ON THE PROBLEM OF SOFTENING OF RADIATION BY MULTIPLE COMPTON SCATTERING IN STELLAR ATMOSPHERES CONTAINING FREE ELECTRONS

by K. K. SEN, *Chandernagore College*

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Chandrasekhar (1948) in his treatment of the transfer of radiation in a plane parallel, electron scattering atmosphere worked out the modification of radiation of a particular wavelength caused by multiple Compton scattering to the first order, by his new method of approximation. The coefficient of scattering was assumed to be independent of the wavelength as in Thomson scattering and the Compton change in wavelength given by

$$\delta\lambda = \gamma(1-\cos\theta), \text{ where } \gamma = \frac{h}{mc},$$

was taken into account. The scattered intensity was expanded in a Taylor's series in powers of γ (eqn. (5)) and only the term proportional to γ in the expansion was taken for subsequent calculation in an isotropic conservative case. The external boundary condition was simply the absence of incident radiation, while the internal boundary in the form of an infinite plane surface was supposed to radiate with a known spectral distribution. The intensity distribution in a spectral line when plotted against the wavelength shift showed a displacement of the maximum in the right direction, but a finite part of the distribution of intensity corresponded to negative values of the wavelength shift. This error was suspected to be due to the approximation involved in using the first term in the Taylor's series.

It was thought worth while to consider the contribution of the second term of the Taylor's series, proportional to γ^2. This has been carried through in the present paper. The method of solving the boundary value problem followed here is, however, different and is dependent on expansion in trigonometrical series. It is found that the use of trigonometrical series in the problem of this kind is quite handy and appropriate, and yields rapidly convergent series for calculation.

The results of the second order calculation considerably reduces the error mentioned above. The correction term is negative all throughout, so that the effect of the second order term is to lower the intensity curve. The shift, however, is of the right type and the shift of maximum is slightly increased. The contribution of the second order term cannot thus be called negligible. It appears quite plausible that a calculation up to third order will again slightly raise the curve, and may give non-negligible contribution.

§ 2. The equation of transfer appropriate to the problem is given by (p. 509, *Proc. Roy. Soc., London*, Vol. **192**, 1948)

$$\mu\frac{\partial I(\tau,\mu,\lambda)}{\partial\tau} = I(\tau,\mu,\lambda) - \frac{1}{4\pi}\int_{-1}^{+1}\int_{0}^{2\pi} I(\tau,\mu',\lambda-\gamma(1-\cos\theta))d\mu' d\phi' \quad .. \quad (1)$$

where γ, the Compton wavelength is given by

$$\gamma = \frac{h}{mc} = \cdot 024\, A, \qquad \ldots \qquad (2)$$

$\mu = \cos \vartheta$ and τ, the optical thickness is given by

$$\tau = \int_z^\infty \rho\sigma\, dz, \qquad \ldots \qquad (3)$$

where ρ is the density and σ the scattering coefficient. $I(\tau, \mu, \lambda)$ is the specific intensity of the radiation of wavelength λ at the optical depth τ and in a direction ϑ to the outward drawn normal. θ is the angle of scattering and

$$\cos\theta = \mu\mu' + (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}} \cos\phi' \qquad \ldots \qquad (4)$$

The source function represented by the second term of the right-hand side of eqn. (1) means that a radiation of wavelength $\lambda-\gamma(1-\cos\theta)$ in the direction (μ', ϕ'), when scattered in the direction $(\mu, 0)$, will have the wavelength λ.

We shall now suppose that $I(\tau, \mu', \lambda-\gamma(1-\cos\theta))$ can be expanded in Taylor's series, so that

$$I(\tau, \mu', \lambda-\gamma(1-\cos\theta)) = I(\tau, \mu', \lambda) - \gamma(1-\cos\theta)\frac{\partial I(\tau, \mu', \lambda)}{\partial\lambda} + \frac{\gamma^2}{2}(1-\cos\theta)^2 \frac{\partial^2 I(\tau, \mu', \lambda)}{\partial\lambda^2} - \ldots \qquad (5)$$

The equation of transfer on substitution of this becomes

$$\mu \frac{\partial I(\tau, \mu, \lambda)}{\partial\tau} = I(\tau, \mu, \lambda) - \tfrac{1}{2}\int_{-1}^{+1}\Big[I(\tau, \mu, \lambda) - \gamma(1-\mu\mu')\frac{\partial I(\tau, \mu', \lambda)}{\partial\lambda} + \gamma^2\left(\tfrac{3}{4} - \mu\mu' + \tfrac{3}{4}\mu^2\mu'^2 - \tfrac{1}{4}\mu^2 - \tfrac{1}{4}\mu'^2\right)\frac{\partial^2 I(\tau, \mu', \lambda)}{\partial\lambda^2}\Big] d\mu' \qquad \ldots \qquad (6)$$

when second order terms of Taylor's expansion are retained.

