A MATRIX TREATMENT OF FOUR-DIMENSIONAL ROTATION IN HYPERSPACE *

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A vector in n-space rotating about a point can undergo all types of $2r$-dimensional rotations, where r admits of all integral values such that $2 \leqslant 2r \leqslant n$. It was shown in a previous paper (Ghosh, 1948) that the general rotation of a vector X about the origin with co-ordinates $(x_1, x_2, \ldots x_n)$ referred to a rectangular system of axes in a Euclidian n-space can be represented in symbolic form by means of the orthogonal transformation $X \to X'$, where

$$X' = e^{\Omega}X = X + \Omega X + \frac{\Omega^2}{2!}X + \ldots . \qquad (1.1)$$

Ω being the skew-symmetric matrix of rotation (ω_{ij}) of rank $2r$ and ΩX denoting the vector Y with components $y_i = \sum_{j=1}^{n} \omega_{ij} x_j$. In the case of a four-dimensional rotation Ω is of rank 4 satisfying the characteristic equation †

$$\Omega^5 + p_1{}^2\Omega^3 + p_2{}^4\Omega = 0, \qquad (1.2)$$

where

$$p_1{}^2 = \sum_{i,j} \omega_{ij}^2 \qquad (1.3)$$

$$p_2{}^4 = \sum_{i,j,k,l} (\omega_{ij}\omega_{kl} - \omega_{ik}\omega_{jl} + \omega_{il}\omega_{jk})^2 \qquad (1.4)$$

On reducing the exponents of Ω in (1.1) below 5 by means of the relation (1.2), we obtain

$$X' = X + f_1(p_1, p_2)\Omega X + f_2(p_1, p_2)\Omega^2 X + f_3(p_1, p_2)\Omega^3 X + f_4(p_1, p_2)\Omega^4 X, \qquad (1.5)$$

where

$$\left.\begin{aligned}
f_1(p_1, p_2) &= 1 - \frac{p_2{}^4}{5!} + \frac{p_1{}^2 p_2{}^4}{7!} - \frac{p_1{}^4 p_2{}^4 - p_2{}^8}{9!} + \ldots \\
f_2(p_1, p_2) &= \frac{1}{2!} - \frac{p_2{}^4}{6!} + \frac{p_1{}^2 p_2{}^4}{8!} - \frac{p_1{}^4 p_2{}^4 - p_2{}^8}{10!} + \ldots \\
f_3(p_1, p_2) &= \frac{1}{3!} - \frac{p_1{}^2}{5!} + \frac{p_1{}^4 - p_2{}^4}{7!} - \frac{p_1{}^6 - 2p_1{}^2 p_2{}^4}{9!} + \ldots \\
f_4(p_1, p_2) &= \frac{1}{4!} - \frac{p_1{}^2}{6!} + \frac{p_1{}^4 - p_2{}^4}{8!} - \frac{p_1{}^6 - 2p_1{}^2 p_2{}^4}{10!} + \ldots
\end{aligned}\right\} \qquad (1.6)$$

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† If ω_{ij} is of the form $\alpha_i\beta_j - \alpha_j\beta_i + \gamma_i\delta_j - \gamma_j\delta_i$, where α's, β's, γ's, δ's, are the components of four arbitrary vectors A, B, C, D, the matrix Ω is of rank 4 and the rotation takes place in the four-space spanned by A, B, C, D.

The object of the present paper is generally to deal with a four-dimensional rotation in n-space, showing that it can always be performed by two suitable plane rotations, and to study some of the properties of the four functions f of two arguments p_1, p_2 involved in (1.5).

2. By a straight-forward calculation it is easy to obtain successive terms in (1.6). We notice that corresponding terms in f_1 and f_2 have the same numerator, as also those in f_3 and f_4. Denoting the numerator of the $(k+1)$th term in either f_1 or f_2 by a_k and that corresponding to either f_3 or f_4 by c_k, it can be verified that they satisfy the following recurrence relations:

$$\left.\begin{aligned} a_k &= -a_{k-1}\,p_1^2 - a_{k-2}\,p_2^4 \quad (k = 3, 4, 5, \ldots), \\ c_k &= -c_{k-1}\,p_1^2 - c_{k-2}\,p_2^4 \quad (k = 2, 3, 4, \ldots), \\ a_k &= -c_{k-1}\,p_2^4 \qquad\qquad\quad (k = 1, 2, 3, \ldots) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.1)$$

with

$$a_0 = c_0 = 1, \quad c_1 = -p_1^2.$$

We make now some further observations on the transformation (1.5):

(i) If the vector X remains invariant, that is, if $X' = X$, then X must satisfy the linear equation

$$\Omega X = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

and since Ω is of rank 4, all these invariant vectors lie on a $(n-4)$-flat.

