INTEGRALS ASSOCIATED WITH HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES

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(Communicated by N. R. Sen, F.N.I.)

(*Received May 11; after revision November 15; read October 20, 1954*)

In a recent paper (Shanti Saran, 1954) I have defined the hypergeometric functions of three variables and deduced the important properties of these functions including some integral representations. In this paper I obtain the single, double and triple integral representations of these functions. The integrals are either of Pochhammer's double loop type or of Mellin-Barne's type. The first type of integrals is useful in the integration of the system of partial differential equations satisfied by these functions and the latter type in Mellin's investigation of the hypergeometric functions.

In Pochhammer's double loop type of integrals $[a_1, a_2, \ldots a_m; b_1, b_2, \ldots . b_n]$ is the notation for the double loop which contains $a_1, a_2, \ldots . a_m$ within one loop and $b_1, b_2 \ldots . b_n$ within the other. It is understood that all other singularities of the integrand are outside the contour of integration. z^α is interpreted as $\exp(\alpha \log z)$, where $\log z$ is real when z is positive, and continuous on the contour of integration.

It will be assumed that x, y and z have such values that the infinite series occurring in the analysis converges. Exceptional values of the parameters which would make some of the gamma functions become infinite are tacitly excluded. The general validity of the results follows by analytic continuation.

The integrals deduced are either of the form

$$\int (-t)^{\mu-1} (t-1)^{\nu-1} f(u)\, g(v)\, dt$$

where u is a function of x and t, and v of y, z and t;

or
$$\int (-t)^{\mu-1} (t-1)^{\nu-1} f(u)\, dt$$

where u is a function of x, y, z and t and the contour is $[1+, 0+; 1-, 0-]$.

In the first, one can say that Euler's transformation factorizes our hypergeometric equations f and g, each of which satisfies the ordinary differential equation (or Appell's hypergeometric differential equation in the case of $g(v)$).

In § 2 single integrals of Pochhammer's type have been deduced and lastly in § 3 single, double and treble integrals of Mellin-Barne's type have been investigated for these functions.

Following the notation given in (Shanti Saran, 1954), we define the hypergeometric functions of three variables as below:

$$F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$
$$= \sum \frac{(\alpha_1, m+n+p)\,(\beta_1, m)\,(\beta_2, n+p)\, x^m y^n z^p}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n)\,(\gamma_3, p)} \quad \ldots \quad \ldots \quad (1.1)$$

$$F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1;\ \gamma_1, \gamma_2, \gamma_2;\ x, y, z)$$
$$= \sum \frac{(\alpha_1, m+n+p)\,(\beta_1, m+p)\,(\beta_2, n)\,x^m y^n z^p}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n+p)} \quad \ldots \quad (1.2)$$

$$F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3;\ \gamma_1, \gamma_2, \gamma_2;\ x, y, z)$$
$$= \sum \frac{(\alpha_1, m+n+p)\,(\beta_1, m)\,(\beta_2, n)\,(\beta_3, p)\,x^m y^n z^p}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n+p)} \quad \ldots \quad (1.3)$$

$$F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1;\ \gamma_1, \gamma_2, \gamma_3;\ x, y, z)$$
$$= \sum \frac{(\alpha_1, m)\,(\alpha_2, n+p)\,(\beta_1, m+p)\,(\beta_2, n)\,x^m y^n z^p}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n)\,(\gamma_3, p)} \quad \ldots \quad (1.4)$$

$$F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1;\ \gamma_1, \gamma_2, \gamma_2;\ x, y, z)$$
$$= \sum \frac{(\alpha_1, m)\,(\alpha_2, n+p)\,(\beta_1, m+p)\,(\beta_2, n)\,x^m y^n z^p}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n+p)} \quad \ldots \quad (1.5)$$

$$F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1;\ \gamma_1, \gamma_2, \gamma_2;\ x, y, z)$$
$$= \sum \frac{(\alpha_1, m)\,(\alpha_2, n)\,(\alpha_3, p)\,(\beta_1, m+p)\,(\beta_2, n)\,x^m y^n z^p}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n+p)} \quad \ldots \quad (1.6)$$