In solving this equation, Chandrasekhar's method of replacing the integrals by sums given by Gauss's formula for numerical quadratures, has been used. Thus in the n-th approximation,

$$\mu_i \frac{\partial I_i(\tau, \lambda)}{\partial\tau} = I_i(\tau, \lambda) - \frac{1}{2}\Big[\sum_j a_j I_j - \gamma \sum_j a_j(1-\mu_i\mu_j)\frac{\partial I_j(\tau, \lambda)}{\partial\lambda} + \gamma^2 \Big\{\sum_j a_j\Big(\frac{3}{4} - \mu_i\mu_j + \frac{3}{4}\mu_i^2\mu_j^2 - \frac{1}{4}\mu_i^2 - \frac{1}{4}\mu_j^2\Big)\Big\}\frac{\partial^2 I_j}{\partial\lambda^2}\Big]$$

where $\qquad i = (\pm 1, \pm 2 \ldots\ldots \pm n), j = (\pm 1, \pm 2 \ldots\ldots \pm n) \qquad \ldots \qquad (7)$

where μ_i's are the zeros of the Legendre polynomial $P_{2n}(\mu)$ and the a_j's are the appropriate Gaussian weights.

$$a_j = a_{-j} \text{ and } j = \pm 1, \pm 2 \ldots\ldots \pm n. \qquad \ldots \qquad (8)$$

Now restricting ourselves to the case of first approximation only, we put

$$a_{+1} = a_{-1} = 1 \text{ and } \mu_{+1} = -\mu_{-1} = \frac{1}{\sqrt{3}}. \qquad \ldots \qquad (9)$$

Eqn. (7) gives the following two equations :—

$$\frac{1}{\sqrt{3}}\frac{\partial I_{+1}}{\partial \tau}-\frac{\gamma}{3}\frac{\partial I_{+1}}{\partial \lambda}-\frac{2\gamma}{3}\frac{\partial I_{-1}}{\partial \lambda}+\frac{\gamma^2}{6}\frac{\partial^2 I_{+1}}{\partial \lambda^2}+\frac{\gamma^2}{2}\frac{\partial^2 I_{-1}}{\partial \lambda^2}=\frac{1}{2}\left(I_{+1}-I_{-1}\right) \quad (10)$$

and

$$\frac{1}{\sqrt{3}}\frac{\partial I_{-1}}{\partial \tau}+\frac{2\gamma}{3}\frac{\partial I_{+1}}{\partial \lambda}+\frac{\gamma}{3}\frac{\partial I_{-1}}{\partial \lambda}-\frac{\gamma^2}{2}\frac{\partial^2 I_{+1}}{\partial \lambda^2}-\frac{\gamma^2}{6}\frac{\partial^2 I_{-1}}{\partial \lambda^2}=\frac{1}{2}\left(I_{+1}-I_{-1}\right) \quad (11)$$

Now introducing the variables

$$x=\frac{3}{2}\tau \text{ and } y=\frac{3}{2\gamma}\left(\lambda-\lambda_0\right) \quad (12)$$

where λ_0 is some suitably chosen wavelength of constant value, equations (10) and (11) can be written in the following forms :—

$$\frac{\sqrt{3}}{2}\frac{\partial I_{+1}}{\partial x}-\frac{1}{2}\frac{\partial I_{+1}}{\partial y}-\frac{\partial I_{-1}}{\partial y}+\frac{3}{8}\frac{\partial^2 I_{+1}}{\partial y^2}+\frac{9}{8}\frac{\partial^2 I_{-1}}{\partial y^2}=\frac{1}{2}\left(I_{+1}-I_{-1}\right) \quad (13)$$

$$\frac{\sqrt{3}}{2}\frac{\partial I_{-1}}{\partial x}+\frac{\partial I_{+1}}{\partial y}+\frac{1}{2}\frac{\partial I_{-1}}{\partial y}-\frac{9}{8}\frac{\partial^2 I_{+1}}{\partial y^2}-\frac{3}{8}\frac{\partial^2 I_{-1}}{\partial y^2}=\frac{1}{2}\left(I_{+1}-I_{-1}\right). \quad (14)$$

Adding (13) and (14) and writing

$$K(x,y)=I_{+1}(x,y)+I_{-1}(x,y) \text{ and } H(x,y)=I_{+1}(x,y)-I_{-1}(x,y) \quad (15)$$

we get

$$\frac{\sqrt{3}}{2}\frac{\partial K}{\partial x}+\frac{1}{2}\frac{\partial H}{\partial y}-\frac{3}{4}\frac{\partial^2 H}{\partial y^2}=H \quad (16)$$

and also subtracting (14) from (13)

$$\frac{\partial H}{\partial x}-\sqrt{3}\frac{\partial K}{\partial x}+\sqrt{3}\frac{\partial^2 K}{\partial x^2}=0. \quad (17)$$

To satisfy (17), we write

$$K=\frac{\partial F(x,y)}{\partial x},\ H=\sqrt{3}\left(\frac{\partial F(x,y)}{\partial y}-\frac{\partial^2 F(x,y)}{\partial y^2}\right). \quad (18)$$

Substituting these in eqn. (16), we get

$$\frac{\partial^2 F}{\partial x^2}+\frac{3}{2}\frac{\partial^4 F}{\partial y^4}-\frac{5}{2}\frac{\partial^3 F}{\partial y^3}+3\frac{\partial^2 F}{\partial y^2}-2\frac{\partial F}{\partial y}=0. \quad (19)$$

From equation (18) we obtain

$$I_{+1}(x,y)=K+H=\frac{1}{2}\left[\frac{\partial F}{\partial x}+\sqrt{3}\left(\frac{\partial F}{\partial y}-\frac{\partial^2 F}{\partial y^2}\right)\right] \quad (20)$$

and

$$I_{-1}(x,y)=K-H=\tfrac{1}{2}\left[\frac{\partial F}{\partial x}-\sqrt{3}\left(\frac{\partial F}{\partial y}-\frac{\partial^2 F}{\partial y^2}\right)\right]. \quad (21)$$