(ii) If $p_2^4 = 0$, the matrix Ω is of rank 2 and satisfies the characteristic equation *

$$\Omega^3 + p_1^2\Omega = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

The reduced expression for X' in (1.5) then becomes (Schwerdtfeger, 1945)

$$X' = X + \frac{\sin p_1}{p_1}\,\Omega X + \frac{1-\cos p_1}{p_1^2}\,\Omega^2 X, \quad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

which corresponds to a plane rotation.

3. To examine more fully the structure of the functions f let us start with the roots of the algebraic equation

$$x^5 + p_1^2 x^3 + p_2^4 x = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

which we denote by

$$0, \quad \pm i\beta_2, \quad \pm i\beta_4 \quad (i = \sqrt{-1}),$$

so that

$$\beta_2^2 + \beta_4^2 = p_1^2, \; \beta_2^2\beta_4^2 = p_2^4. \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

Since now the relation

$$e^x = 1 + xf_1(p_1, p_2) + x^2 f_2(p_1, p_2) + x^3 f_3(p_1, p_2) + x^4 f_4(p_1, p_2) \quad \ldots \quad (3.3)$$

holds good only when x satisfies the equation (3.1), we get the following set of identities

$$e^{\pm i\beta_2} = (1 - \beta_2^2 f_2 + \beta_2^4 f_4) \pm i(\beta_2 f_1 - \beta_2^3 f_3), \quad \ldots \quad \ldots \quad (3.4)$$

$$e^{\pm i\beta_4} = (1 - \beta_4^2 f_2 + \beta_4^4 f_4) \pm i(\beta_4 f_1 - \beta_4^3 f_3), \quad \ldots \quad \ldots \quad (3.5)$$

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* If ω_{ij} is of the form $\alpha_i\beta_j - \alpha_j\beta_i$, where α's and β's are the components of two arbitrary vectors A and B, the rank of the matrix Ω is 2 and the corresponding rotation takes place in the plane of A, B.

whence

$$\left.\begin{aligned}\sin\beta_2 &= \beta_2 f_1-\beta_2^3 f_3, \quad \cos\beta_2 = 1-\beta_2^2 f_2+\beta_2^4 f_4,\\ \sin\beta_4 &= \beta_4 f_1-\beta_4^3 f_3, \quad \cos\beta_4 = 1-\beta_4^2 f_2+\beta_4^4 f_4.\end{aligned}\right\} \quad \ldots \quad (3.6)$$

Solving these equations, we get

$$\left.\begin{aligned}f_1(p_1, p_2) &= \frac{\beta_4^3\sin\beta_2-\beta_2^3\sin\beta_4}{\beta_2\beta_4(\beta_4^2-\beta_2^2)},\\ f_2(p_1, p_2) &= \frac{\beta_4^4(1-\cos\beta_2)-\beta_2^4(1-\cos\beta_4)}{\beta_2^2\beta_4^2(\beta_4^2-\beta_2^2)},\\ f_3(p_1, p_2) &= \frac{\beta_4\sin\beta_2-\beta_2\sin\beta_4}{\beta_2\beta_4(\beta_4^2-\beta_2^2)},\\ f_4(p_1, p_2) &= \frac{\beta_4^2(1-\cos\beta_2)-\beta_2^2(1-\cos\beta_4)}{\beta_2^2\beta_4^2(\beta_4^2-\beta_2^2)}.\end{aligned}\right\} \quad \ldots \quad \ldots \quad (3.7)$$

The equations (3.6) and (3.7) show the connection of f-functions with the circular functions. It should be noticed that the exponents of β_2 or β_4 greater than 4 are reducible by means of the relations

$$\left.\begin{aligned}\beta^{2k+3} &= (-1)^k(c_k\beta^3-a_k\beta)\\ \beta^{2k+4} &= (-1)^k(c_k\beta^4-a_k\beta^2)\end{aligned}\right\} \quad (k = 1, 2, 3 \ldots) \quad \ldots \quad (3.8)$$

where β stands for either β_2 or β_4. An explicit expression for the coefficient c_k is given by

$$(-1)^k c_k = \frac{\beta_4^{2k+2}-\beta_2^{2k+2}}{\beta_4^2-\beta_2^2}. \quad \ldots \quad \ldots \quad \ldots \quad (3.9)$$