$$F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \beta_2;\ \gamma_1, \gamma_2, \gamma_2;\ x, y, z)$$
$$= \sum \frac{(\alpha_1, m+p)\,(\alpha_2, n)\,(\beta_1, m+n)\,(\beta_2, n)\,x^m y^n z^p}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n+p)} \quad \ldots \quad (1.7)$$

$$F_R(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1;\ \gamma_1, \gamma_2, \gamma_2;\ x, y, z)$$
$$= \sum \frac{(\alpha_1, m+p)\,(\alpha_2, n)\,(\beta_1, m+p)\,(\beta_2, n)\,x^m y^n z^p}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n+p)} \quad \ldots \quad (1.8)$$

$$F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3;\ \gamma_1, \gamma_1, \gamma_1;\ x, y, z)$$
$$= \sum \frac{(\alpha_1, m)\,(\alpha_2, n+p)\,(\beta_1, m)\,(\beta_2, n)\,(\beta_3, p)\,x^m y^n z^p}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m+n+p)} \quad \ldots \quad (1.9)$$

and

$$F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1;\ \gamma_1, \gamma_1, \gamma_1;\ x, y, z)$$
$$= \sum \frac{(\alpha_1, m)\,(\alpha_2, n+p)\,(\beta_1, m+p)\,(\beta_2, n)\,x^m y^n z^p}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m+n+p)} \quad \ldots \quad (1.10)$$

The triple summation in the above series extends over all positive integral values of m, n, p from zero to infinity.

As proved in (Shanti Saran, 1954), the domains of convergence of the above series are given by

$$F_E : r+(\sqrt{s}+\sqrt{t})^2 = 1 \qquad F_F : rs = (1-s)(s-t)$$

$$F_G : \left.\begin{array}{l} r+s = 1 \\ r+t = 1 \end{array}\right\} \qquad F_K : t = (1-r)(1-s)$$

$$F_M : \left.\begin{array}{l} r+t = 1 \\ s = 1 \end{array}\right\} \qquad F_N : s(1-r)+t(1-s) = 0$$

$$F_P : 4\,rst = (st-s-t)^2 \qquad F_R : s\,(1-\sqrt{r})^2+t\,(1-s) = 0$$

$$F_S : \frac{1}{r}+\frac{1}{s} = 1, \qquad F_T = t = r-rs+s$$

where $|x| < r, \quad |y| < s \quad \text{and} \quad |z| < t.$

§ 2. We know the following integral representations (Erdélyi, 1953) for F_2, F_3 and F_4, viz.

$$F_2\,(\rho+\rho_1-1\,;\ \beta, \beta_1\,;\ \gamma, \gamma_1\,;\ x, y) = \frac{\Gamma(\rho)\,\Gamma(\rho_1)\,\Gamma(2-\rho-\rho_1)}{(2\pi i)^2}\times$$

$$\times\int(-t)^{-\rho}\,(t-1)^{-\rho_1}\,{}_2F_1\left(\rho, \beta\,;\ \gamma\,;\ \frac{x}{t}\right)\,{}_2F_1\left(\rho_1, \beta_1\,;\ \gamma_1\,;\ \frac{y}{1-t}\right)dt \quad .. \quad (2.1)$$

where $|t| > |x|$, $|1-t| > |y|$ along the contour.*

$$F_3\,(\alpha, \alpha_1, \beta, \beta_1\,;\ \rho+\rho_1\,;\ x, y) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho_1)\,\Gamma(\rho+\rho_1)}{(2\pi i)^2}\times$$

$$\times\int(-t)^{\rho-1}\,(t-1)^{\rho_1-1}\,{}_2F_1\,(\alpha, \beta\,;\ \rho\,;\ tx)\,{}_2F_1\,(\alpha_1, \beta_1\,;\ \rho_1\,;\ (1-t)y)\,dt \quad (2.2)$$

and

$$F_4\,(\alpha, \beta\,;\ \gamma, \gamma_1\,;\ x, y) = \frac{\Gamma(\gamma)\,\Gamma(\gamma_1)\,\Gamma(2-\gamma-\gamma_1)}{(2\pi i)^2}\times$$

$$\times\int(-t)^{-\gamma}\,(t-1)^{-\gamma_1}\,{}_2F_1\left(\alpha, \beta\,;\ \gamma+\gamma_1-1\,;\ \frac{x}{t}+\frac{y}{1-t}\right)dt \quad .. \quad (2.3)$$

where $\left|\frac{x}{t}+\frac{y}{1-t}\right| < 1$ along the contour.*

From the definition of F_E, we get

$$F_E = \sum_{m=0}^{\infty}\frac{(\alpha_1, m)\,(\beta_1, m)}{(1, m)\,(\gamma_1, m)}\,F_4(\alpha_1+m, \beta_2\,;\ \gamma_2, \gamma_3\,;\ y, z)x^m.$$

