THE EFFECT OF EXCHANGE POTENTIAL ON THE MASS-RADIUS RELATION FOR COLD BODIES

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(Communicated by F. C. Auluck, F.N.I.)

(*Received November 1, 1954*; *read January 2, 1955*)

1. In recent years, the internal constitution of the planets and white dwarf stars has received considerable attention. An excellent review has been made by Wildt (1947). The work has been further extended by Ramsey (1948, 1950 and 1951), Bullen (1949*a*, 1949*b*, and 1950), Brown (1950), Mestel (1952*a* and 1952*b*) and others.

We, in this paper, propose to investigate the mass-radius relationship of spherical *cold* * bodies, taking into account the electron-exchange potential.

Kothari (1938), on the basis of the theory of pressure-ionization, calculated the mass-radius relationship of cold bodies. He assumed that the material is composed of atoms of one kind only. Regarding the chemical composition of the material, the following alternative assumptions were made:

(i) The material is hydrogen: referred to as assumption H.
(ii) The material is iron: referred to as assumption F.

It is satisfactory to note that the known mass-radius values of the planets are in the region bounded by the curves drawn on the assumptions H and F. The theory led to a maximum radius for cold bodies and it also seemed to indicate that the two major planets, Saturn and Jupiter, are predominantly hydrogen planets. However, Kothari's theory does not take into account the exchange interaction between the free electrons and it also neglects the internal temperatures of the cold bodies.

2. Let us consider a completely degenerate (i.e. temperature is zero) electron gas with nV electrons in volume V, half of them of each spin. It is assumed that the volume is filled with a uniform distribution of positive charge to make it electrically neutral. The maximum momentum of the Fermi-distribution is given by:

$$P_0 = h\left(\frac{3n}{8\pi}\right)^{\frac{1}{3}}$$

where h is Planck's constant.

The exchange potential can be conveniently stated in terms of the ratio $\eta = P/P_0$, P being the magnitude of the momentum of the electron. Dirac (1930) found that:

$$\text{the exchange potential energy} = -e^2(4P_0/h)\,F(\eta) \quad \ldots \quad (1)$$

where

$$F(\eta) = \tfrac{1}{2} + \frac{1-\eta^2}{4\eta}\, ln\left[(1+\eta)/(1-\eta)\right]. \quad \ldots \quad (1')$$

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* Matter will be considered *cold* or *degenerate* whenever the free electrons present form a degenerate gas.

Slater (1951) suggested that we may use this result to produce a good approximation to the exchange term in the Hartree-Fock equation by averaging over the momenta P of the various electrons. This gives,

$$\text{the exchange potential energy} = -3e^2\left(\frac{3n}{8\pi}\right)^{\frac{1}{3}}. \qquad (2)$$

For a degenerate electron gas at temperature θ, Zirin (1953) found that the exchange potential energy is given by:

$$\text{the exchange potential} = -3e^2\beta\left(\frac{3n}{8\pi}\right)^{\frac{1}{3}}, \qquad (3)$$

where β is a function of $(k\theta/\epsilon_0)$, k being the Boltzmann constant and ϵ_0 the energy of the top of the Fermi band.

We will use the expressions given by equations (2) and (3) later on.

3. The Virial Theorem states that for an assembly of particles interacting according to the inverse square law of force,

$$2T+W = 3pV, \qquad (4)$$

where T is the total kinetic energy of all the particles, W the total potential energy, V the volume and p the external pressure to which the assembly is subject.

Let us consider materials composed of atomic weight A and atomic number Z, compressed to such an extent that pressure-ionization takes place. Dividing the material into similar spherical cells, each containing a nucleus of charge Ze at its centre and Z electrons uniformly distributed throughout the cell, we have the radius of the cell a in terms of the density ρ,

$$a = \left(\frac{\gamma_1 A m_H}{\rho}\right)^{\frac{1}{3}}, \qquad (5)$$

where m_H is the mass of an hydrogen atom, and γ_1 is a factor of the order of unity. Its exact value depends on many factors—particularly on lattice structure.

Now, we will derive, first the mass-radius relationship of a spherical aggregate of cold matter, taking into account exchange energy at zero temperature and then applying second approximation, we will derive the above relationship for higher temperatures.

4. *Mass-Radius relation at zero-temperature.*—Let us estimate the total kinetic energy T. The kinetic energy of a degenerate gas, for zero-temperature, is given by the usual relation:

$$E_0 = \frac{3}{10}\frac{h^2}{m}\left(\frac{3n^*}{8\pi}\right)^{\frac{2}{3}} N, \qquad (6)$$

where n^* is the total electron concentration, m the mass of an electron, and N the total number of electrons. Substituting $Z/\gamma_2 a^3$ for n^*, Z for N and eliminating a with the help of equation (5), the kinetic energy per cell is given by:

$$T' = \frac{3}{10}\frac{h^2}{m}\left(\frac{3}{8\pi}\frac{Z\rho}{\gamma_1\gamma_2 A m_H}\right)^{\frac{2}{3}} Z. \qquad (7)$$

Here γ_2 is a factor of the order of unity. Multiplying T' by $(\rho V/Am_H)$, the total number of cells, the total kinetic energy is given by:

$$T = \frac{3}{10}\frac{h^2}{m}\left(\frac{3}{8\pi}\frac{Z\rho}{\gamma_1\gamma_2 A m}\right)^{\frac{2}{3}} Z\frac{\rho V}{A m_H}. \qquad (8)$$

