ON 'SIMILARITY' SOLUTIONS OF PRANDTL'S BOUNDARY LAYER EQUATIONS

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INTRODUCTION

In two recent papers, Goldstein (1939) and Mangler (1943) have examined the question of the existence of 'similarity' solutions of two-dimensional boundary layer equations of an incompressible viscous fluid. By assuming the longitudinal velocity profile $u(x, y)$ to be affine, i.e., assuming that the values of the longitudinal component $u(x, y)$ of the velocity for two different values of x are the same when suitable scale factors depending on x are introduced in u and y, they have found that the partial differential equations of the boundary layer reduce to an ordinary differential equation. They have naturally taken the known velocity $U(x)$ of the potential flow outside the boundary layer as the scale factor for the longitudinal component $u(x, y)$ of the velocity within the boundary layer and have deduced from this assumption that, in order that 'similarity' solutions of the boundary layer equations may exist, $U(x)$ must be proportional to x^m or to e^{px} where m and p are constants.

It has been shown in this paper that under more general conditions 'similarity' solutions of the boundary layer equations can be proved to exist. This has been done by discarding the scale factor $U(x)$ for the longitudinal component $u(x, y)$ of the velocity in the boundary layer and introducing, in its place, an unknown scale factor $h(x)$ of the dimensions of a velocity. This scale factor $h(x)$ and also the scale factor $g(x)$ for y are then determined by examining under what conditions the boundary layer equations reduce to an ordinary differential equation. The velocity $U(x)$ of the potential flow outside the boundary layer is then easily determined and it is found that the value of $U(x)$ is more general than that found by Goldstein or Mangler. The 'similarity' solutions obtained in this paper have therefore a wider range of applications.

DIMENSIONLESS STREAM FUNCTION

In the steady two-dimensional flow in the boundary layer of an incompressible viscous fluid along a curved boundary, the equation of motion is

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}, \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

where x is measured along the boundary of the curve, y perpendicular to it, and $U(x)$ is the velocity in the irrotational flow just outside the boundary layer. The equation of continuity in the boundary layer is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

If there be a free stream line in the liquid along which the boundary layer approximations are valid, the equation (1) is the equation of motion and the equation (2) is the equation of continuity in a thin layer about that stream line, x being now measured along the stream line. In particular, these equations hold in the flow in a two-dimensional jet and in two-dimensional wakes behind a cylinder.

Introducing the stream function ψ, we can write

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad \ldots \qquad (3)$$

Taking l to be a suitable reference length, V a suitable reference velocity, $R = \frac{Vl}{\nu}$ the Reynolds number and $g(x)$ a dimensionless scale factor, we introduce dimensionless variables

$$\xi = \frac{x}{l}, \quad \eta = \frac{y\sqrt{R}}{lg(x)}. \quad \ldots \qquad (4)$$

We further introduce a dimensionless stream function $f(\xi, \eta)$ defined by

$$f(\xi, \eta) = \frac{\psi(x, y)\sqrt{R}}{lh(x)}, \quad \ldots \qquad (5)$$

where $h(x)$ has the dimensions of a velocity. The components of velocity are then

$$u = \frac{\partial \psi}{\partial y} = \frac{h}{g} f',$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{l}{\sqrt{R}}\left[f\frac{dh}{dx} + h\left(\frac{1}{l}\frac{\partial f}{\partial \xi} - \frac{\eta f'}{g}\frac{dg}{dx}\right)\right].$$

We then have

$$\frac{\partial u}{\partial x} = f'\frac{d}{dx}\left(\frac{h}{g}\right) + \frac{h}{gl}\frac{\partial f'}{\partial \xi} - \frac{\eta h f''}{g^2}\frac{dg}{dx},$$

$$\frac{\partial u}{\partial y} = \frac{h\sqrt{R}}{lg^2} f'',$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{R}{l^2 g^3} hf''',$$

$$\nu\frac{\partial^2 u}{\partial y^2} = \frac{hV}{lg^3} f''',$$

where the dash denotes differentiation of f with respect to η.