Using (2.3), we get

$$F_E = \sum_{m=0}^{\infty}\frac{(\alpha_1, m)\,(\beta_1, m)x^m}{(1, m)\,(\gamma_1, m)}\,\frac{\Gamma(\gamma_2)\,\Gamma(\gamma_3)\,\Gamma(2-\gamma_2-\gamma_3)}{(2\pi i)^2}\times$$

$$\times\int(-t)^{-\gamma_2}\,(t-1)^{-\gamma_3}\,{}_2F_1\left(\alpha_1+m, \beta_2\,;\ \gamma_2+\gamma_3-1\,;\ \frac{y}{t}+\frac{z}{1-t}\right)dt$$

$$= \frac{\Gamma(\gamma_2)\,\Gamma(\gamma_3)\,\Gamma(2-\gamma_2-\gamma_3)}{(2\pi i)^2}\int(-t)^{-\gamma_2}\,(t-1)^{-\gamma_3}\times$$

$$\times\sum_{m=0}^{\infty}\frac{(\alpha_1, m)\,(\beta_1, m)x^m}{(1, m)\,(\gamma_1, m)}\times\,{}_2F_1\left(\alpha_1+m, \beta_2\,;\ \gamma_2+\gamma_3-1\,;\ \frac{y}{t}+\frac{z}{1-t}\right)dt.$$

* The contour of integration is a Pochhammer's double loop type (1+, 0+, 1−, 0−) and t^{ρ}, etc., have their principal values.

Since the series for m is absolutely convergent for $|x|+\left|\frac{y}{t}+\frac{z}{1-t}\right|<1$, the change in the order of integration and summation is permissible and we have

$$F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z)=\frac{\Gamma(\gamma_2)\Gamma(\gamma_3)\Gamma(2-\gamma_2-\gamma_3)}{(2\pi i)^2}\times$$

$$\times\int(-t)^{-\gamma_2}(t-1)^{-\gamma_3}F_2\left(\alpha_1;\ \beta_1, \beta_2;\ \gamma_1, \gamma_2+\gamma_3-1;\ x, \frac{y}{t}+\frac{z}{1-t}\right)dt \qquad (2.4)$$

where $|x|+\left|\frac{y}{t}+\frac{z}{1-t}\right|<1$ along the contour.*

Also, rewriting F_E as

$$\sum_{n=0}^{\infty}\frac{(\alpha_1, n)\,(\beta_2, n)}{(1, n)\,(\gamma_2, n)}F_2(\alpha_1+n;\ \beta_1, \beta_2+n;\ \gamma_1, \gamma_3;\ x, z)y^n,$$

and using (2.1), we get

$$F_E=\sum_{n=0}^{\infty}\frac{(\alpha_1, n)\,(\beta_2, n)y^n}{(1, n)\,(\gamma_2, n)}\times\frac{\Gamma(\rho)\,\Gamma(\rho_1)\,\Gamma(2-\rho-\rho_1)}{(2\pi i)^2}\times$$

$$\times\int(-t)^{-\rho}(t-1)^{-\rho_1}\,{}_2F_1\left(\rho, \beta_1;\ \gamma_1;\ \frac{x}{t}\right)\,{}_2F_1\left(\rho_1, \beta_2+n;\ \gamma_3;\ \frac{z}{1-t}\right)dt$$

where $|t|>|x|$, $|1-t|>|z|$ along the contour and $\alpha_1=\rho+\rho_1-1$.