The total potential energy W is the sum of two parts—W_P being the electrostatic potential energy and W_E the potential energy due to the exchange of electrons. We will assume that the total potential energy is obtained by multiplying the energy per cell by the total number of cells. It is not easy to calculate W_P accurately. Approximately, we get:

$$-W'_P = \int_0^a \frac{\left(Ze - \frac{4\pi}{3} x^3 n^* e\right)}{x} 4\pi x^2 n^* e \, dx$$

$$= \frac{9}{10} \frac{Z^2 e^2}{a}, \quad \ldots \quad (9)$$

and hence, eliminating a with the help of equation (5) and multiplying by $(\rho V/Am_H)$, we get,

$$-W_P = \frac{9}{10} Z^2 e^2 \left(\frac{\rho}{\gamma_1 A m_H}\right)^{\frac{1}{3}} \frac{\rho V}{A m_H}. \quad \ldots \quad (10)$$

The expression for the electron exchange potential energy at zero temperature, as calculated by Slater, is given by equation (2):

$$\text{exchange potential energy} = -3e^2 \left(\frac{3n}{8\pi}\right)^{\frac{1}{3}} \quad \ldots \quad (2)$$

For the sake of simplicity, we will take $n^* = Z / \frac{4\pi}{3} a^3$, the concentration of total electrons in place of n, the concentration of free electrons, in the above expression. Substituting this in the above expression and multiplying it with $(\rho V/Am_H)$, the total number of cells, the total exchange energy is given by

$$-W_E = 3e^2 \left(\frac{9}{32\pi^2} \frac{Z\rho}{\gamma_1 A m_H}\right)^{\frac{1}{3}} \frac{\rho V}{A m_H}. \quad \ldots \quad (11)$$

Comparing W_P and W_E we find that,

$$\frac{W_E}{W_P} = \frac{5}{(12\pi^2)^{\frac{1}{3}}} \frac{1}{Z^{\frac{5}{3}}}. \quad \ldots \quad (12)$$

Thus, we see that in the case of hydrogen the ratio W_E/W_P will be maximum, i.e. it will be nearly equal to unity and will be smaller the higher the atomic number of the element is. In the case of iron, it will be equal to about 1/228. Hence, exchange potential energy cannot be neglected for hydrogen and other low atomic number elements.

The pressure in degenerate matter effectively depends on the free-electron concentration and is given, at zero temperature, by the expression:

$$p = \frac{8\pi}{15} \frac{h^2}{m} \left(\frac{3n}{8\pi}\right)^{\frac{5}{3}} \quad \ldots \quad (13)$$

where n is the number of free-electrons per unit volume.

Let us define μ, the mean molecular weight per free electron, by the relation:

$$n = \frac{\rho}{\mu m_H}. \quad \ldots \quad (14)$$

If the material is r times ionized,

$$n = \frac{r\rho}{Am_H}, \text{ and } \mu = \frac{A}{r}. \quad \ldots \quad (15)$$

Thus, the value of μ gives a measure of the degree of ionization. If the matter is fully ionized, $\mu = \mu_0 = A/Z$. Eliminating n between equations (13) and (14) we have:

$$p = K\frac{\rho^{\frac{5}{3}}}{\mu^{\frac{5}{3}}} \quad \ldots \quad (16)$$

where

$$K = \frac{8\pi}{15}\frac{h^2}{m}\left(\frac{3}{8\pi m_H}\right)^{\frac{5}{3}}. \quad \ldots \quad (16')$$

Substituting equations (8), (10), (11) and (16) in equation (4), we get:

$$\mu = \frac{\mu_0(\gamma_1\gamma_2)^{\frac{2}{5}}}{\left[1 - \frac{(\gamma_1\gamma_2)^{\frac{2}{3}}}{\gamma_1^{\frac{1}{3}}}\left(\frac{\Delta AZ}{\rho}\right)^{\frac{1}{3}}\eta_Z\right]^{\frac{3}{5}}}, \quad \ldots \quad (17)$$

where

$$\Delta = \left[2\,.\,3^{\frac{1}{3}}\pi^{\frac{2}{3}}me^2m_H^{\frac{1}{3}}/h^2\right]^3, \quad \ldots \quad (17')$$

$$\mu_0 = A/Z, \quad \ldots \quad (17'')$$

and

$$\eta_Z = 1 + \frac{5}{(12\pi^2Z^5)^{\frac{1}{3}}}. \quad \ldots \quad (17''')$$

As a rough approximation, put γ_1 and γ_2 equal to unity. Then equation (17) reduces to:

$$\mu = \frac{\mu_0}{\left[1 - \left(\frac{\Delta AZ}{\rho}\right)^{\frac{1}{3}}\eta_Z\right]^{\frac{3}{5}}}. \quad \ldots \quad (18)$$

If ρ^* denotes the density of the material in a singly ionized state, then

$$\rho^* = \frac{(\Delta AZ)\eta_Z{}^3}{(1-1/Z^{\frac{5}{3}})^3}. \quad \ldots \quad (19)$$

Hence, we may write the equation (18) as,

$$\mu = \frac{\mu_0}{\left[1 - \left(\frac{\rho^*}{\rho}\right)^{\frac{1}{3}}\left(1 - \frac{1}{Z^{\frac{5}{3}}}\right)\right]^{\frac{3}{5}}}. \quad \ldots \quad (20)$$

Now we will consider the case of pressure-ionization in hydrogen and in iron. For a given value of μ the density ρ can be calculated by the help of equation (18), and knowing the value of μ and ρ, the pressure p can easily be found out by equation (16). Figure 1 gives the plot of log ρ—log p in the case of hydrogen and iron. The dotted curve gives the plot of log ρ—log p in the case of hydrogen, when exchange potential is neglected, i.e. when $\eta_Z = 1$; the corresponding curve for iron does not differ appreciably from the case when exchange potential is taken into account as is evident from Table I. The ladder of μ values has been marked on the curves. The straight lines $\mu = 1$ and $\mu = 56/26$, would give the relation between log ρ—log p at complete ionization, i.e., when $\mu = A/Z$, for hydrogen and iron, respectively. The log p—log ρ values for *atomic* or *metallic* hydrogen as obtained by Ramsey (1951) are indicated by crosses (×).

