 & 1 \\ \frac{3}{4} & 0 \end{vmatrix}$$

and $D_{n1}(\omega_{p,\,p+1}) = \frac{1}{2}$ for all n. This proves that the irreducible representations of A_n are given by (7) and in case n is odd, we have an additional representation given by (8)

We observe that the representation matrices are chosen in such a way that their elements are rational numbers. If, however, we want them to be symmetric matrices as is generally required in Quantum Mechanics we can take

$$a_{23} = a_{32} = \sqrt{\frac{(n-2r+4)^2-1}{(n-2r+4)^2}}.$$

As an illustration, we give below the matrices of the irreducible representations for ω_{12}, ω_{23}, ... ω_{56} of the algebra A_5. By taking the anti-commutators of these $\omega_{p,\,p+1}$ repeatedly we can compute the matrices for $\xi_{r-1} = \omega_{1r}$,

The algebra A_5 has 4 irreducible representations $D_{51}, D_{52}, D_{53}, D_{54}$ of orders 1, 5, 9, 5 respectively.

(i) D_{51} :—

$$\omega_{r,\,r+1} = \frac{1}{2}\,;\quad r = 1, 2, 3, 4, 5.$$

(ii) D_{52} :—

$$\omega_{12} = \frac{1}{2}E_4 + -\frac{3}{2}E_1$$

$$\omega_{23} = \frac{1}{2}E_3 + \begin{vmatrix} -1 & 1 \\ \frac{3}{4} & 0 \end{vmatrix}$$

$$\omega_{34} = \frac{1}{2}E_2 + \begin{vmatrix} -\frac{5}{6} & 1 \\ \frac{8}{9} & -\frac{1}{6} \end{vmatrix} + \frac{1}{2}E_1$$

$$\omega_{45} = \frac{1}{2}E_1 + \begin{vmatrix} -\frac{3}{4} & 1 \\ \frac{15}{16} & -\frac{1}{4} \end{vmatrix} + \frac{1}{2}E_2$$

$$\omega_{56} = \begin{vmatrix} -\frac{7}{10} & 1 \\ \frac{24}{25} & -\frac{3}{10} \end{vmatrix} + \frac{1}{2}E_3$$

(iii) D_{53} :—

$$\omega_{12} = \frac{1}{2}E_3 + -\frac{3}{2}E_1 + \frac{1}{2}E_2 + -\frac{3}{2}E_1 + \frac{1}{2}E_1 + -\frac{3}{2}E_1$$

$$\omega_{23} = \frac{1}{2}E_2 + \begin{vmatrix} -1 & 1 \\ \frac{3}{4} & 0 \end{vmatrix} + \frac{1}{2}E_1 + \begin{vmatrix} -1 & 1 \\ \frac{3}{4} & 0 \end{vmatrix} \times E_2$$

$$\omega_{34} = \frac{1}{2}E_1 + \begin{vmatrix} -\frac{5}{6} & 1 \\ \frac{8}{9} & -\frac{1}{6} \end{vmatrix} + \frac{1}{2}E_1 + \begin{vmatrix} -\frac{5}{6} & 1 \\ \frac{8}{9} & -\frac{1}{6} \end{vmatrix} + \frac{1}{2}E_2 + -\frac{3}{2}E_1$$

$$\omega_{45} = \begin{vmatrix} -\frac{3}{4} & 1 \\ \frac{15}{16} & -\frac{1}{4} \end{vmatrix} + \frac{1}{2}E_3 + \begin{vmatrix} -1 & 1 \\ \frac{3}{4} & 0 \end{vmatrix} \times E_2$$

$$\omega_{56} = \frac{1}{2}E_1 + \begin{vmatrix} -\frac{5}{6} & 1 \\ \frac{8}{9} & -\frac{1}{6} \end{vmatrix} \times E_3 + \frac{1}{2}E_2$$