Substituting in (1), we get after a little simplification

$$\frac{lg^2}{V}\frac{d}{dx}\left(\frac{h}{g}\right)f'^2 + \frac{gh}{V}\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right) - \frac{lg}{V}ff''\frac{dh}{dx} = \frac{lg^3}{hV}U\frac{dU}{dx} + f'''. \quad \ldots \qquad (6)$$

Putting

$$\left.\begin{aligned} \alpha &= \frac{lg}{V}\frac{dh}{dx}, \\ \beta &= \frac{lg^3}{hV}U\frac{dU}{dx}, \\ \gamma &= \frac{lg^2}{V}\frac{d}{dx}\left(\frac{h}{g}\right), \end{aligned}\right\} \quad \ldots \qquad (7)$$

we can write the equation (6) as

$$f'''+\alpha ff''+\beta-\gamma f'^2 = \frac{gh}{V}\left(f'\frac{\partial f'}{\partial \xi}-f''\frac{\partial f}{\partial \xi}\right). \qquad (8)$$

In order that the differential equation (8) may reduce to an ordinary differential equation in f and η only, it is necessary that f and f' should be independent of ξ, and that the coefficients α, β, γ appearing in (8) should be independent of x. Hence, when 'similarity' solutions of the boundary layer equations exist, the dimensionless stream function f satisfies the ordinary differential equation

$$f'''+\alpha ff''+\beta-\gamma f'^2 = 0. \qquad (9)$$

This equation is more general than the corresponding equation of the dimensionless stream function obtained by Goldstein and Mangler. When α, β, γ are constants, the equations (7) determine $g(x)$, $h(x)$ and $U(x)$.

Determination of scale factors and the velocity of irrotational flow

From (7), we get at once

$$(2\alpha-\gamma) = \frac{l}{V}\frac{d}{dx}(gh),$$

which gives, on integration

$$gh = \frac{V}{l}\Big[(2\alpha-\gamma)x+A\Big], \qquad (10)$$

where A is a constant.

Eliminating g between (10) and the first equation of (7), we get

$$\frac{1}{h}\frac{dh}{dx} = \frac{\alpha}{(2\alpha-\gamma)x+A}. \qquad (11)$$

Assuming that $(2\alpha-\gamma)\neq 0$, we get from (11) after integration

$$h = k\left[(2\alpha-\gamma)x+A\right]^{\frac{\alpha}{2\alpha-\gamma}}, \qquad (12)$$

where k is a constant.

From (10) and (12), we then get

$$g = \frac{V}{lk}\Big[(2\alpha-\gamma)x+A\Big]^{\frac{(\alpha-\gamma)}{(2\alpha-\gamma)}}. \qquad (13)$$

The second equation of (7) then gives

$$U\frac{dU}{dx} = \frac{\beta hV}{lg^3} = \frac{\beta k^4l^2}{V^2}\Big[(2\alpha-\gamma)x+A\Big]^{\frac{(3\gamma-2\alpha)}{(2\alpha-\gamma)}}.$$

If $\gamma\neq 0$, we get on integration

$$U^2 = \frac{\beta k^4l^2}{\gamma V^2}\Big[(2\alpha-\gamma)x+A\Big]^{\frac{2\gamma}{(2\alpha-\gamma)}}+B, \qquad (14)$$

where B is a constant.

The equations (12) and (13) determine the scale factors $h(x)$ and $g(x)$ for $u\ (x, y)$ and y and the equation (14) determines the flow outside the boundary layer.

Particular cases

Case I

When $B = 0, \beta \neq 0, \gamma \neq 0$ and $2\alpha - \gamma \neq 0$, we have a 'similarity' solution for which

$$U = \frac{k^2 l}{V}\sqrt{\frac{\beta}{\gamma}}\left[(2\alpha-\gamma)x+A\right]^{\frac{\gamma}{2\alpha-\gamma}},$$

$$g = \frac{V}{lk}\left[(2\alpha-\gamma)x+A\right]^{\frac{(\alpha-\gamma)}{(2\alpha-\gamma)}},$$

$$h = k\left[(2\alpha-\gamma)x+A\right]^{\frac{\alpha}{(2\alpha-\gamma)}},$$

so that

$$h = \sqrt{\frac{\beta}{\gamma}}\,(gU).$$

If there is a stagnation point on the boundary, we measure x along the boundary from this point. Then $U = 0$, when $x = 0$. This condition requires that

$$A = 0 \text{ and } \frac{\gamma}{2\alpha-\gamma} > 0.$$

Hence we have for this 'similarity' solution

$$\left.\begin{aligned} U &= \frac{k^2 l}{V}\sqrt{\frac{\beta}{\gamma}}\left[(2\alpha-\gamma)x\right]^{\frac{\gamma}{(2\alpha-\gamma)}}, \\ g &= \frac{V}{lk}\left[(2\alpha-\gamma)x\right]^{\frac{(\alpha-\gamma)}{(2\alpha-\gamma)}}, \\ h &= \sqrt{\frac{\beta}{\gamma}}\,(gU). \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (15)$$

In this case we can take, without loss of generality, $\beta = \gamma$. This particular case of the general 'similarity' solution obtained here has been discussed by Mangler.