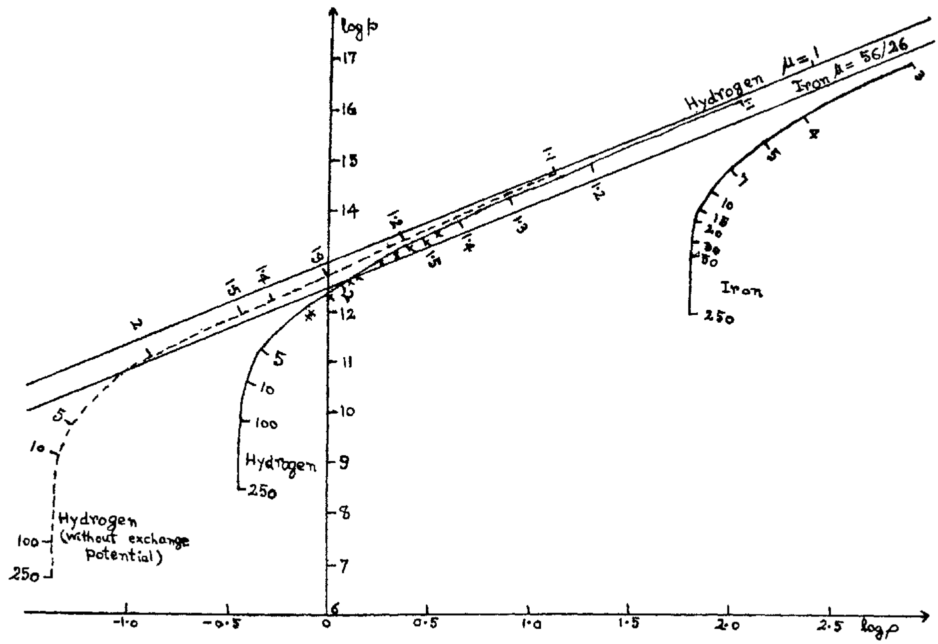

FIG. 1. The continuous curves show the relation between pressure and density of hydrogen and of iron, when exchange potential energy is taken into account. The dotted curve gives pressure-density relation of hydrogen, when exchange term is neglected. The ladder of μ values has been marked along the curves.

The straight lines in the figure would represent the pressure-density relation if the ionization did not vary with density but was complete (i.e. $\mu = A/Z$) at all densities, for the case of hydrogen and iron.

Ramsey's (1951) calculated values of pressure-density for metallic hydrogen, have been marked by crosses (×).

TABLE I

Hydrogen					Iron				
μ	Without exchange potential		With exchange potential		μ	Without exchange potential		With exchange potential	
	log ρ	log p	log ρ	log p		log ρ	log p	log ρ	log p
250	$\bar{2}$·637	6·728	$\bar{1}$·552	8·252	250	1·800	12·000	1·806	12·010
100	$\bar{2}$·637	7·392	$\bar{1}$·552	8·916	50	1·807	13·176	1·813	13·185
10	$\bar{2}$·665	9·105	$\bar{1}$·580	10·629	30	1·816	13·561	1·822	13·571
5	$\bar{2}$·729	9·713	$\bar{1}$·644	11·238	20	1·832	13·881	1·838	13·891
2	$\bar{1}$·130	11·044	0·044	12·568	15	1·852	14·123	1·858	14·132
1·5	$\bar{1}$·563	11·974	0·478	13·499	10	1·905	14·504	1·911	14·514
1·4	$\bar{1}$·739	12·151	0·654	13·842	7	1·997	14·916	2·003	14·925
1·3	$\bar{1}$·990	12·789	0·905	14·314	5	2·167	15·443	2·173	15·453
1·2	0·381	13·500	1·296	15·024	4	2·374	15·949	2·380	15·959
1·1	1·136	14·820	2·050	16·344	3	2·917	17·062	2·923	17·072

We may approximately regard,

$$\rho = \frac{3M}{4\pi R^3}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

where M is the mass and R the radius of the spherical aggregate of matter.

Then equation (18) reduces to

$$\mu = \frac{\mu_0}{\left[1 - \left(\frac{4\pi}{3\odot}\right)^{\frac{1}{3}} (\Delta AZ)^{\frac{1}{3}} \left(\frac{\odot}{M}\right)^{\frac{1}{3}} R\eta_Z\right]^{\frac{3}{5}}}, \qquad \ldots \quad \ldots \quad (22)$$

where $\odot$ is the mass of the sun.

The radius of a spherical aggregate of cold matter of mass M in equilibrium under its own gravitational forces is given by the well-known relation due to Milne (1932), (neglecting the relativistic mechanics effect, which is justified so long as $M < \odot$),

$$R = \frac{l}{\mu_0^{\frac{5}{3}}} \left(\frac{\odot}{M}\right)^{\frac{1}{3}}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

where

$$l = \frac{5\left(\omega^0_{3/2}\right)^{\frac{1}{3}} K}{2^{\frac{7}{3}} \pi^{\frac{2}{3}} G\odot} = 2{\cdot}79 \times 10^9 \text{ cm.} \qquad \ldots \quad \ldots \quad \ldots \quad (23')$$

Here $\omega^0_{3/2}$ is a constant (2·1219) characteristic of Emden's solution of Emden's equation of index 3/2 and G is the constant of Gravitation.