The boundary conditions are—

(i) existence of a known spectral distribution at the lower boundary denoted by $\tau=\tau_1$ or $x=x_1$; and

(ii) absence of inward radiation at the upper boundary denoted by $\tau=0$ or $x=0$. These boundary conditions are equivalent to

$$\frac{1}{2}\left[\frac{\partial F}{\partial x}+\sqrt{3}\left(\frac{\partial F}{\partial y}-\frac{\partial^2 F}{\partial y^2}\right)\right]_{x=x_1} = a \text{ known function of } y = \psi(y) \quad .. \quad (22)$$

and

$$\left[\frac{\partial F}{\partial x}-\sqrt{3}\left(\frac{\partial F}{\partial y}-\frac{\partial^2 F}{\partial y^2}\right)\right]_{x=0} = 0. \quad .. \quad .. \quad .. \quad (23)$$

The problem before us is to solve eqn. (19) under the boundary conditions (22) and (23).

§ 3. The boundary value problem formulated in § 2 can be solved by the method of expansion in trigonometrical series.

Let us take as trial solution of (19) *

$$F(x,y) = A_0(x^2+y)+B_0x+Ae^{mx+iny}. \quad .. \quad .. \quad .. \quad (24)$$

Substituting in (19) we obtain

$$m^2 = 3n^2(1-\tfrac{1}{2}n^2)+in(2-\tfrac{5}{2}n^2)$$
$$= (\alpha_n+i\beta_n)^2 \quad \text{(say)}.$$

Hence

$$m = \pm(\alpha_n+i\beta_n) \quad .. \quad .. \quad .. \quad .. \quad .. \quad (25)$$

where

$$\alpha_n{}^2-\beta_n{}^2 = 3n^2(1-\tfrac{1}{2}n^2) \quad .. \quad .. \quad .. \quad .. \quad (26)$$

and

$$2\alpha_n\beta_n = n(2-\tfrac{5}{2}n^2) \quad .. \quad .. \quad .. \quad .. \quad (27)$$

From (27) it is clear that when n is a positive integer, α_n and β_n will be of opposite signs and when n is a negative integer α_n and β_n are of the same sign. It should also be noted that as n approaches zero, α_n and β_n will be of the same sign.

Now from (26) and (27) we have

$$\alpha_n{}^2+\beta_n{}^2 = \sqrt{\frac{9}{4}n^8-\frac{11}{4}n^6-n^4+4n^2} \quad .. \quad .. \quad .. \quad (28)$$

and from (26) and (28)

$$\alpha_n = \sqrt{\frac{3}{2}n^2\left(1-\frac{1}{2}n^2\right)+\tfrac{1}{2}\sqrt{\frac{9}{4}n^8-\frac{11}{4}n^6-n^4+4n^2}} \quad .. \quad .. \quad (29)$$

$$\beta_n = -\sqrt{\frac{1}{2}\sqrt{\frac{9}{4}n^8-\frac{11}{4}n^6-n^4+4n^2}-\frac{3}{2}n^2\left(1-\frac{1}{2}n^2\right)} \quad .. \quad .. \quad (30)$$

when n is a positive integer.

We see that the solution of (19) can be expanded as the sum of the following series

$$F(x,y) = A_0(x^2+y)+B_0x+a_0+\sum_{n=1}\Big\{A_ne^{(\alpha_n+i\beta_n)x+iny}+B_ne^{(\alpha_n-i\beta_n)x-iny}$$
$$+C_ne^{-(\alpha_n+i\beta_n)x+iny}+D_ne^{-(\alpha_n-i\beta_n)x-iny}\Big\} \quad .. \quad .. \quad (31)$$

* The first two terms in eqn. (24) have been introduced to adjust the boundary conditions of our problem. This will be shown in § 4 and § 7.

and n having only integral values. Writing the solution in the real form

$$F(x,y) = A_0(x^2+y)+B_0x+a_0+\sum_{n=1}^{\infty}\Big[e^{\alpha_n x}\Big\{a_n\cos(\beta_n x+ny)+b_n\sin(\beta_n x+ny)\Big\}$$
$$+e^{-\alpha_n x}\Big\{c_n\cos(\beta_n x-ny)+d_n\sin(\beta_n x-ny)\Big\}\Big] \quad \ldots \quad \ldots \quad (32)$$

§ 4. Now substituting in the boundary condition (23), the values of $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ and $\frac{\partial^2 F}{\partial y^2}$ derived from (32), we get

$$B_0-\sqrt{3}A_0+\sum_{n=1}^{\infty}\Big[\Big\{a_n(\alpha_n-\sqrt{3}n^2)+b_n(\beta_n-\sqrt{3}n)-c_n(\alpha_n+\sqrt{3}n^2)$$
$$+d_n(\beta_n+\sqrt{3}n)\Big\}\cos ny$$
$$+\Big\{b_n(\alpha_n-\sqrt{3}n^2)-a_n(\beta_n-\sqrt{3}n)+c_n(\beta_n+\sqrt{3}n)+d_n(\alpha_n+\sqrt{3}n^2)\Big\}\sin ny\Big] = 0 \quad \ldots \quad (33)$$