There cannot be a stagnation point on the boundary, when

$$\frac{\gamma}{2\alpha-\gamma} < 0.$$

Taking $A = 0$ in this case, so that U, g and h are given by (15), we see that U becomes infinite when $x = 0$. Therefore the point on the boundary from which x is measured corresponds to a singularity of the irrotational flow outside the boundary layer.

If $\alpha \neq 0$, we can take $\alpha = 1$, without any loss of generality. Putting

$$m = \frac{\gamma}{2-\gamma},$$

so that

$$\gamma = \frac{2m}{m+1},$$

we find that in both the above cases (when $x = 0$ is a stagnation point or a singularity of the flow outside the boundary layer) U is proportional to x^m, where m is any number, positive or negative, except 0 and -1. Here also we can take $\beta = \gamma$, without any loss of generality. This case has been considered by Falkner and Skan (1930) and the corresponding equation for the dimensionless stream function f has been considered by Hartree (1937). If, on the other hand, $\alpha = 0$, U is proportional to $\frac{1}{x}$, which means that there is a source or a sink at $x = 0$. g is then proportional to x and h is a constant. This 'similarity' solution corresponds to the two-dimensional flow in the boundary layer along a wall of a converging or diverging channel with straight boundaries. An exact solution of the equation satisfied by the dimensionless stream function has been given by Pohlhausen (1921).

Case II

When $2\alpha - \gamma = 0$, $\alpha \neq 0$, the equation (11) becomes

$$\frac{1}{h}\frac{dh}{dx} = \frac{\alpha}{A},$$

which gives, on integration

$$h = C \exp\left(\frac{\alpha}{A}x\right),$$

where C is a constant. The equation (10) then gives

$$g = \frac{VA}{lC}\exp\left(-\frac{\alpha}{A}x\right). \qquad \text{(16)}$$

Substituting in (7) and integrating, we get

$$U^2 = \frac{\beta}{2\alpha}\frac{l^2C^4}{V^2A^2}\exp\left(\frac{4\alpha}{A}x\right) + D. \qquad \text{(17)}$$

This case has been considered by Goldstein (1939). Assuming $D = 0$ and making use of the conditions to be satisfied on the boundaries of the boundary layer, Goldstein has shown that there cannot be any 'similarity' solution of the boundary layer equations in this case, unless $\frac{\alpha}{A}$ is positive.

Case III

When $B \neq 0$, $2\alpha - \gamma \neq 0$, $\beta = 0$ we get from (12), (13) and (14)

$$U = \text{constant},$$

$$g = \frac{V}{lk}\Big[(2\alpha-\gamma)x + A\Big]^{\frac{(\alpha-\gamma)}{(2\alpha-\gamma)}},$$

$$h = k\Big[(2\alpha-\gamma)x + A\Big]^{\frac{\alpha}{(2\alpha-\gamma)}}.$$

If, in addition, $\gamma = 0$ and $A = 0$ we have

$$\left.\begin{aligned} U &= \text{constant}, \\ g &= \frac{V}{lk}(2\alpha x)^{\frac{1}{2}}, \\ h &= k(2\alpha x)^{\frac{1}{2}}. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

This case corresponds to the 'similarity' solution in the boundary layer along a flat plate placed in a uniform stream parallel to the plane of the plate and perpendicular to its edge. This case has been discussed by Blasius (1908).

Case IV

When $B = 0$, $\beta = 0$, $\gamma \neq 0$ and $2\alpha - \gamma \neq 0$, we have a 'similarity' solution for which

$$U = 0,$$

$$g = \frac{V}{lk}\Big[(2\alpha-\gamma)x + A\Big]^{\frac{(\alpha-\gamma)}{(2\alpha-\gamma)}},$$

$$h = k\left[(2\alpha-\gamma)x + A\right]^{\frac{\alpha}{(2\alpha-\gamma)}}.$$

This solution cannot be obtained from the 'similarity' solution discussed by Goldstein and Mangler. The reason for this is that both Goldstein and Mangler have taken $U(x)$ as the scale factor for the longitudinal component $u(x, y)$ of the velocity in the boundary layer. The success achieved here is due to the fact that a scale factor $h(x)$ different from $U(x)$ has been assumed for $u\ (x, y)$ in this paper. It is found that this scale factor can be so determined as to yield a 'similarity' solution of the boundary layer equations.

Taking $A = 0$, $\alpha = 1$, $\gamma = -1$ we have

$$\left.\begin{aligned} U &= 0, \\ g &= \frac{V}{lk}(3x)^{\frac{2}{3}}, \\ h &= k(3x)^{\frac{1}{3}}. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

This case corresponds to the flow in a two-dimensional jet issuing from a thin slit in an incompressible fluid at rest. The corresponding equation satisfied by the dimensionless stream function has been considered by Schlichting (1933) and Bickley (1937).

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