Eliminating μ from equations (22) and (23) we get

$$R = \frac{\frac{l}{\mu_0^{\frac{5}{3}}} \left(\frac{\odot}{M}\right)^{\frac{1}{3}}}{1 + \frac{l}{\mu_0^{\frac{5}{3}}} \left(\frac{4\pi}{3\odot}\right)^{\frac{1}{3}} (\Delta AZ)^{\frac{1}{3}} \left(\frac{\odot}{M}\right)^{\frac{2}{3}} \eta_Z}, \qquad \ldots \quad \ldots \quad (24a)$$

$$= \frac{\frac{l}{\mu_0^{\frac{5}{3}}} \left(\frac{\odot}{M}\right)^{\frac{1}{3}}}{1 + \frac{l}{\mu_0^{\frac{5}{3}}} \left(\frac{4\pi}{3\odot}\right)^{\frac{1}{3}} \rho^{*\frac{1}{3}} \left(1 - \frac{1}{Z^{\frac{5}{3}}}\right) \left(\frac{\odot}{M}\right)^{\frac{2}{3}}}. \qquad \ldots \quad \ldots \quad (24b)$$

Equation (24) shows that there is a maximum radius $R_{\max}$ at a particular mass M_0. By differentiating with respect to M, we get,

$$R_{\max} = \frac{l^{\frac{1}{2}}}{2\mu_0^{\frac{5}{6}} \eta_Z^{\frac{1}{2}}} \left(\frac{3\odot}{4\pi}\right)^{\frac{1}{6}} \frac{1}{(\Delta AZ)^{\frac{1}{6}}}, \qquad \ldots \quad \ldots \quad \ldots \quad (25a)$$

$$= \frac{l^{\frac{1}{2}}}{2\mu_0^{\frac{5}{6}} \left(1 - \frac{1}{Z^{\frac{5}{3}}}\right)^{\frac{1}{2}}} \left(\frac{3\odot}{4\pi}\right)^{\frac{1}{6}} \frac{1}{\rho^{*\frac{1}{6}}}, \qquad \ldots \quad \ldots \quad (25b)$$

and

$$\frac{M_0}{\odot} = \frac{l^{\frac{3}{2}}}{\mu_0^{\frac{5}{2}}} \left(\frac{4\pi}{3\odot}\right)^{\frac{1}{2}} (\Delta AZ)^{\frac{1}{2}} \eta_Z^{\frac{3}{2}}, \qquad \ldots \quad \ldots \quad \ldots \quad (26a)$$

$$= \frac{l^{\frac{3}{2}}}{\mu_0^{\frac{5}{2}}} \left(\frac{4\pi}{3\odot}\right)^{\frac{1}{2}} \rho^{*\frac{1}{2}} \left(1 - \frac{1}{Z^{\frac{5}{3}}}\right)^{\frac{3}{2}}. \qquad \ldots \quad \ldots \quad \ldots \quad (26b)$$

We now proceed to numerical work. We have to make some assumptions regarding the chemical composition of the cold bodies. We make the following alternative assumptions:

(i) The cold bodies are composed of hydrogen: referred to as assumption H.
(ii) The cold bodies are composed of iron: referred to as assumption F.

In the assumption F, we can either take ρ^* as given by equation (19) or can identify it with the density of the ordinary metal (7·86 g./cm.[3]); we will refer to them as assumptions F_a and F_b respectively. Figure 2 gives the calculated values of the radii at different masses. If we neglect the exchange potential (as done by Kothari, 1938) we get the dotted curve $KH-H$ in the case of hydrogen, but there is no appreciable difference in the case of assumption F_a, whereas in the case of F_b, the results are identical as is evident from Table II. This is what we had concluded earlier as well. The observed mass-radius values of the planets and white dwarf stars are indicated by crosses (×).

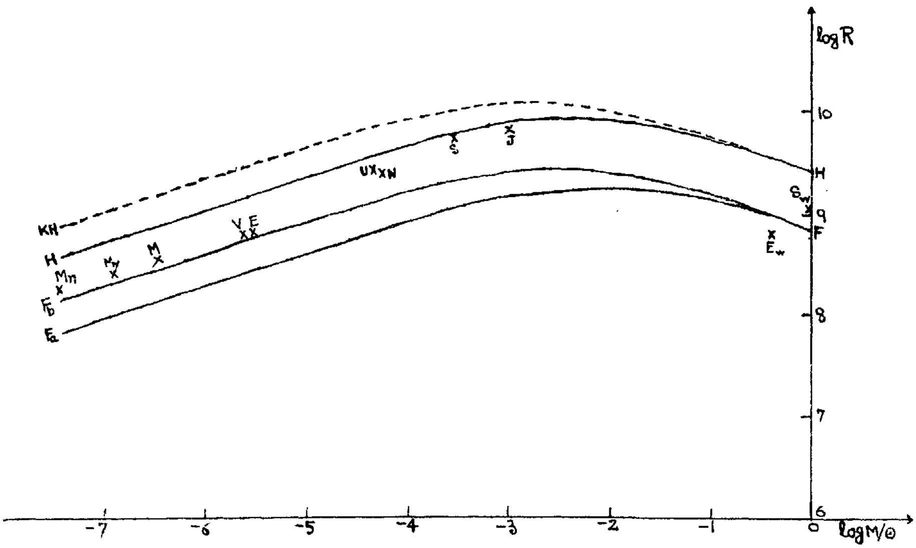

FIG. 2. The plot shows the theoretical relation between mass and radius. The continuous curves $H-H$, F_a-F, and F_b-F are for hydrogen, iron-assumption F_a, and iron-assumption F_b respectively, when exchange potential energy is taken into account. The dotted curve $KH-H$ is for hydrogen, when exchange potential energy is neglected.