From this it is clear that the individual coefficients of $\cos ny$ and $\sin ny$ are zeros.

$$\therefore \quad B_0 = \sqrt{3}\,A_0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

and

$$\left.\begin{aligned} a_n &= \frac{(\alpha_n^2+\beta_n^2-3n^2-3n^4)c_n+2\sqrt{3}n(n\beta_n-\alpha_n)d_n}{(\beta_n-\sqrt{3}n)^2+(\alpha_n-\sqrt{3}n^2)^2} \\ b_n &= \frac{2\sqrt{3}n(n\beta_n-\alpha_n)c_n-(\alpha_n^2+\beta_n^2-3n^2-3n^4)d_n}{(\beta_n-\sqrt{3}n)^2+(\alpha_n-\sqrt{3}n^2)^2} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (35)$$

Now substituting in the boundary condition at the lower bound of the atmosphere given by equation (22) the values of $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ and $\frac{\partial^2 F}{\partial y^2}$ derived from equation (32) and simplifying, we find that

$$\tfrac{1}{2}\{B_0+2A_0x_1+\sqrt{3}A_0\}+\sum_{n=1}^{\infty}\Big[\tfrac{1}{2}e^{\alpha_n x_1}\Big\{\big(a_n(\alpha_n+\sqrt{3}n^2)+b_n(\beta_n+\sqrt{3}n)\big)\cos\beta_n x_1$$
$$+\big(b_n(\alpha_n+\sqrt{3}n^2)-a_n(\beta_n+\sqrt{3}n)\big)\sin\beta_n x_1\Big\}$$
$$+\tfrac{1}{2}e^{-\alpha_n x_1}\Big\{\big(d_n(\beta_n-\sqrt{3}n)-c_n(\alpha_n-\sqrt{3}n^2)\big)\cos\beta_n x_1$$
$$-\big(d_n(\alpha_n-\sqrt{3}n^2)+c_n(\beta_n-\sqrt{3}n)\big)\sin\beta_n x_1\Big\}\Big]\cos ny$$
$$+\sum_{n=1}^{\infty}\Big[\tfrac{1}{2}e^{\alpha_n x_1}\Big\{\big(b_n(\alpha_n+\sqrt{3}n^2)-a_n(\beta_n+\sqrt{3}n)\big)\cos\beta_n x_1$$
$$-\big(a_n(\alpha_n+\sqrt{3}n^2)+b_n(\beta_n+\sqrt{3}n)\big)\sin\beta_n x_1\Big\}$$
$$+\tfrac{1}{2}e^{-\alpha_n x_1}\Big\{\big(d_n(\alpha_n-\sqrt{3}n^2)+c_n(\beta_n-\sqrt{3}n)\big)\cos\beta_n x_1$$
$$+\big(d_n(\beta_n-\sqrt{3}n)-c_n(\alpha_n-\sqrt{3}n^2)\big)\sin\beta_n x_1\Big\}\Big]\sin ny$$
$$=\psi(y) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

We now assume that the distribution at the base of the atmosphere is capable of Fourier's expansion in the form,

$$\psi(y) = A'_0 + \sum_{n=1}^{\infty} A_n' \cos ny + \sum_{n=1}^{\infty} B_n' \sin ny \quad \ldots \quad \ldots \quad (37)$$

Thus comparing the coefficients of cos ny and sin ny in (36) and (37), and replacing in them values of B_0, a_n and b_n by (34) and (35) it can be shown that

$$A_n' = M_n c_n + N_n d_n \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

$$B_n' = N_n c_n - M_n d_n \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

$$A_0' = A_0(x_1+\sqrt{3}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

where

$$M_n = \tfrac{1}{2} e^{\alpha_n x_1} \left\{ \frac{(\alpha_n^2+\beta_n^2-3n^2-3n^4)(\alpha_n+\sqrt{3}n^2)+2\sqrt{3}n(n\beta_n-\alpha_n)(\beta_n+\sqrt{3}n)}{(\alpha_n-\sqrt{3}n^2)^2+(\beta_n-\sqrt{3}n)^2} \cos \beta_n x_1 \right.$$

$$\left. - \frac{(\alpha_n^2+\beta_n^2-3n^2-3n^4)(\beta_n+\sqrt{3}n)-2\sqrt{3}n(n\beta_n-\alpha_n)(\alpha_n+\sqrt{3}n^2)}{(\alpha_n-\sqrt{3}n^2)^2+(\beta_n-\sqrt{3}n)^2} \sin \beta_n x_1 \right\}$$

$$-\tfrac{1}{2} e^{-\alpha_n x_1} \{(\alpha_n-\sqrt{3}n^2) \cos \beta_n x_1 + (\beta_n-\sqrt{3}n) \sin \beta_n x_1\} \quad \ldots \quad \ldots \quad (41)$$

and

$$N_n = \tfrac{1}{2} e^{\alpha_n x_1} \left\{ \frac{2\sqrt{3}n(n\beta_n-\alpha_n)(\alpha_n+\sqrt{3}n^2)-(\alpha_n^2+\beta_n^2-3n^2-3n^4)(\beta_n+\sqrt{3}n)}{(\alpha_n-\sqrt{3}n^2)^2+(\beta_n-\sqrt{3}n)^2} \cos \beta_n x_1 \right.$$

$$\left. - \frac{(\alpha_n^2+\beta_n^2-3n^2-3n^4)(\alpha_n+\sqrt{3}n^2)+2\sqrt{3}n(n\beta_n-\alpha_n)(\beta_n+\sqrt{3}n)}{(\alpha_n-\sqrt{3}n^2)^2+(\beta_n-\sqrt{3}n)^2} \sin \beta_n x_1 \right\}$$

$$+\tfrac{1}{2} e^{-\alpha_n x_1} \{(\beta_n-\sqrt{3}n) \cos \beta_n x_1 - (\alpha_n-\sqrt{3}n^2) \sin \beta_n x_1\}. \quad \ldots \quad (42)$$