As before,

$$F_E=\frac{\Gamma(\rho)\Gamma(\rho_1)\Gamma(2-\rho-\rho_1)}{(2\pi i)^2}\times$$

$$\times\int(-t)^{-\rho}(t-1)^{-\rho_1}\,{}_2F_1\left(\rho, \beta_1;\ \gamma_1;\ \frac{x}{t}\right)F_4\left(\rho_1, \beta_2;\ \gamma_2, \gamma_3;\ \frac{y}{1-t}, \frac{z}{1-t}\right)dt \qquad (2.5)$$

where $\alpha_1=\rho+\rho_1-1$ and $|t|>|x|$, and $\left|\sqrt{\frac{y}{1-t}}\right|+\left|\sqrt{\frac{z}{1-t}}\right|<1$ along the contour.

Using similar methods we can easily prove the following formulae:—

$$F_F(\alpha_1\ \alpha_1, \alpha_1,\ \beta_1, \beta_2, \beta_1;\ \gamma_1, \gamma_2, \gamma_2;\ x, y, z)=\frac{\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(2-\gamma_1-\gamma_2)}{(2\pi i)^2}\times$$

$$\times\int(-t)^{-\gamma_1}(t-1)^{-\gamma_2}F_1\left(\alpha_1;\ \beta_2, \beta_1;\ \gamma_1+\gamma_2-1;\ \frac{y}{1-t}, \frac{x}{t}+\frac{z}{1-t}\right)dt \qquad (2.6)$$

where $|1-t|>|y|$ and $\left|\frac{x}{t}+\frac{z}{1-t}\right|<1$ along the contour.

$$F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3;\ \gamma_1, \gamma_2, \gamma_2;\ x, y, z)=\frac{\Gamma(\rho)\,\Gamma(\rho_1)\,\Gamma(2-\rho-\rho_1)}{(2\pi i)^2}\times$$

$$\times\int(-t)^{-\rho}(t-1)^{-\rho_1}\,{}_2F_1\left(\rho, \beta_1;\ \gamma_1;\ \frac{x}{t}\right)F_1\left(\rho_1;\ \beta_2, \beta_3;\ \gamma_2;\ \frac{y}{1-t}, \frac{z}{1-t}\right)dt \quad (2.7)$$

where $\alpha_1=\rho+\rho_1-1$, $|t|>|x|$, $|1-t|>|y|$ and $|1-t|>|z|$ along the contour.

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* The contour of integration is a Pochhammer's double loop type $(1+, 0+, 1-, 0-)$ and t^ρ, etc., have their principal values.

$$F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_3 ; x, y, z) = \frac{\Gamma(\rho)\, \Gamma(\rho_1)\, \Gamma(2-\rho-\rho_1)}{(2\pi i)^2} \times$$

$$\times \int (-t)^{-\rho} (t-1)^{-\rho_1} {}_2F_1\left(\rho, \alpha_1 ; \gamma_1 ; \frac{x}{t}\right) F_2\left(\alpha_2 ; \beta_2, \rho_1 ; \gamma_2, \gamma_3 ; y, \frac{z}{1-t}\right) dt \quad (2.8)$$

where $|t| > |x|$ and $\left|\frac{z}{1-t}\right| < 1-|y|$ along the contour and $\beta_1 = \rho+\rho_1-1$.

$$F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_2 ; x, y, z) = \frac{\Gamma(\rho)\, \Gamma(\rho_1)\, \Gamma(2-\rho-\rho_1)}{(2\pi i)^2} \times$$

$$\times \int (-t)^{-\rho} (t-1)^{-\rho_1} {}_2F_1\left(\rho, \alpha_1 ; \gamma_1 ; \frac{x}{t}\right) F_1\left(\alpha_2 ; \beta_2, \rho_1 ; \gamma_2 ; y, \frac{z}{1-t}\right) dt \quad (2.9)$$

where $\beta_1 = \rho+\rho_1-1$ and $|t| > |x|$ and $|1-t| > |z|$ along the contour.

$$F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_2 ; x, y, z) = \frac{\Gamma(1-\rho)\, \Gamma(1-\rho_1)\, \Gamma(\rho+\rho_1)}{(2\pi i)^2} \times$$

$$\times \int (-t)^{\rho-1} (t-1)^{\rho_1-1} {}_2F_1(\alpha_2, \beta_2 ; \rho ; ty)\, F_2(\beta_1 ; \alpha_1, \alpha_3 ; \gamma_1, \rho_1 ; x, (1-t)z)\, dt \quad (2.10)$$

where $\gamma_2 = \rho+\rho_1$ and $|x| + |(1-t)z| < 1$ and $|yt| < 1$ along the contour.