M_n = Moon	E = Earth	J = Jupiter
M_y = Mercury	U = Uranus	E_w = O_2 Eridani B
M = Mars	N = Neptune	S_w = Sirius B
V = Venus	S = Saturn	

It will be observed that the two major planets, Saturn and Jupiter, are predominantly hydrogen planets. Uranus and Neptune lies nearer to hydrogen curve than to iron curve. The terrestrial planets lie near to the iron curve, indicating that they are composed of heavier substances like iron.

The values of R_{max} and $M_0/\odot$ on different assumptions can be readily obtained from equation (25) and equation (26) respectively, and are given below. The values under the heading 'observed' have been estimated from the run of the observed mass-radius values. The values of R_{max} and $M_0/\odot$ when exchange potential is not taken into account, is also given for the sake of comparison.

TABLE II

	$\log\left(\frac{M}{\odot}\right)$	Calculated log R					
		Without exchange potential			With exchange potential		
		H	F_a	F_b	H	F_a	F_b
Moon	$\bar{8}$·558	8·865	7·811	8·114	8·561	7·809	8·114
Mercury	$\bar{7}$·081	9·039	7·985	8·288	8·734	7·983	8·288
Mars	$\bar{7}$·520	9·185	8·131	8·434	8·881	8·130	8·434
Venus	$\bar{6}$·387	9·469	8·419	8·721	9·168	8·147	8·721
Earth	$\bar{6}$·479	9·499	8·450	8·750	9·198	8·447	8·750
Uranus	$\bar{5}$·646	9·854	8·827	9·117	9·569	8·825	9·117
Neptune	$\bar{5}$·714	9·872	8·848	9·137	9·590	8·847	9·137
Saturn	$\bar{4}$·456	10·036	9·066	9·328	9·792	9·064	9·328
Jupiter	$\bar{4}$·980	10·091	9·190	9·415	9·894	9·189	9·415
O_2 Eridani B ..	$\bar{1}$·644	9·555	8·980	8·995	9·546	8·980	8·995
Sirius B	$\bar{1}$·982	9·446	8·879	8·888	9·440	8·879	8·888

TABLE III

	Without exchange potential		With exchange potential	
	$R_{\max}$ in cm.	$M_0/\odot$	$R_{\max}$ in cm.	$M_0/\odot$
'Observed' ..	8×10^9	$1{\cdot}6\ \times10^{-3}$	8×10^9	$1{\cdot}6\ \times10^{-3}$
Assumption H ..	$12{\cdot}5\ \times10^9$	$1{\cdot}41\times10^{-3}$	$8{\cdot}76\times10^9$	$4{\cdot}04\times10^{-3}$
Assumption F_a ..	$1{\cdot}95\times10^9$	$7{\cdot}90\times10^{-3}$	$1{\cdot}95\times10^9$	$7{\cdot}95\times10^{-3}$
Assumption F_b ..	$2{\cdot}77\times10^9$	$2{\cdot}77\times10^{-3}$	$2{\cdot}77\times10^9$	$2{\cdot}77\times10^{-3}$

5. *Mass-Radius relation at Non-Zero Temperature.*—Now we will apply the second approximation. We will denote the quantities in the case of non-zero temperature by a bar over the symbols, that differ from zero-temperature case.

Let us calculate the total kinetic energy. The kinetic energy of a degenerate electron gas to the second approximation is given by

$$E_1 = \tfrac{3}{5} N\zeta_0 \left[1+\frac{5\pi^2}{12}\left(\frac{k\theta}{\zeta_0}\right)^2\right], \quad \ldots \quad (27)$$

where

$$\zeta_0 = \frac{h^2}{2m}\left(\frac{3n^*}{8\pi}\right)^{\frac{2}{3}}. \quad \ldots \quad (27')$$

Here θ is the temperature in absolute units. Substituting $Z/\gamma_2 a^3$ for n^*, Z for N and eliminating a by the help of equation (5) and multiplying it by $(\bar{\rho}\bar{V}/Am_H)$, the total number of cells, the total kinetic energy is given by the following expression:

$$\bar{T} = \frac{3}{10}\frac{h^2}{m}\left(\frac{3}{8\pi}\frac{Z\bar{\rho}}{\gamma_1\gamma_2 A m_H}\right)^{\frac{2}{3}} Z\left\{1+\left(\frac{A\Gamma}{Z\bar{\rho}}\right)^{\frac{4}{3}}(\gamma_1\gamma_2)^{\frac{4}{3}}\,\bar{\theta}^2\right\}\frac{\bar{\rho}\bar{V}}{Am_H}, \quad .. \quad (28)$$

where

$$\Gamma = \frac{8\pi}{3}\left[\frac{5}{3}\frac{\pi^2 k^2 m^2 m_H^{\frac{4}{3}}}{h^4}\right]^{\frac{3}{4}}. \quad .. \quad .. \quad .. \quad (28')$$

The total electrostatic potential, as before, is given by:

$$-\bar{W}_P = \frac{9}{10}Z^2e^2\left(\frac{\bar{\rho}}{\gamma_1 A m_H}\right)^{\frac{1}{3}}\frac{\bar{\rho}\bar{V}}{Am_H}. \quad .. \quad .. \quad .. \quad (29)$$

Introducing Zirin's modification in the Slater's formula for exchange potential, the total exchange potential energy at non-zero temperatures is given by:

$$-\bar{W}_E = 3e^2\beta\left(\frac{9}{32\pi^2}\frac{Z\bar{\rho}}{\gamma\, Am_H}\right)^{\frac{1}{3}}\frac{\bar{\rho}\bar{V}}{Am_H}, \quad .. \quad .. \quad (30)$$

where β is a function of $k\bar{\theta}/\epsilon_0$, ϵ_0 being the energy of the top of the Fermi band.