4. The four functions f satisfy among themselves two identical relations which may be derived as follows:

Squaring (3.6) and adding, we have

$$\sin^2\beta_2+\cos^2\beta_2 = 1 = (\beta_2 f_1-\beta_2^3 f_3)^2+(1-\beta_2^2 f_2+\beta_2^4 f_4)^2, \quad \ldots \quad (4.1)$$

$$\sin^2\beta_4+\cos^2\beta_4 = 1 = (\beta_4 f_1-\beta_4^3 f_3)^2+(1-\beta_4^2 f_2+\beta_4^4 f_4)^2. \quad \ldots \quad (4.2)$$

Simplifying (4.1) and making use of (3.8) to reduce the exponents of β_2 below 5, we get finally

$$0 = \{f_1^2-2f_2+a_1(f_3^2-2f_2f_4)-a_2f_4^2\}+\beta_2^2\{f_2^2+2f_4-2f_1f_3-c_1(f_3^2-2f_2f_4)+c_2f_4^2\}.$$

Similarly, from (4.2), we get the same equation with β_2^2 replaced by β_4^2. Hence, the required identities are

$$\left.\begin{aligned}f_1^2-2f_2+a_1(f_3^2-2f_2f_4)-a_2f_4^2 &= 0,\\ f_2^2+2f_4-2f_1f_3-c_1(f_3^2-2f_2f_4)+c_2f_4^2 &= 0.\end{aligned}\right\} \quad \ldots \quad \ldots \quad (4.3)$$

5. We now proceed to show how the general four-dimensional rotation is obtained by two suitable plane rotations, the rotation in either plane being independent of the rotation in the other.

From (1.5) we see that the four-dimensional rotation operator Δ is of the form

$$\Delta = E+f_1\Omega+f_2\Omega^2+f_3\Omega^3+f_4\Omega^4 \quad \ldots \quad \ldots \quad \ldots \quad (5.1)$$

where E denotes the unit matrix. Utilizing the explicit expressions (3.7) for the functions f, the operator Δ takes the form

$$\Delta = E + \frac{\sin\beta_2}{\beta_2}\left\{\frac{\beta_4^2\Omega+\Omega^3}{\beta_4^2-\beta_2^2}\right\} + \frac{1-\cos\beta_2}{\beta_2^2}\left\{\frac{\beta_4^2\Omega^2+\Omega^4}{\beta_4^2-\beta_2^2}\right\}$$
$$-\frac{\sin\beta_4}{\beta_4}\left\{\frac{\beta_2^2\Omega+\Omega^3}{\beta_4^2-\beta_2^2}\right\} - \frac{1-\cos\beta_4}{\beta_4^2}\left\{\frac{\beta_2^2\Omega^2+\Omega^4}{\beta_4^2-\beta_2^2}\right\}. \quad \ldots \quad (5.2)$$

Let Γ_2 and Γ_4 denote the matrices $(\beta_4^2\Omega+\Omega^3)/(\beta_4^2-\beta_2^2)$ and $(\beta_2^2\Omega+\Omega^3)/(\beta_2^2-\beta_4^2)$ respectively; then $\Gamma_2+\Gamma_4 = \Omega$,

$$\Gamma_2 \,.\, \Gamma_4 = -\frac{\Omega^6+(\beta_2^2+\beta_4^2)\Omega^4+\beta_2^2\beta_4^2\Omega^2}{(\beta_2^2-\beta_4^2)^2} = \Gamma_4 \,.\, \Gamma_2 = 0. \quad \ldots \quad (5.3)$$

Consequently,

$$\Gamma_2^2 = \Omega\Gamma_2, \quad \Gamma_4^2 = \Omega\Gamma_4, \quad \Gamma_2^2+\Gamma_4^2 = \Omega^2.$$

Written in terms of Γ_2 and Γ_4 the operator (5.2) therefore becomes

$$\Delta = E + \frac{\sin\beta_2}{\beta_2}\Gamma_2 + \frac{1-\cos\beta_2}{\beta_2^2}\Gamma_2^2 + \frac{\sin\beta_4}{\beta_4}\Gamma_4 + \frac{1-\cos\beta_4}{\beta_4^2}\Gamma_4^2. \quad \ldots \quad (5.4)$$