Also,

$$F_N = \frac{\Gamma(\rho)\, (\Gamma\rho_1)\, \Gamma(2-\rho-\rho_1)}{(2\pi i)^2} \int (-t)^{-\rho} (t-1)^{-\rho_1} {}_2F_1\left(\rho, \alpha_1 ; \gamma_1 ; \frac{x}{t}\right) \times$$

$$\times F_3\left(\alpha_2, \alpha_3 ; \beta_2, \rho_1 ; \gamma_2 ; y, \frac{z}{1-t}\right) dt \quad (2.11)$$

where $\beta_1 = \rho+\rho_1-1$ and $|t| > |x|$ and $|1-t| > |z|$ along the contour.

$$F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \beta_2 ; \gamma_1, \gamma_2, \gamma_2 ; x, y, z) = \frac{\Gamma(1-\rho)\, \Gamma(1-\rho_1)\, \Gamma(\rho+\rho_1)}{(2\pi i)^2} \times$$

$$\times \int (-t)^{\rho-1} (t-1)^{\rho_1-1} F_K(\beta_2, \beta_1, \beta_1, \alpha_1, \alpha_2, \alpha_1 ; \rho_1, \rho, \gamma_1 ; (1-t)\, z, ty, x) dt \quad (2.12)$$

where $\gamma_2 = \rho+\rho_1$ and $u = (1-r)\,(1-s)$ with $|x| < u, |ty| < s, |z(1-t)| < r$ along the contour.

Also,

$$F_R(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_2 ; x, y, z) = \frac{\Gamma(\rho)\, \Gamma(\rho_1)\, \Gamma(2-\rho-\rho_1)}{(2\pi i^2)} \times$$

$$\times \int (-t)^{-\rho} (t-1)^{-\rho_1} F_N\left(\rho, \alpha_2, \rho_1, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_2 ; \frac{x}{t}, y, \frac{z}{1-t}\right) dt \quad (2.13)$$

where $\alpha_1 = \rho+\rho_1-1$, $s(1-r)+u(1-s) = 0$ with $\left|\frac{x}{t}\right| < r, \left|\frac{z}{1-t}\right| < u, |y| < s$ along the contour.

$$F_R = \frac{\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(2-\gamma_1-\gamma_2)}{(2\pi i)^2} \times$$

$$\times \int (-t)^{\gamma_1}\,(t-1)^{-\gamma_2}\,F_3\left(\alpha_1, \alpha_2, \beta_1, \beta_2\,;\ \gamma_1+\gamma_2-1\,;\ \frac{y}{1-t}, \frac{x}{t}+\frac{z}{1-t}\right)dt \quad \ldots \quad (2.14)$$

where $\left|\frac{y}{1-t}\right| < 1$ and $\left|\frac{x}{t}+\frac{z}{1-t}\right| < 1$ along the contour.

Also,

$$F_R = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho_1)\,\Gamma(\rho+\rho_1)}{(2\pi i)^2} \times \int (-t)^{\rho-1}\,(t-1)^{\rho_1-1}\,{}_2F_1(\alpha_2, \beta_2\,;\ \rho\,;\ ty) \times$$

$$\times F_4(\alpha_1, \beta_1\,;\ \gamma_1, \rho_1\,;\ x, (1-t)z)dt \quad \ldots \quad (2.15)$$

where $\gamma_2 = \rho+\rho_1$ and $|\sqrt{x}| + |\sqrt{(1-t)z}| < 1$ and $|yt| < 1$ along the contour.

$$F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3\,;\ \gamma_1, \gamma_1, \gamma_1\,;\ x, y, z) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho_1)\,\Gamma(\rho+\rho_1)}{(2\pi i)^2} \times$$

$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho_1-1}\,{}_2F_1(\alpha_1, \beta_1\,;\ \rho\,;\ tx)\,F_1(\alpha_2\,;\ \beta_2, \beta_3\,;\ \rho_1\,;\ y(1-t), z(1-t))dt \quad \ldots \quad (2.16)$$

where $\gamma_1 = \rho+\rho_1$ and $|tx| < 1$, $|y(1-t)| < 1$ and $|z(1-t)| < 1$ along the contour.

$$F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1\,;\ \gamma_1, \gamma_1, \gamma_1\,;\ x, y, z) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho_1)\,\Gamma(\rho+\rho_1)}{(2\pi i)^2} \times$$

$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho_1-1}\,F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1\,;\ \rho, \rho_1, \rho_1\,;\ tx, (1-t)\,y, (1-t)z)dt \quad \ldots \quad (2.17)$$

where $\gamma_1 = \rho+\rho_1, r+u=1, s=1$; and $|tx| < r$, $|(1-t)y| < s$, $|(1-t)z| < t$ along the contour.