The pressure for a Fermi-Dirac electron gas, to the second approximation, is given by:

$$\bar{p} = \frac{8\pi}{15}\frac{h^2}{m}\left(\frac{3n}{8\pi}\right)^{\frac{5}{3}}\left[1+\frac{5\pi^2}{12}\left(\frac{k\theta}{\zeta_0}\right)^2\right], \quad .. \quad .. \quad .. \quad (31)$$

where

$$\zeta_0 = \frac{h^2}{2m}\left(\frac{3\eta}{8\pi}\right)^{\frac{2}{3}}. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (31')$$

Here we are taking n instead of n^*, because in degenerate matter the pressure effectively depends on free electron concentration.

Eliminating n from equation (31) by the help of equation (14), we get,

$$\bar{p} = K\frac{\bar{\rho}^{\frac{5}{3}}}{\bar{\mu}^{\frac{5}{3}}}+L\frac{\bar{\rho}^{\frac{1}{3}}}{\bar{\mu}^{\frac{1}{3}}}\bar{\theta}^2, \quad .. \quad .. \quad .. \quad .. \quad (32)$$

where

$$K = \frac{8\pi}{15}\frac{h^2}{m}\left(\frac{3}{8\pi m_H}\right)^{\frac{5}{3}}, \quad .. \quad .. \quad .. \quad (32')$$

and

$$L = \frac{4\pi^{\frac{8}{3}}k^2 m}{3^{\frac{5}{3}}h^2 m_H^{\frac{1}{3}}}, \quad .. \quad .. \quad .. \quad .. \quad (32'')$$

Substituting equations (28), (29), (30) and (32) in equation (4), we obtain

$$\bar{\mu} = \frac{\mu_0(\gamma_1\gamma_2)^{\frac{2}{5}}\Big/\left\{1+\left(\frac{\mu_0\Gamma}{\ }\right)^{\frac{4}{3}}(\gamma_1\gamma_2)^{\frac{4}{3}}\bar{\theta}^2\right\}^{\frac{3}{5}}}{\left[1-\dfrac{\dfrac{(\gamma_1\gamma_2)^{\frac{2}{3}}}{\gamma_1^{\frac{1}{3}}}\left(\dfrac{\Delta AZ}{\bar{\rho}}\right)^{\frac{1}{3}}\bar{\eta}_Z+(\gamma_1\gamma_2)^{\frac{2}{3}}\left(\dfrac{\mu_0\Omega}{\bar{\rho}}\right)^{\frac{5}{3}}\left(\dfrac{\bar{\rho}}{\bar{\mu}}\right)^{\frac{1}{3}}\bar{\theta}^2}{1+\left(\dfrac{\mu_0\Gamma}{\bar{\rho}}\right)^{\frac{4}{3}}(\gamma_1\gamma_2)^{\frac{4}{3}}\bar{\theta}^2}\right]^{\frac{3}{5}}}, \quad .. \quad (33)$$

where

$$\Omega = \left(\frac{80\pi^{\frac{10}{3}}k^2m^2m_H^{\frac{4}{3}}}{3^{\frac{7}{3}}h^4}\right)^{\frac{3}{5}}, \quad .. \quad .. \quad .. \quad (33')$$

and

$$\bar{\eta}_Z = 1+\frac{5\beta}{(12\pi^2Z^5)^{\frac{1}{3}}}. \quad .. \quad .. \quad .. \quad .. \quad (33'')$$

As a rough approximation, we replace γ_1 and γ_2 by unity. Then we get,

$$\bar{\mu} = \frac{\mu_0 \Big/ \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}} \right)^{\frac{4}{3}} \bar{\theta}^2 \right\}^{\frac{3}{5}}}{\left[1 - \frac{\left(\frac{\Delta A Z}{\bar{\rho}} \right)^{\frac{1}{3}} \bar{\eta}_Z + \left(\frac{\mu_0 \Omega}{\bar{\rho}} \right)^{\frac{5}{3}} \left(\frac{\bar{\rho}}{\bar{\mu}} \right)^{\frac{1}{3}} \bar{\theta}^2}{1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}} \right)^{\frac{4}{3}} \bar{\theta}^2} \right]^{\frac{3}{5}}}. \quad \ldots \quad \ldots \quad (34)$$

If $\bar{\rho}^*$ denotes the density of the material in singly ionized state, then,

$$\bar{\rho}^* = \frac{(\Delta A Z) \bar{\eta}_Z^{\,3}}{\left[\left\{ 1 - \frac{1}{Z^{\frac{5}{3}} \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}^*} \right)^{\frac{4}{3}} \bar{\theta}^2 \right\}} \right\} \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}^*} \right)^{\frac{4}{3}} \bar{\theta}^2 \right\} - \left(\frac{\mu_0 \Omega}{\bar{\rho}^*} \right)^{\frac{5}{3}} \left(\frac{\bar{\rho}^*}{A} \right)^{\frac{1}{3}} \bar{\theta}^2 \right]^3}, \quad \ldots \quad (35)$$