From (38), (39) and (40), it can be shown that

$$A_0 = \frac{A_0'}{x_1+\sqrt{3}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (43)$$

and

$$\left. \begin{aligned} c_n &= \frac{M_n A_n' + N_n B_n'}{M_n^2+N_n^2} \\ d_n &= \frac{N_n A_n' - M_n B_n'}{M_n^2+N_n^2} \end{aligned} \right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (44)$$

From (43) and (44) it is clear that A_0, c_n and d_n can be evaluated when the values of the Fourier coefficients are known. (M_n and N_n can be calculated from (41) and (42) for different values of n). And from (35), we can calculate the values of a_n and b_n once the values of c_n and d_n are known.

§ 5. Thus completing the determination of constants a_n, b_n, c_n and d_n as in §4, it is easy to find out the values of $K(x, y)$ and $H(x, y)$ at any level of the atmosphere (equation (18)). But we are mainly concerned with the distribution of the emergent radiation at the outer boundary of the atmosphere. This from equation (32), and equations (18), (21) and (20) is given by

$$I_{+1}(0.\,y) = \sqrt{3}A_0 + \sum_{n=1}^{\infty}\Big[\,\{(a_n-c_n)\alpha_n+(b_n+d_n)\beta_n\}\cos ny$$
$$+\{\,(b_n+d_n)\alpha_n-(a_n-c_n)\beta_n\,\}\sin ny\Big] \quad .. \quad (45)$$

where a_n, b_n, c_n and d_n are given by the method described in § 4.

§ 6. It has been supposed that an infinite plane surface is radiating outwards uniformly with a known spectral distribution. Above such a radiating surface there exists an atmosphere of free electrons which modifies the distribution at the base. The expression for the modified emergent radiation at the upper boundary of the atmosphere has been given in § 5.

Let us suppose now that the spectral distribution at the lower bound of the atmosphere is given by

$$\psi(y) = \frac{2}{\sqrt{\pi}}e^{-y^2} \quad .. \quad .. \quad .. \quad .. \quad (46)$$

We expand this in a Fourier series of cosines between $-\pi$ and $+\pi$ as follows

$$\psi(y) = \frac{1}{\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}e^{-n^2/4}\cos ny \quad .. \quad .. \quad .. \quad (47)$$

This range $(-\pi, \pi)$ practically covers the significant part of the function $\psi(y)$.

Then in equation (37)

$$A_n' = \frac{2}{\pi}e^{-n^2/4}, \; A_0' = \frac{1}{\pi} \text{ and } B_n' = 0. \quad .. \quad .. \quad (48)$$

From (43)

$$A_0 = \frac{1}{\pi(x_1+\sqrt{3})} \quad .. \quad .. \quad .. \quad .. \quad (49)$$

and from (44),

$$\left.\begin{aligned} c_n &= \frac{2}{\pi}\frac{M_n}{M_n^2+N_n^2}e^{-n^2/4} \\ \text{and } d_n &= \frac{2}{\pi}\frac{N_n}{M_n^2+N_n^2}e^{-n^2/4} \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad (50)$$

and a_n and b_n are given by (35).

For a particular value x_1 or τ_1 (the optical thickness of the atmosphere) the value of M_n and N_n are obtained from (41) and (42) for different values of n. These values are used for determining the constants a_n, b_n, c_n and d_n. Substituting these in equation (45), we can find out the value of $I_{+1}(0, y, \psi)$, the emergent radiation from the outer surface of the atmosphere, when the distribution at the base is given by (46). The values of $I_{+1}(0, y, \psi)$ obtained for $x_1 = 1$ or $\tau_1 = \frac{2}{3}$ are shown in the third column of Table 1.

§ 7. To compare the effect of retaining the second order term of Taylor's series in the representation of $I(\tau, \mu', \lambda-\gamma(1-\cos\theta))$ it was thought worth while to repeat the calculations by the present method of expansion in trigonometrical series for the case obtained by retaining only the first order term of Taylor's series and compare the results with those obtained by the use of Green's function by Chandrasekhar. In this case of first order calculation the equation corresponding to equation (19) will be (cf. Eq. (23), (20), (21), p. 511, *Proc. Roy. Soc. London*, Vol. 192, 1948)

$$\frac{\partial^2 s(x, y)}{\partial x^2}+\frac{\partial^2 s(x, y)}{\partial y^2} = 2\frac{\partial s}{\partial y} \quad \ldots \quad (51)$$

where

$$I_{+1}(x, y)-I_{-1}(x, y) = \sqrt{3}\,\frac{\partial s(x, y)}{\partial y} \quad \ldots \quad (52)$$

$$I_{+1}(x, y)+I_{-1}(x, y) = \frac{\partial s(x, y)}{\partial x}. \quad \ldots \quad (53)$$