Since $\Gamma_2 . \Gamma_4 = 0$, the above admits of being written in the product form

$$\Delta = \left(E + \frac{\sin\beta_4}{\beta_4}\Gamma_4 + \frac{1-\cos\beta_4}{\beta_4^2}\Gamma_4^2\right)\left(E + \frac{\sin\beta_2}{\beta_2}\Gamma_2 + \frac{1-\cos\beta_2}{\beta_2^2}\Gamma_2^2\right) \ldots \quad (5.5)$$

where the order of rotations due to Γ_2 and Γ_4 is immaterial. It may be seen that Γ_2 and Γ_4 are of rank 2, satisfying the characteristic equations

$$\Gamma_2^3+\beta_2^2\Gamma_2 = 0, \quad \Gamma_4^3+\beta_4^2\Gamma_4 = 0 \quad \text{respectively.} \quad \ldots \quad \ldots \quad (5.6)$$

Let us represent the rotation of X due to Γ_2 by X_2' and that due to Γ_4 by X_4' as in (2.4). Then $X_2'-X$ will represent all vectors lying in the plane of rotation of Γ_2, while $X_4'-X$ will represent those corresponding to Γ_4. Now from (2.4)

$$X_2'-X = \frac{\sin\beta_2}{\beta_2}\Gamma_2 X + \frac{1-\cos\beta_2}{\beta_2^2}\Gamma_2^2 X. \quad \ldots \quad \ldots \quad (5.7)$$

Therefore

$$\Gamma_4(X_2'-X) = 0.$$

Thus all vectors lying in the plane of rotation of Γ_2 belong to the invariant $(n-2)$-flat of Γ_4. Similarly we prove that all vectors lying in the plane of rotation of Γ_4 belong to the invariant $(n-2)$-flat of Γ_2. It must be noted the $(n-4)$-flat defined in (2.2) remains invariant for both Γ_2 and Γ_4.

Let $\bar{\Delta}$ denote the transposed of Δ, then from (5.4) we have

$$\frac{\Delta-\bar{\Delta}}{2} = \frac{\sin\beta_2}{\beta_2}\Gamma_2 + \frac{\sin\beta_4}{\beta_4}\Gamma_4, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.8)$$

$$\frac{\Delta+\bar{\Delta}}{2} = E + \frac{1-\cos\beta_2}{\beta_2^2}\Gamma_2^2 + \frac{1-\cos\beta_4}{\beta_4^2}\Gamma_4^2. \quad \ldots \quad \ldots \quad (5.9)$$

Representing $(\Delta-\bar{\Delta})\,\frac{1}{2}$ by ϕ, it may be proved that ϕ satisfies the characteristic equation

$$\phi^5+q_1^2\phi^3+q_2^4\phi = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.10)$$

where

$$q_1^2 = \sin^2\beta_2+\sin^2\beta_4, \quad q_2^4 = \sin^2\beta_2 \sin^2\beta_4.$$

Thus corresponding to a pair of characteristic roots $\pm\, i \sin \beta_{2k}$ of ϕ there is a pair $\pm\, i\, \beta_{2k}$ of Ω. To express Ω in terms of ϕ we may, however, use the determinantal equation

$$\begin{vmatrix} \Omega & \beta_2 & \beta_4 \\ \phi & \sin\beta_2 & \sin\beta_4 \\ -\phi^3 & \sin^3\beta_2 & \sin^3\beta_4 \end{vmatrix} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (5.11)$$

The symmetric operators $\phi^2, \phi^4, \ldots$ are all expressible in terms of $\Gamma_2{}^2$ and $\Gamma_4{}^2$. We observe that in (5.9) the operator $\dfrac{\Delta+\overline{\Delta}}{2}-E$, which we denote by ψ, is symmetric and involves $\Gamma_2{}^2$ and $\Gamma_4{}^2$. To express ψ in terms of ϕ^2 and ϕ^4, we make use of the following determinantal equation

$$\begin{vmatrix} \psi & 1 & 1 \\ \phi^2 & 1+\cos\beta_2 & 1+\cos\beta_4 \\ -\phi^4 & \sin^2\beta_2(1+\cos\beta_2) & \sin^2\beta_4(1+\cos\beta_4) \end{vmatrix} = 0. \quad \ldots \quad (5.12)$$

The above yields a notable property of the orthogonal matrix Δ of order n and rank 4.