As this equation cannot be exactly solved for $\bar{\rho}^*$, we replace $\bar{\rho}^*$ by ρ^* as given by equation (19), on the right-hand-side of equation (35), as an approximation. Then,

$$\bar{\rho}^* = \frac{(\Delta A Z) \bar{\eta}_Z^{\,3}}{\left[\left\{ 1 - \frac{1}{Z^{\frac{5}{3}} \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\rho^*} \right)^{\frac{4}{3}} \bar{\theta}^2 \right\}} \right\} \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\rho^*} \right)^{\frac{4}{3}} \bar{\theta}^2 \right\} - \left(\frac{\mu_0 \Omega}{\rho^*} \right)^{\frac{5}{3}} \left(\frac{\rho^*}{A} \right)^{\frac{1}{3}} \bar{\theta}^2 \right]}. \quad \ldots \quad (36)$$

Hence we may write equation (34) as:

$$\bar{\mu} = \frac{\mu_0 \Big/ \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}} \right)^{\frac{4}{3}} \bar{\theta} \right\}^{\frac{3}{5}}}{\left[1 - \frac{\left(\frac{\bar{\rho}^*}{\bar{\rho}} \right)^{\frac{1}{3}} \left[\left\{ 1 - \frac{1}{Z^{\frac{5}{3}} \left\{ 1 + (\mu_0 \Gamma / \bar{\rho}^*)^{\frac{4}{3}} \bar{\theta}^2 \right\}} \right\} \left\{ 1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}^*} \right)^{\frac{4}{3}} \bar{\theta}^2 \right\} + \left(\frac{\mu_0 \Omega}{\bar{\rho}} \right)^{\frac{5}{3}} \left(\frac{\bar{\rho}}{\bar{\mu}} \right)^{\frac{1}{3}} \bar{\theta}^2 \right]^{\frac{3}{5}}}{1 + \left(\frac{\mu_0 \Gamma}{\bar{\rho}} \right)^{\frac{4}{3}} \bar{\theta}^2} \right]} \quad \ldots \quad (37)$$

We may regard, approximately that,

$$\bar{\rho} = \frac{3M}{4\pi \bar{R}^3}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

The radius $\bar{R}$ of a spherical aggregate composed of cold matter is given by (Milne, 1932; see equation (23)):

$$\bar{R} = \frac{l}{\bar{\mu}^{\frac{5}{3}}} \left(\frac{\odot}{M} \right)^{\frac{1}{3}}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

Substituting the value of $\bar{\rho}$ from equation (38) and then eliminating $\bar{\mu}$ from equation (34) by the help of equation (39) we get,

$$\bar{R} = \frac{\frac{l}{\mu_0^{\frac{5}{3}}} \left(\frac{\odot}{M} \right)^{\frac{1}{3}} \left\{ 1 + \left(\frac{4\pi}{3\odot} \right)^{\frac{4}{3}} (\mu_0 \Gamma)^{\frac{4}{3}} \left(\frac{\odot}{M} \right)^{\frac{4}{3}} \bar{R}^4 \bar{\theta}^2 \right\}}{1 + \frac{l}{\mu_0^{\frac{5}{3}}} \left(\frac{4\pi}{3\odot} \right)^{\frac{1}{3}} (\Delta A Z)^{\frac{1}{3}} \left(\frac{\odot}{M} \right)^{\frac{2}{3}} \bar{\eta}_Z + l^{\frac{4}{5}} \Omega^{\frac{5}{3}} \left(\frac{4\pi}{3\odot} \right)^{\frac{4}{3}} \left(\frac{\odot}{M} \right)^{\frac{8}{5}} \bar{R}^{\frac{16}{5}} \bar{\theta}^2}, \quad \ldots \quad (40a)$$

$$\bar{R}=\frac{\frac{l}{\mu_0^{\frac{5}{3}}}\left(\frac{\odot}{M}\right)^{\frac{1}{3}}\left\{1+\left(\frac{4\pi}{3\odot}\right)^{\frac{4}{3}}(\mu_0\Gamma)^{\frac{4}{3}}\left(\frac{\odot}{M}\right)^{\frac{4}{3}}\bar{R}^4\bar{\theta}^2\right\}}{1+\frac{l}{\mu_0^{\frac{5}{3}}}\left(\frac{4\pi}{3\odot}\right)^{\frac{1}{3}}\left(\frac{\odot}{M}\right)^{\frac{2}{3}}\bar{\rho}^{*\frac{1}{3}}\left[\left\{1-\frac{1}{Z^{\frac{5}{3}}\left\{1+\left(\frac{\mu_0\Gamma}{\bar{\rho}^*}\right)^{\frac{4}{3}}\bar{\theta}^2\right\}}\right\}\left\{1+\left(\frac{\mu_0\Gamma}{\bar{\rho}^*}\right)^{\frac{4}{3}}\bar{\theta}^2\right\}-\left(\frac{\mu_0\Omega}{\bar{\rho}^*}\right)^{\frac{5}{3}}\left(\frac{\bar{\rho}^*}{A}\right)^{\frac{1}{3}}\bar{\theta}^2\right]+l^{\frac{4}{5}}\Omega^{\frac{5}{3}}\left(\frac{4\pi}{3\odot}\right)^{\frac{4}{3}}\left(\frac{\odot}{M}\right)^{\frac{8}{5}}\bar{R}^{\frac{16}{5}}\bar{\theta}^2}. \qquad (40b)$$