The boundary conditions are representable in the present case as

$$\frac{1}{2}\left[\frac{\partial s}{\partial x}+\sqrt{3}\,\frac{\partial s}{\partial y}\right]_{x=x_1} = \psi(y) = \text{a known distribution in } y \quad \ldots \quad (54)$$

and

$$\left[\frac{\partial s}{\partial x}-\sqrt{3}\,\frac{\partial s}{\partial y}\right]_{x=0} = 0. \quad \ldots \quad (55)$$

Taking a trial solution of the type

$$s(x, y) = A_0(x^2+y)+B_0x+Ae^{mx+iny}$$

as before, the general solution can be written in the form by a similar type of arguments as

$$s(x, y) = A_0(x^2+y)+B_0x+a_0+\sum_{n=1}^{\infty}\Big[e^{\alpha_n' x}\left\{a_n'\cos(\beta_n'x+ny)+b_n'\sin(\beta_n'x+ny)\right\}$$

$$+e^{-\alpha_n' x}\left\{c_n'\cos(\beta_n'x-ny)+d_n'\sin(\beta_n'x-ny)\right\}\Big] \quad \ldots \quad (56)$$

where

$$\alpha_n' = \sqrt{\tfrac{1}{2}\left(n^2+\sqrt{n^4+4n^2}\right)} \quad \ldots \quad (57)$$

$$\beta_n' = \sqrt{\tfrac{1}{2}\left(\sqrt{n^4+4n^2}-n^2\right)}. \quad \ldots \quad (58)$$

Relations corresponding to equations (26) and (27) are now found to be

$$\alpha_n'^2-\beta_n'^2 = n^2 \quad \ldots \quad (59)$$

$$2\alpha'_n\beta_n' = 2n \quad \ldots \quad (60)$$

for n positive, α_n' and β_n' will be of the same sign, and for n negative, α_n' and β_n' will be of opposite signs.

Now applying the boundary condition (55), and putting the coefficients of cos ny and sin ny individually equal to zero, we get two relations between the constants of the type,

$$B_0 = \sqrt{3}A_0 \quad \ldots \quad (61)$$

$$\left.\begin{aligned} a'_n &= \frac{(\alpha_n'^2+\beta_n'^2-3n^2)c_n'-2\sqrt{3}n\alpha_n'd_n'}{\alpha_n'^2+(\beta_n'-\sqrt{3}n)^2} \\ b'_n &= -\frac{2\sqrt{3}n\alpha_n'c_n'+(\alpha_n'^2+\beta_n'^2-3n^2)d_n'}{\alpha_n'^2+(\beta_n'-\sqrt{3}n)^2} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (62)$$

Now from the boundary condition (54), we get

$$\begin{aligned} \tfrac{1}{2}\{B_0+2A_0x_1+\sqrt{3}A_0\} &+ \sum_{n=1}^{\infty}\Big[\tfrac{1}{2}e^{\alpha_n'x_1}\Big\{\big(a_n'\alpha_n'+b_n'(\beta_n'+\sqrt{3}n)\big)\cos\beta_n'x_1 \\ &\qquad +\big(b_n'\alpha_n'-a_n'(\beta_n'+\sqrt{3}n)\big)\sin\beta_n'x_1\Big\} \\ &\quad +\tfrac{1}{2}e^{-\alpha_n'x_1}\Big\{\big(d_n'(\beta_n'-\sqrt{3}n)-c_n'\alpha_n'\big)\cos\beta_n'x_1 \\ &\qquad -\big(d_n'\alpha_n'+c_n'(\beta_n'-\sqrt{3}n)\big)\sin\beta_n'x_1\Big\}\Big]\cos ny \\ &+ \sum_{n=1}^{\infty}\Big[\tfrac{1}{2}e^{\alpha_n'x_1}\Big\{\big(b_n'\alpha_n'-a_n'(\beta_n'+\sqrt{3}n)\big)\cos\beta_n'x_1 \\ &\qquad -\big(a_n'\alpha_n'+b_n'(\beta_n'+\sqrt{3}n)\big)\sin\beta_n'x_1\Big\} \\ &\quad +\tfrac{1}{2}e^{-\alpha_n'x_1}\Big\{\big(d_n'(\beta_n'-\sqrt{3}n)-c_n'\alpha_n'\big)\sin\beta_n'x_1 \\ &\qquad +\big(d_n'\alpha_n'+c_n'(\beta_n'-\sqrt{3}n)\big)\cos\beta_n'x_1\Big\}\Big]\sin ny \\ &= \psi(y) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (63) \end{aligned}$$

As before, the distribution at the base is supposed to be capable of Fourier expansion

$$\psi(y) = A_0'+\sum_{n=1}^{\infty}A_n'\cos ny+\sum_{n=1}^{\infty}B_n'\sin ny \quad \ldots \quad \ldots \quad (64)$$