6. Let us next consider the case of a four-dimensional rotation in 4-space. Ω is now a matrix of order 4 and rank 4. The structure of the functions f remaining unchanged, the values of the arguments are given by

$$p_1^2 = \omega_{12}{}^2+\omega_{13}{}^2+\omega_{14}{}^2+\omega_{23}{}^2+\omega_{24}{}^2+\omega_{34}{}^2, \quad \ldots \quad \ldots \quad (6.1)$$

$$p_2^4 = (\omega_{12}\omega_{34}-\omega_{13}\omega_{24}+\omega_{14}\omega_{23})^2. \quad \ldots \quad \ldots \quad \ldots \quad (6.2)$$

Introducing the conjugate matrix Ω^*, we have the following well-known relations

$$\left.\begin{aligned} \Omega\Omega^* &= -p_2{}^2E, \\ \Omega^2+\Omega^{*2} &= -p_1{}^2E, \\ \Omega^3 &= -p_1{}^2\Omega+p_2{}^2\Omega^*, \\ \Omega^{*3} &= -p_1{}^2\Omega^*+p_2{}^2\Omega. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (6.3)$$

The four-dimensional rotation operator (5.1) now reduces to the form

$$(1-p_2{}^4f_4)E+(f_1-p_1{}^2f_3)\Omega+p_2{}^2f_3\Omega^*+(f_2-p_1{}^2f_4)\Omega^2, \quad \ldots \quad (6.4)$$

which, on substitution from (3.7) becomes $1/(\beta_4{}^2-\beta_2{}^2)$ times

$$\{(\beta_4{}^2\cos\beta_2-\beta_2{}^2\cos\beta_4)E+(\beta_4\sin\beta_4-\beta_2\sin\beta_2)\Omega$$
$$+(\beta_4\sin\beta_2-\beta_2\sin\beta_4)\Omega^*+(\cos\beta_2-\cos\beta_4)\Omega^2\}. \quad \ldots \quad (6.5)$$

The matrices Γ_2 and Γ_4 are now expressible as

$$\Gamma_2 = \frac{\beta_2}{\beta_4{}^2-\beta_2{}^2}(\beta_4\Omega^*-\beta_2\Omega), \quad \ldots \quad \ldots \quad \ldots \quad (6.6)$$

$$\Gamma_4 = \frac{\beta_4}{\beta_2{}^2-\beta_4{}^2}(\beta_2\Omega^*-\beta_4\Omega), \quad \ldots \quad \ldots \quad \ldots \quad (6.7)$$

and being of rank 2, they determine the component plane rotations (Cole, 1890) as in (5.5).

7. We conclude this paper by proving that f-functions with multiple arguments (mp_1, mp_2) are in general expressible in terms of f-functions with arguments (p_1, p_2).

Consider the equation

$$x^5+(mp_1)^2x^3+(mp_2)^4x = 0, \quad \quad (7.1)$$

where m is a positive integer. The roots of this equation are related to those of (3.1) and are 0, $\pm im\beta_2$, $\pm im\beta_4$. Reducing e^x by means of (7 1) it follows that the relation

$$e^x = 1+\sum_{s=1}^{4} x^s f_s(mp_1, mp_2) \quad \quad (7.2)$$

holds good if x satisfies the equation (7.1). Substituting a root $im\beta_2$ for x in the above and referring to the relation (3.4), the left-hand side af (7.2) can be written in the form

$$\{(1-\beta_2^2f_2+\beta_2^4f_4)+i(\beta_2f_1-\beta_2^3f_3)\}^m. \quad \quad (7.3)$$

Reducing the above to the form which does not involve powers of β_2 greater than 4, by means of the relations (3.8) and computing similar coefficients together in (7.2), we obtain the required formulae.

Thus, for $m = 2$, we have

$$\left.\begin{aligned} 2f_1(2p_1, 2p_2) &= 2f_1+2a_1(f_1f_4+f_2f_3)+2a_2f_3f_4, \\ 2^2f_2(2p_1, 2p_2) &= 2f_2+f_1^2+a_1(f_3^2+2f_2f_4)+a_2f_4^2, \\ 2^3f_3(2p_1, 2p_2) &= 2f_3+2f_1f_2+2c_1(f_1f_4+f_2f_3)+2c_2f_3f_4, \\ 2^4f_4(2p_1, 2p_2) &= 2f_4+f_2^2+2f_1f_3+c_1(f_3^2+2f_2f_4)+c_2f_4^2. \end{aligned}\right\} \quad (7.4)$$

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