ON THE INTERNAL BALLISTICS OF LEAKING GUNS

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1. Introduction

Corner's (1947 and 1950) theory of leaking guns, which is applicable to worn orthodox guns, smooth bore mortars and recoilless guns has, in addition to the usual assumptions in the theory of internal ballistics of orthodox guns, made the assumption that the setting up of the gas flow through convergent and divergent nozzle can be represented by the classical one-dimensional approach and the flow starts at a certain pressure called the nozzle-start pressure. The system of equations formulated on these assumptions can always be integrated numerically. Corner gave a simple solution for the case where the rate of burning is proportional to the pressure, having the same accuracy as Hunt-Hinds. A still simpler method consists in a reduction to an 'isothermal' model, which is closely related to Crow's method. Dr. Thiruvenkatachar and Mr. Venkatesan, on account of the values of Ψ involved being small, successfully attempted to solve the general equations for a leaking gun for tubular charge, i.e. $\theta = 0$, by a method of successive approximation based on an expansion in powers of Ψ. This method gives reasonable simple means of obtaining all the ballistic information of interest, i.e. the velocity, the maximum pressure, and the relation between shot-start and nozzle-start pressure.

In the present paper the author has extended these results (for tubular propellant) to the case of charges of any shape, i.e. for any value of θ.

2. The Fundamental Equations for a Leaking Gun

In formulating the fundamental equations for a leaking gun we make the following assumptions :—

(i) the rate of burning is proportional to the pressure,
(ii) the initial resistance is represented by a shot-start pressure,
(iii) the setting up of the nozzle-flow can be represented by the use of the equations for quasi-steady nozzle-flow beginning instantaneously at a certain pressure called the nozzle-start pressure,
(iv) no unburnt propellant is lost through the nozzle.

During burning, when the nozzle is open and shot is in motion, we shall have the following equations :—

$$p\left(K_0+Ax-\frac{C}{\delta}\right) = CNRT\left(1+\frac{kCN}{6W}\right) \quad \ldots \quad (1a)$$

$$W_1\frac{d^2x}{dt^2} = Ap \quad \ldots \quad (1b)$$

where $$W_1 = W+\tfrac{1}{2}kCN \quad \ldots \quad (1c)$$

$$D\frac{df}{dt} = -\beta p \qquad (1d)$$

$$Z = (1-f)(1+\theta f) \qquad (1e)$$

$$\frac{dN}{dt} = \frac{dZ}{dt} - \frac{\psi Sp}{C(RT)^{\frac{1}{2}}} \qquad (1f)$$

$$\frac{d(NT)}{dt} = -(\bar{\gamma}-1)\frac{Ap}{CR}\frac{dx}{dt} + T_0\frac{dZ}{dt} - \frac{\gamma\psi Sp(RT)^{\frac{1}{2}}}{CR} \qquad (1g^*)$$

These equations are different from those of corresponding conventional gun in three respects. Firstly, the altered Lagrange's correction factor occurs in (1*a*) and (1*c*). Secondly, an additional term appears on the right of the energy equation (1*g*) to account for the loss of energy by the gas-flow through the nozzle. And lastly (1*f*) is the nozzle-flow equation.

For the reduction of these equations to non-dimensional parameters the following transformations have been used :—

$$Al = K_0 - \frac{C}{\delta} \qquad (2a)$$

$$\xi = 1 + \frac{x}{l} \qquad (2b)$$

$$\tau = \left(\frac{\beta CRT_0}{ADl}\right) t \qquad (2c)$$

$$\zeta = \left(\frac{Al}{CRT_0}\right) p \qquad (2d)$$

$$\eta = \frac{d\xi}{d\tau} = \frac{AD}{C\beta RT_0}\frac{dx}{dt} \qquad (2e)$$

$$T' = \frac{T}{T_0} \qquad (2f)$$

$$M' = \frac{A^2D^2}{\beta^2 CRT_0 W} \qquad (2g)$$

$$\Psi = \frac{\psi SD}{\beta C(RT_0)^{\frac{1}{2}}} \qquad (2h)$$

Thus equations (1) reduce to

$$\zeta\xi = NT'\left(1 + \frac{kCN}{6W}\right) \qquad (3a)$$

$$\eta\frac{d\eta}{d\xi} = \left(M\Big/1 + \frac{kCN}{2W}\right)\zeta \qquad (3b)$$