Comparing the coefficients, we find that

$$A_n' = M_n'c_n'+N_n'd_n' \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (65)$$

$$B_n' = N_n'c_n'-M_n'd_n' \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (66)$$

$$A_0' = A_0(x_1+\sqrt{3}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (67)$$

where

$$\begin{aligned} M_n' = \tfrac{1}{2}e^{\alpha_n'x_1}\Big\{&-\frac{2\sqrt{3}n\alpha_n'^2+(\alpha_n'^2+\beta_n'^2-3n^2)(\beta_n'+\sqrt{3}n)}{\alpha_n'^2+(\beta_n'-\sqrt{3}n)^2}\sin\beta_n'x_1 \\ &+\frac{(\alpha_n'^2+\beta_n'^2-3n^2)\alpha_n'-2\sqrt{3}n\alpha_n'(\beta_n'+\sqrt{3}n)}{\alpha_n'^2+(\beta_n'-\sqrt{3}n)^2}\cos\beta_n'x_1\Big\} \\ &-\tfrac{1}{2}e^{-\alpha_n'x_1}\{\alpha_n'\cos\beta_n'x_1+(\beta_n'-\sqrt{3}n)\sin\beta_n'x_1\} \quad \ldots \quad \ldots \quad \ldots \quad (68) \end{aligned}$$

and

$$\begin{aligned} N_n' = \tfrac{1}{2}e^{\alpha_n'x_1}\Big\{&-\frac{(\alpha_n'^2+\beta_n'^2-3n^2)\alpha_n'-2\sqrt{3}n\alpha_n'(\beta_n'+\sqrt{3}n)}{\alpha_n'^2+(\beta_n'-\sqrt{3}n)^2}\sin\beta_n'x_1 \\ &-\frac{2\sqrt{3}n\alpha_n'^2+(\alpha_n'^2+\beta_n'^2-3n^2)(\beta_n'+\sqrt{3}n)}{\alpha_n'^2+(\beta_n'-\sqrt{3}n)^2}\cos\beta_n'x_1\Big\} \\ &+\tfrac{1}{2}e^{-\alpha_n'x_1}\{(\beta_n'-\sqrt{3}n)\cos\beta_n'x_1-\alpha_n'\sin\beta_n'x_1\}. \quad \ldots \quad \ldots \quad (69) \end{aligned}$$

From (65) and (66) it is seen that

$$c_n' = \frac{M_n'A_n' + N_n'B_n'}{M_n'^2 + N_n'^2} \quad \cdots \quad (70)$$

and

$$d_n' = \frac{N_n'A_n' - M_n'B_n'}{M_n'^2 + N_n'^2} \quad \cdots \quad (71)$$

It is clear from (70) and (71) that c_n' and d_n' can be determined, when the values of A_n' and B_n' are known. We can calculate the values of a_n' and b_n' from (62).

It is now easy to find out the values of $I_{+1}(x, y)$ and $I_{-1}(x, y)$ at any depth (equations (52) and (53)). But we are mainly concerned with the intensity distribution at the outer surface of the stellar atmosphere and this from (52), (53), (56), is given by

$$I_{+1}(0, y) = \sqrt{3}A_0 + \sum_{n=1}^{\infty}\Big[\big\{(a_n' - c_n')\alpha_n' + (b_n' + d_n')\beta_n'\big\}\cos ny + \big\{(b_n' + d_n')\alpha_n' - (a_n' - c_n')\beta_n'\big\}\sin ny\Big] \quad \cdots \quad (72)$$

Again supposing that $\psi(y) = \dfrac{2}{\sqrt{\pi}}e^{-y^2}$, and expanding in a Fourier series in cosines as before and comparing the coefficients we get from equations (67), (70) and (71)

$$A_0' = \frac{1}{\pi}, \text{ and hence } A_0 = \frac{1}{\pi(x_1 + \sqrt{3})} \quad \cdots \quad (73)$$

and

$$\left.\begin{aligned} c_n' &= \frac{2}{\pi}\frac{M_n'e^{-n^2/4}}{M_n'^2 + N_n'^2} \\ d_n' &= \frac{2}{\pi}\frac{N_n'e^{-n^2/4}}{M_n'^2 + N_n'^2} \end{aligned}\right\} \quad \cdots \quad (74)$$

The constants a_n' and b_n' are given by (62).

For a given value of x_1 the values of M_n' and N_n' have been calculated, and these values are used to determine a_n', b_n', c_n', d_n'. From (72), we obtain the values of emergent intensity at the outer boundary of the stellar atmosphere, when the intensity distribution at the base is given by $\psi(y) = \dfrac{2}{\sqrt{\pi}}e^{-y^2}$. The results for $x_1 = 1$ or $\tau_1 = \frac{2}{3}$ are shown in the first column of Table 1.

§ 8. The second column of Table 1 contains the values of the intensity obtained by the method of Chandrasekhar (1948) in the case discussed in § 7. Chandrasekhar calculated the values of $I_{+1}(0, y, \delta)$ for different values of y (Radiative Transfer, p. 332) supposing the distribution at the lower boundary to be $\delta(y)$. It was also mentioned that the solution for any other distribution $\psi(y)$ at the lower bound of the atmosphere could be obtained by

$$I_{+1}(0, y, \psi) = \int_{-\infty}^{\infty} I_{+1}(0, y - \eta, \delta)\,\psi(\eta)\,d\eta \quad \cdots \quad (75)$$

In the present paper we take

$$\psi(\eta) = \frac{2}{\sqrt{\pi}} e^{-\eta^2}$$

and
$$I_{+1}(0, y, \delta) = \frac{\sqrt{3}}{\pi} e^y \int_0^\infty \frac{(p \cos \beta y + \beta q \sin \beta y)\sqrt{1+\beta^2}}{(p^2+\beta^2 q^2)} d\beta \quad .. \quad (76)$$

(cf. eqn. (104), p. 333, Radiative Transfer.)