(iv) D_{54} :—

$$\omega_{12} = \tfrac{1}{2} E_2 + - \tfrac{3}{2} E_1 + \tfrac{1}{2} E_1 + - \tfrac{3}{2} E_1$$

$$\omega_{23} = \tfrac{1}{2} E_1 + E_2 \times \begin{vmatrix} -1 & 1 \\ \tfrac{3}{4} & 0 \end{vmatrix}$$

$$\omega_{34} = \begin{vmatrix} -\tfrac{5}{6} & 1 \\ \tfrac{8}{9} & -\tfrac{1}{6} \end{vmatrix} + \tfrac{1}{2} E_2 + - \tfrac{3}{2} E_1$$

$$\omega_{45} = \tfrac{1}{2} E_1 + \begin{vmatrix} -1 & 1 \\ \tfrac{3}{4} & 0 \end{vmatrix} \times E_2$$

$$\omega_{56} = \tfrac{1}{2} E_3 + - \tfrac{3}{2} E_2 .$$

4. THE DIMENSION OF THE ALGEBRA A_n

We now prove that the dimension of the algebra A_n is given by the simple

expression $\dfrac{2}{n+2} \dbinom{2n+1}{n}$; i.e. we show that

$$\sum_{r=1}^{m+1} f_{nr}^2 = \sum_{r=1}^{m+1} \left(\frac{n-2r+4}{n+2} \right)^2 \binom{n+2}{r-1}^2 = \frac{2}{n+2} \binom{2n+1}{n}$$

for $n = 2m$ or $2m-1$

(i) Let $n = 2m$; to show that

$$\sum_0^m (m-p+1)^2 \binom{2m+2}{p}^2 = (m+1) \binom{4m+1}{2m}$$

Proof :—

$$(1+x)^{2m+2} = \sum_0^{2m+2} \binom{2m+2}{2m+2-p} x^{2m+2-p} .$$

$$\therefore \quad x^{-m-1} (1+x)^{2m+2} = \sum_0^{2m+2} \binom{2m+2}{2m+2-p} x^{m-p+1}$$

$$\therefore \quad \frac{d}{dx} \left\{ x^{-m-1} (1+x)^{2m+2} \right\} = \sum_0^{2m+2} (m-p+1) \binom{2m+2}{2m+2-p} x^{m-p}$$

Also

$$(1+x)^{2m+2} = \sum_0^{2m+2} \binom{2m+2}{p} x^p .$$

$$\therefore \quad \frac{d}{dx} \left\{ x^{-m-1} (1+x)^{2m+2} \right\} = - \sum_0^{2m+2} (m-p+1) \binom{2m+2}{p} x^{p-m-2}$$

Hence, in

$$-\left\{\sum_0^{2m+2}(m-p+1)\binom{2m+2}{2m+2-p}x^{m-p}\right\}\left\{\sum_0^{2m+2}(m-p+1)\binom{2m+2}{p}x^{p-m-2}\right\}$$

the coefficient of

$$\frac{1}{x^2}=-\sum_0^{2m+2}(m-p+1)^2\binom{2m+2}{p}^2$$

$$=-2\sum_0^m(m-p+1)^2\binom{2m+2}{p}^2.$$

This must therefore be equal to the coefficient of $\dfrac{1}{x^2}$ in

$$\left[\frac{d}{dx}\left\{x^{-m-1}(1+x)^{2m+2}\right\}\right]^2,$$

i.e. to the coefficient of $\dfrac{1}{x^2}$ in

$$\left\{(2m+2)(1+x)^{2m+1}x^{-m-1}-(m+1)(1+x)^{2m+2}x^{-m-2}\right\}^2$$

$$=\frac{(m+1)^2(1+x)^{4m+2}(1-x)^2}{x^{2m+4}}.$$

Coefficient of $\dfrac{1}{x^2}=(m+1)^2\left\{\binom{4m+2}{2m+2}-2\binom{4m+2}{2m+1}+\binom{4m+2}{2m}\right\}$

$$=-2(m+1)\binom{4m+1}{2m}\text{ on simplification.}$$

Hence

$$\sum_0^m(m-p+1)^2\binom{2m+2}{p}^2=(m+1)\binom{4m+1}{2m}.$$

When $n=2m-1$, one can similarly prove the result by considering the expansion for $(1+x^2)^{2m+1}$. We thus obtain that the dimension of the Lie-algebra of the orthogonal group with spin $\frac{3}{2}$ is

$$\frac{2^{n+1}}{n+2}\binom{2n+1}{n}.$$

5. The Centre of the Algebra A_n

Let $P_r=\Sigma\,\xi_{p_1p_2\ldots p_r}$ where $\xi_{p_1p_2\ldots p_r}=\xi_{p_1}\xi_{p_2}\cdots\xi_{p_r}$ and the summation extends over nP_r permutations.