Since equation (40) cannot be exactly solved for $\bar{R}$, we, as an approximation, replace $\bar{R}$ by R, as given by equation (24) on the right-hand-side of the above equation. Then we get:

$$\bar{R}=\frac{\frac{l}{\mu_0^{\frac{5}{3}}}\left(\frac{\odot}{M}\right)^{\frac{1}{3}}\left\{1+\left(\frac{4\pi}{3\odot}\right)^{\frac{4}{3}}(\mu_0\Gamma)^{\frac{4}{3}}\left(\frac{\odot}{M}\right)^{\frac{4}{3}}R^4\bar{\theta}^2\right\}}{1+\frac{l}{\mu_0^{\frac{5}{3}}}\left(\frac{4\pi}{3\odot}\right)^{\frac{1}{3}}(\Delta AZ)^{\frac{1}{3}}\left(\frac{\odot}{M}\right)^{\frac{2}{3}}\bar{\eta}_Z+l^{\frac{4}{5}}\Omega^{\frac{5}{3}}\left(\frac{4\pi}{3\odot}\right)^{\frac{4}{3}}\left(\frac{\odot}{M}\right)^{\frac{8}{5}}R^{\frac{16}{5}}\bar{\theta}^2}, \quad .. \qquad (41a)$$

$$=\frac{\frac{l}{\mu_0^{\frac{5}{3}}}\left(\frac{\odot}{M}\right)^{\frac{1}{3}}\left\{1+\left(\frac{4\pi}{3\odot}\right)^{\frac{4}{3}}(\mu_0\Gamma)^{\frac{4}{3}}\left(\frac{\odot}{M}\right)^{\frac{4}{3}}R^4\bar{\theta}^2\right\}}{1+\frac{l}{\mu_0^{\frac{5}{3}}}\left(\frac{4\pi}{3\odot}\right)^{\frac{1}{3}}\left(\frac{\odot}{M}\right)^{\frac{2}{3}}\bar{\rho}^{*\frac{1}{3}}\left[\left\{1-\frac{1}{Z^{\frac{5}{3}}\left\{1+\left(\frac{\mu_0\Gamma}{\bar{\rho}^*}\right)^{\frac{4}{3}}\bar{\theta}^2\right\}}\right\}\left\{1+\left(\frac{\mu_0\Gamma}{\bar{\rho}^*}\right)^{\frac{4}{3}}\bar{\theta}^2\right\}-\left(\frac{\mu_0\Omega}{\bar{\rho}^*}\right)^{\frac{5}{3}}\left(\frac{\bar{\rho}^*}{A}\right)^{\frac{1}{3}}\bar{\theta}^2\right]+l^{\frac{4}{5}}\Omega^{\frac{5}{3}}\left(\frac{4\pi}{3\odot}\right)^{\frac{4}{3}}\left(\frac{\odot}{M}\right)^{\frac{8}{5}}R^{\frac{16}{5}}\bar{\theta}^2}. \quad .. \quad .. \qquad (41b)$$

We now proceed to numerical work. We will make similar assumptions as in the zero-temperature case, regarding the chemical composition of the cold bodies. The calculated values of radii of cold bodies at temperatures 0°A., 10^4 °A. and 10^5 °A are tabulated below:

TABLE IV

	$\log \frac{M}{\odot}$	Assumption H			Assumption F_a			Assumption F_b		
		θ = 0°A	θ = 10^4°A	θ = 10^5°A	θ = 0°A	θ = 10^4°A	θ = 10^5°A	θ = 0°A	θ = 10^4°A	θ = 10^5°A
Moon	$\bar{8}$·558	8·561	8·578	8·975	7·809	7·809	7·811	8·114	8·114	8·119
Mercury ..	$\bar{7}$·081	8·734	8·751	9·122	7·983	7·983	7·985	8·288	8·288	8·291
Mars	$\bar{7}$·520	8·881	8·897	9·244	8·130	8·130	8·131	8·434	8·434	8·436
Venus	$\bar{6}$·387	9·168	9·183	9·477	8·417	8·417	8·419	8·721	8·721	8·718
Earth	$\bar{6}$·479	9·198	9·213	9·501	8·447	8·447	8·449	8·750	8·750	8·748
Uranus ..	$\bar{5}$·646	9·569	9·581	9·777	8·825	8·825	8·826	9·117	9·117	9·108
Neptune ..	$\bar{5}$·714	9·590	9·601	9·792	8·847	8·847	8·847	9·137	9·137	9·128
Saturn	$\bar{4}$·456	9·792	9·798	9·938	9·064	9·064	9·065	9·328	9·328	9·315
Jupiter ..	$\bar{4}$·980	8·894	9·898	9·967	9·189	9·189	9·189	9·415	9·415	9·401
O_2 Eridani B ..	$\bar{1}$·644	9·546	9·546	9·546	8·980	8·980	8·980	8·995	8·995	8·994
Sirius B ..	$\bar{1}$·982	9·440	9·440	9·440	8·879	8·879	8·879	8·888	8·888	8·887

Thus we see that the radii of the cold bodies increase with increase in temperature (excluding the case of assumption F_b which we will consider presently). The increase in the radii of cold bodies is appreciable in the case of assumption H but in the case of assumption F_a the increase is very small, in the temperature-range considered. This is due to the fact that the value of β (see equation (30)) varies appreciably at the three temperatures considered in the case of assumption H, whereas it remains more or less constant (i.e. equal to unity) in the case of assumption F_a at these temperatures. Coming to the case of assumption F_b, we notice that the radii of the cold bodies, taken in order of their increasing mass, increases for smaller bodies but decreases for heavier bodies, with increase in temperature. This is due to the fact that whereas the value $\bar{p}^*$ in the case of assumption F_a assumes different values at different temperatures, it is constant in the case of assumption F_b.

Concluding, I take this opportunity to express my sincere thanks to Prof. D. S. Kothari and Dr. F. C. Auluck for suggesting the problem and for constant help and encouragement throughout the work. I am also thankful to the Government of India for the award of a scholarship.

Abstract

The mass-radius relationship for cold bodies, has been calculated, based on the theory of pressure-ionization, taking into account the exchange potential of the electron. The theory has been developed for zero-temperature as well as for non-zero temperatures.

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Issued April 28, 1955.