The values of $I_{+1}(0, y, \psi)$ have been calculated for values of y ranging from $y = 1$, to $y = 3$, using the values of $I_{+1}(0, y-\eta, \delta)$ from Chandrasekhar's Table 1, (*Proc. Roy. Soc.* (London), Vol. 192, p. 516). The values of $I_{+1}(0, y, \psi)$ for y's beyond this range are not calculated, as these cannot be obtained from the data of the Table 1 mentioned above. It is found that within the range allowed by the table, the values obtained by the method of trigonometrical series followed here, agree completely with those obtained by Chandrasekhar's method.

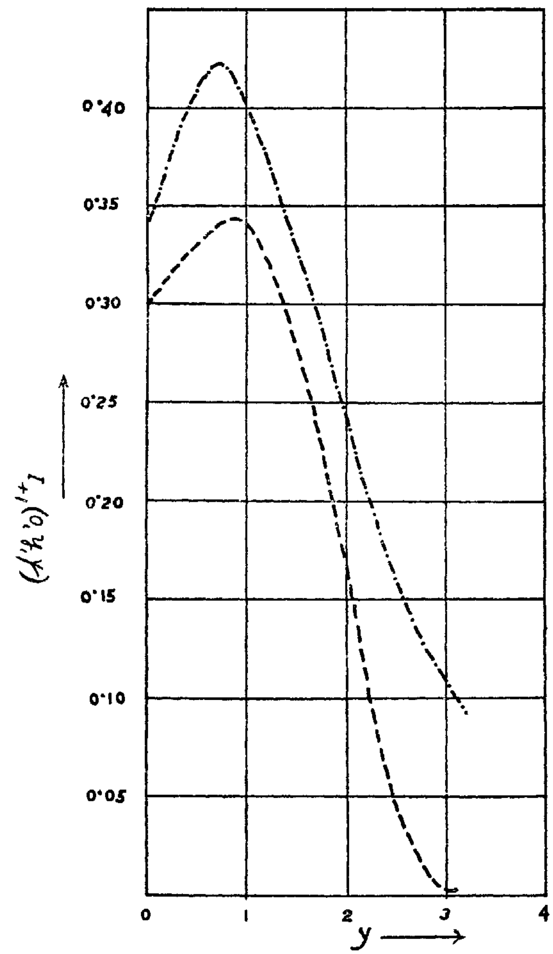

Fig. 1.

The abscissae y denote wavelength shifts in units of $\frac{4}{3}$ Compton wavelength and the ordinates I_{+1} (0, y, ψ), the emergent intensities from the outer boundary of the atmosphere.

TABLE 1

y		$I_{+1}(0, y, \psi)$ First approx. (with first order term in Taylor's series).	$I_{+1}(0, y, \psi)$ First approx. (by Chandrasekhar's method).	$I_{+1}(0, y, \psi)$ Second approx. (with 2nd order term in Taylor's series).
0	..	0·34	....	0·30
0·5	..	0·41	....	0·33
1·0	..	0·40	0·40	0·34
1·5	..	0·33	0·33	0·28
2·0	..	0 24	0·24	0·17
2·5	..	0·16	0·16	0·05
3·0	..	0·11	0·11	0·003
3·142	..	0·09	....	0·003

§ 9. The comparative results of the first and second order calculations are illustrated in Fig. 1. It should be remembered that y means wavelength shift in units of $\frac{4}{3}$ Compton wavelength. It is clear from Fig. 1 that in passing from the first to the second approximation the shift and intensity distribution are affected to a marked degree. The second approximation curve is drawn in dashes and the first approximation results are traced as dots and dashes. The additional term of Taylor's series which we have included for the second approximation makes negative contribution to the intensity. Thus the reduction of intensity for different values of y is evident, and this overall lowering of intensity reduces the error of the first order calculation noticed by Chandrasekhar. There is possibility of the intensity curve being again raised a little, if we include the third order term Taylor's series in our consideration.

It is clear from the above treatment that the trigonometrical series gives us a very effective method of treating problems of this type. The agreement between the results obtained by the present method and that of Green's function used by Chandrasekhar, show that the accuracies obtained by the two methods are practically the same. The series representing $I_{+1}(0, y, \psi)$ is highly convergent and hence easy to calculate numerically.

The problem of transfer, allowing for the partial polarisation of the scattered radiation is also being considered by the same method. The results will be published shortly.

In conclusion, I have much pleasure to acknowledge my indebtedness to Prof. N. R. Sen for many helpful discussions and encouragement during the preparation of this work.

ABSTRACT

The problem of softening of radiation by multiple Compton scattering in stellar atmospheres containing free electrons, has been solved in the first approximation (in Chandrasekhar's method of solution by Gaussian approximation) by the method of trigonometrical series. The intensity distribution at the outer surface has been calculated by retaining the first and the second order terms of Taylor's expansion of scattered intensity. The first order calculation by the method of trigonometrical series gives result which is identical with that found by Chandrasekhar's method with the aid of Green's function. The second order calculation considerably reduces the error which was noticed by Chandrasekhar in the first order calculation.

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