Thus $\xi_1P_{2m+1}=\Sigma\,\xi_1\xi_{q_1q_2\ldots q_{2m+1}}+\Sigma\,\xi_1\xi_{1q_2,\ldots q_{2m+1}}+\Sigma\,\xi_1\xi_{q_2q_3\ldots q_{2m+1}}$

$$+\ldots+\ldots+\Sigma\,\xi_1\xi_{q_2\ldots q_{2m+1},1}$$

where the q_r are to be summed up over all indices excepting 1.

Since $\left[\xi_1, \{\xi_{q_1}, \xi_{q_2}\}\right] = 0$, we obtain

$$\xi_1 P_{2m+1} = \xi_{1q_1 \ldots q_{2m+1}} + (m+1)\, \xi_{11q_2 \ldots q_{2m+1}} + m\, \xi_{1q_21q_3 \ldots q_{2m+1}},$$

the q_r being summed up over all indices excepting 1.

Now
$$2\,\xi_{1q1} + \xi_{q11} = \xi_q - \xi_{11q}$$

$$\therefore \quad \xi_1 P_{2m+1} = \xi_{1q_1 q_2 \ldots q_{2m+1}} + \frac{m}{2}\,\xi_{q_2 q_3 \ldots q_{2m+1}}$$

$$-\frac{m}{2}\,\xi_{q_211q_3 \ldots q_{2m+1}} + \left(\frac{m}{2}+1\right)\xi_{11q_2 \ldots q_{2m+1}}.$$

From $\quad \xi_{11} = \frac{3}{4} - \xi_1 \quad$ we obtain

$$\xi_1 P_{2m+1} = \xi_{1q_1 \ldots q_{2m+1}} + \frac{2m+3}{4}\,\xi_{q_2 \ldots q_{2m+1}}$$

$$-\frac{m+2}{2}\,\xi_{1q_2 \ldots q_{2m+1}} + \frac{m}{2}\,\xi_{q_21q_3 \ldots q_{2m+1}}$$

we have similarly

$$P_{2m+1}\,\xi_1 = \xi_{q_1 q_2 \ldots q_{2m+1},\,1} + (m+1)\,\xi_{11q_2q_3 \ldots q_{2m+1}}$$

$$+ m\,\xi_{1q_21q_3 \ldots q_{2m+1}} \text{ and hence}$$

$$\left[\xi_1, P_{2m+1}\right] = \left[\xi_1, \xi_{q_1q_2 \ldots q_{2m+1}}\right].$$

Forming in the same way $\xi_2 P_{2m+1}, \ldots \xi_n P_{2m+1}, P_{2m+1}\xi_2, \ldots P_{2m+1}\xi_n$, we obtain.

(13) $$P_1 P_{2m+1} = P_{2m+2} - P_{2m+1} + \frac{(n-2m)(2m+3)}{4}\,P_{2m} = P_{2m+1}P_1.$$

It can be seen similarly that

$$\xi_1 P_{2m} = \xi_{1\,q_1 q_2 \ldots q_{2m}} + \frac{m}{2}\,\xi_{q_2 \ldots q_{2m}} - \frac{m}{2}\,\xi_{1q_2 \ldots q_{2m}} + \frac{m}{2}\,\xi_{q_21\,q_3 \ldots q_{2m}}.$$

and

$$P_{2m}\,\xi_1 = \xi_{1\,q_1 q_2 \ldots q_{2m}} + \frac{m}{2}\,\xi_{q_2 \ldots q_{2m}} + \frac{m}{2}\,\xi_{1q_2 \ldots q_{2m}} - \frac{m}{2}\,\xi_{q_21\,q_3 ,\,,\,,\, q_{2m}}$$

from which we have

$$\left[\xi_1, P_{2m}\right] = m\left(\xi_{q_21q_3 \ldots q_{2m}} - \xi_{1q_2 \ldots q_{2m}}\right) = -m\left[\xi_1, \xi_{q_2 \ldots q_{2m}}\right],$$

i.e. $\qquad P_{2m} + m\,P_{2m-1}$ is an element of the centre.