We know that *

$$F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1\,;\ \gamma_1, \gamma_1, \gamma_1\,;\ x, y, z)$$

$$= (1-z)^{-\alpha_2}\,F_S\left(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \gamma_1-\beta_1-\beta_2\,;\ \gamma_1, \gamma_1, \gamma_1\,;\ x, \frac{y-z}{1-z}, \frac{-z}{1-z}\right).$$

Using (2.16) we get

$$F_T = (1-z)^{-\alpha_2}\,\frac{\Gamma(1-\rho)\,\Gamma(1-\rho_1)\,\Gamma(\rho+\rho_1)}{(2\pi i)^2} \times$$

$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho_1-1}\,{}_2F_1(\alpha_1, \beta_1\,;\ \rho\,;\ tx) \times$$

$$F_1\left(\alpha_2\,;\ \beta_2, \rho+\rho_1-\beta_1-\beta_2, \rho_1\,;\ \frac{y-z}{1-z}\,(1-t), \frac{-z}{1-z}\,(1-t)\right)dt$$

where $|tx| < 1$ and $\left|\frac{y-z}{1-z}\,(1-t)\right| < 1$, $\left|\frac{z}{1-z}(1-t)\right| < 1$ along the contour and $\gamma_1 = \rho+\rho_1$.

* (Shanti Saran, 1954). Use (5.15).

§ 3. Mellin-Barne's Contour Integrals

In this section I have deduced single, double and treble integral representations for these functions. I give the detailed deductions for F_E only and similar methods give the integral representations for the other nine functions also.

(1) *Single Integral representation*

From the definition of F_E, we have

$$F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2 ; \gamma_1, \gamma_2, \gamma_3, x, y, z)$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha_1\ m)\ (\beta_1, m)}{(1, m)\ (\gamma_1, m)} F_4(\alpha_1+m, \beta_2 ; \gamma_2, \gamma_3 ; y, z)x^m.$$

Using the relation (Appell, P. et Kampè, J. de Ferìet, 1926), namely

$$\frac{\Gamma(\alpha)\ \Gamma(\beta)}{\Gamma(\gamma_1)} F_4(\alpha, \beta ; \gamma, \gamma_1 ; x, y)$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} {}_2F_1(\alpha+t, \beta+t ; \gamma ; x)\frac{\Gamma(\alpha+t)\ \Gamma(\beta+t)}{\Gamma(\gamma_1+t)}\Gamma(-t)\ (-y)^t\, dt$$

we obtain

$$\frac{\Gamma(\alpha_1)\ \Gamma(\beta_2)}{\Gamma(\gamma_2)}F_E = \frac{1}{2\pi i}\sum_{m=0}^{\infty} \frac{(\beta_1, m)x^m}{(1, m)\ (\gamma_1, m)}\int_{-i\infty}^{+i\infty}\frac{\Gamma(\alpha_1+m+t)\ \Gamma(\beta_2+t)}{\Gamma(\gamma_3+t)} \times$$

$$\times\, {}_2F_1(\alpha_1+m+t, \beta_2+t ; \gamma_2 ; y)\ \Gamma(-t)\ (-z)^t\, dt.$$

Changing the order of integration and summation which is easily justifiable for $|x|+|y| < 1$, we get

$$\frac{\Gamma(\alpha_1)\ \Gamma(\beta_2)}{\Gamma(\gamma_2)}F_E = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} F_2(\alpha_1+t, \beta_1, \beta_2+t ; \gamma_1, \gamma_2 ; x, y)\times$$

$$\times\frac{\Gamma(\alpha_1+t)\ \Gamma(\beta_2+t)}{\Gamma(\gamma_2+t)}\Gamma(-t)\ (-z)^t\, dt \qquad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