[We take $P_0 = 1$ and $P_r = 0$ for $r > n$ and $r < 0$; we notice that there will be $(k+1)$ elements of this form for $n = 2k$ or $2k-1$.]
We also obtain

(14) $$P_1 P_{2m} = P_{2m+1} + \frac{m(n-2m+1)}{2}\,P_{2m-1} = P_{2m}P_1.$$

From (13) we have $P_1{}^2 = P_2 - P_1 + \dfrac{3n}{4}$

$$\therefore \quad P_1 + P_2 = P_1{}^2 + 2P_1 - \frac{3n}{4}\,.$$

$$\therefore \quad (P_1 + P_2)\,P_{2m} = P_1 P_{2m+1} + \frac{m(n-2m+1)}{2}\,P_1 P_{2m-1}$$

$$+2P_1 P_{2m} - \frac{3n}{4}\,P_{2m.}$$

$$(P_1 + P_2)\,P_{2m-1} = P_1 P_{2m} + P_1 P_{2m-1}$$

$$+\frac{(n-2m+2)(2m+1)}{4}\,P_1 P_{2m-2} - \frac{3n}{4}\,P_{2m-1}\,.$$

We thus obtain

$$\left(P_1 + P_2\right)\left(P_{2m} + mP_{2m-1}\right) + \frac{3n}{4}\left(P_{2m} + mP_{2m-1}\right) =$$

$$P_1 P_{2m+1} + (m+2)\,P_1 P_{2m} + \frac{m(n-2m+3)}{2}\,P_1 P_{2m-1}$$

$$+\frac{m(n-2m+2)(2m+1)}{4}\,P_1 P_{2m-2}$$

$$= P_{2m+2} - P_{2m+1} + \frac{(n-2m)(2m+3)}{4}\,P_{2m}$$

$$+\frac{m(n-2m+3)}{2}\,P_{2m} + (m+2)P_{2m+1}$$

$$+\frac{m(m+2)(n-2m+1)}{2}\,P_{2m-1} - \frac{m(n-2m+3)}{2}\,P_{2m-1}$$

$$+\frac{m(n-2m+3)(n-2m+2)(2m+1)}{8}\,P_{2m-2}$$

$$+\frac{m(n-2m+2)(2m+1)}{4}\,P_{2m-1}$$

$$+\frac{m(m-1)(2m+1)(n-2m+2)(n-2m+3)}{3}\,P_{2m-3}$$

or we obtain the recurrent relation

$$(15) \quad (P_1 + P_2)\left(P_{2m} + mP_{2m-1}\right) = \left(P_{2m+2} + \overline{m+1}\,P_{2m+1}\right) + m(n-2m)\left(P_{2m} + mP_{2m-1}\right)$$

$$+\frac{m(2m+1)(n-2m+3)(n-2m+2)}{8}\left(P_{2m-2}P + \overline{m-1}P_{2m-3}\right).$$

It is evident that on utilising (15) in succession, we can express a central element

$P_{2m} + m\,P_{2m-1}$ as a polynomial in $\theta = P_1 + P_2 = \displaystyle\sum_{r=1}^{n} \xi_r + \sum_{r,s=1}^{n} \{\xi_r, \xi_s\}\,; \; r \neq s.$

We obtain the minimal equation that $\theta = P_1 + P_2$ satisfies indirectly as follows:

Since   spur $\xi_r =$ spur $\xi_s =$ spur $\{ \xi_r, \xi_s \}$,

$$\text{spur of } \theta = \frac{n(n+1)}{2} S_{nr} \text{ in the irreducible representation } D_{nr},$$

i.e. $$\text{spur } \theta = \frac{n(n+1)}{2} f_{nr} \frac{n^2 + (5-4r)n + 4(r-1)(r-3)}{2n.(n+1)} .$$

Hence the roots of θ are $\dfrac{n^2 + (5-4r)n + 4(r-1)(r-3)}{4}$

$$r = 1, 2, 3, \ldots (k+1) \text{ where } n = 2k \text{ or } 2k-1$$

or the minimal equation that θ satisfies is

(16) $$\prod_{r=1}^{k+1} \left\{ \theta - \frac{n^2 + (5-4r)n + 4(r-1)(r-3)}{4} \right\} = 0.$$

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SUMMARY

In this paper we determine explicitly the matrices of all the finite-dimensional representations of the Lie-algebra of the orthogonal group with any number of symbols with spin $= \frac{3}{2}$. For this purpose we use the direct product resolution of such an algebra into that of a Dirac-algebra and a ξ-algebra due to Madhava Rao and others. We find first of all the matrices for the representations of the ξ-algebra; since those of the Dirac-algebra are known one can work out the same for the Lie-algebra. We determine finally the centre of the ξ-algebra.

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