Similarly, by rewriting F_E as

$$\sum_{m=0}^{\infty} \frac{(\alpha_1, m)\ (\beta_2, m)}{(1, m)\ (\gamma_2, m)} F_2(\alpha_1+m ; \beta_1, \beta_2+m ; \gamma_1, \gamma_3 ; x, z)y^m$$

and using the relation (Appell, P. et Kampè, J. de Ferìet, 1926)

$$\frac{\Gamma(\alpha)\ \Gamma(\beta_1)}{\Gamma(\gamma_1)}F_2\ (\alpha ; \beta, \beta_1 ; \gamma, \gamma_1 ; x, y)$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} {}_2F_1\ (\alpha+t, \beta ; \gamma ; x)\ \frac{\Gamma(\alpha+t)\ \Gamma(\beta_1+t)}{\Gamma(\gamma_1+t)}\Gamma(-t)\ (-y)^t\, dt,$$

we obtain

$$\frac{\Gamma(\alpha_1)\,\Gamma(\beta_1)}{\Gamma(\gamma_1)}F_E = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} F_4(\alpha_1+t,\,\beta_2\,;\,\gamma_1,\gamma_3\,;\,x,\,z)\times$$

$$\times\frac{\Gamma(\alpha_1+t)\,\Gamma(\beta_1+t)}{\Gamma(\gamma_1+t)}\,\Gamma(-t)\,(-y)^t\,dt \qquad .. \quad (3.2)$$

(2) Double Integral representation

Using the above integral for F_2 in (3.1) we get

$$\frac{\Gamma(\alpha_1)\,\Gamma(\beta_1)\,\Gamma(\beta_2)}{\Gamma(\gamma_2)\,\Gamma(\gamma_3)}F_E = \frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} {}_2F_1\,(\alpha_1+s+t,\,\beta_1+t\,;\,\gamma_1\,;\,x)\times$$

$$\times\frac{\Gamma(\alpha_1+s+t)\,\Gamma(\beta_1+s)\,\Gamma(\beta_2+t)}{\Gamma(\gamma_1+s)\,\Gamma(\gamma_3+t)}\,\Gamma(-s)\,\Gamma(-t)\,(-x)^s\,(-z)^t\,dt \qquad .. \quad (3.3)$$

A similar double integral follows by using the relation of F_4 in (3.1).

(3) Triple Integral representation

From (3.3) we get

$$\frac{\Gamma(\alpha_1)\,\Gamma(\beta_1)\,\Gamma(\beta_2)}{\Gamma(\gamma_2)\,\Gamma(\gamma_3)}F_E = \frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} {}_2F_1(\alpha_1+s+t,\,\beta_2+t\,;\,\gamma_2\,;\,y)\times$$

$$\times\frac{\Gamma(\alpha_1+s+t)\,\Gamma(\beta_1+s)\,\Gamma(\beta_2+t)}{\Gamma(\gamma_1+s)\,\Gamma(\gamma_3+t)}\,\Gamma(-s)\,\Gamma(-t)\,(-x)^s\,(-z)^t\,ds\,dt.$$

Using the relation (Appell, P. et Kampè, J. de Feriet)

$$\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma)}\,{}_2F_1(\alpha,\,\beta\,;\,\gamma\,;\,x) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma(\alpha+s)\,\Gamma(\beta+s)}{\Gamma(\gamma+s)}\,\Gamma(-s)\,(-x)^s\,ds$$

we get

$$\frac{\Gamma(\alpha_1)\,\Gamma(\beta_1)\,\Gamma(\beta_2)}{\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(\gamma_3)}F_E$$

$$= \frac{1}{(2\pi i)^3}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\frac{\Gamma(\alpha_1+r+s+t)\,\Gamma(\beta_1+r)\,\Gamma(\beta_2+s+t)}{\Gamma(\gamma_1+r)\,\Gamma(\gamma_2+s)\,\Gamma(\gamma_3+t)}\times$$

$$\times\,\Gamma(-r)\,\Gamma(-s)\,\Gamma(-t)\,(-x)^r\,(-y)^s\,(-z)^t\,dr\,ds\,dt.$$

I am thankful to Dr. S. C. Mitra and Dr. R. P. Agarwal for the interest taken during the preparation of this paper and to the Government of India for a research grant.

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Issued July 25, 1955.