$$\frac{df}{d\tau} = -\zeta \qquad (3c)$$

$$Z = (1-f)(1+\theta f) \qquad (3d)$$

* Heat loss to the barrel is taken into account by changing γ into $\bar{\gamma}$ in the first term.

$$\frac{dN}{d\tau}=\frac{dZ}{d\tau}-\Psi\zeta(T')^{-\frac{1}{2}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3e)$$

$$\frac{d(NT')}{d\tau}=-(\bar{\gamma}-1)\zeta\frac{d\xi}{d\tau}+\frac{dZ}{d\tau}-\gamma\Psi\zeta(T')^{\frac{1}{2}} \quad \ldots \quad \ldots \quad (3f)$$

Denoting the shot-start pressure and nozzle-start pressure by Z_0 and Z_N respectively, we have

$$\eta=0 \text{ for } Z=Z_0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4a)$$

$$N=Z \text{ for } Z\leqslant Z_N \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4b)$$

Combining (3*b*), (3*c*) and (2*e*) we get

$$\eta\frac{d\eta}{d\xi}=\frac{d\eta}{d\tau}=-\frac{M}{1+\frac{kCN}{2W}}\frac{df}{d\tau} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Since the factor $1+\frac{kCN}{2W}$ has a value close to unity throughout the period of burning, we may replace it by average value σ ($\sigma\sim1$).

Thus

$$\frac{d\eta}{d\tau}=\frac{M}{\sigma}\frac{df}{d\tau}.$$

Integrating the above differential equation and using the initial condition $f=f_0$ when $\eta=0$, we have

$$\eta=\frac{M}{\sigma}(f_0-f) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

From equation (1*e*) we get

$$f=\frac{(\theta-1)\pm\sqrt{(\theta+1)^2-4\theta Z}}{2\theta} \quad \ldots \quad \ldots \quad \ldots \quad (7a)$$

$$\therefore \quad f_0=\frac{(\theta-1)\pm\sqrt{(\theta+1)^2-4\theta Z_0}}{2\theta} \quad \ldots \quad \ldots \quad \ldots \quad (7b)$$

∴ Equation (6) becomes

$$\eta=\pm\frac{M}{2\theta\sigma}\left[\sqrt{(\theta+1)^2-4\theta Z_0}-\sqrt{(\theta+1)^2-4\theta Z}\right] \quad \ldots \quad \ldots \quad (8)$$

Since η is always positive we have to choose the positive sign in equation (8), whether θ is positive or negative.

Hence

$$\eta=\frac{M}{2\theta\sigma}\left[\sqrt{(\theta+1)^2-4\theta Z_0}-\sqrt{(\theta+1)^2-4\theta Z}\right] \quad \ldots \quad \ldots \quad (8a)$$

Again from (3*c*) and (3*e*) we get

$$\frac{dN}{d\tau}=\frac{dZ}{d\tau}+\Psi(T')^{-\frac{1}{2}}\frac{df}{d\tau}.$$

Substituting for $\frac{df}{d\tau}$ from equation (3*d*) we have

$$\frac{dN}{d\tau}=\frac{dZ}{d\tau}+\Psi(T')^{\frac{1}{2}}\left[-\frac{1}{\sqrt{(\theta+1)^2-4\theta Z}}\frac{dZ}{d\tau}\right] \quad \ldots \quad \ldots \quad (9)$$

Integrating the above equation,

$$N = Z - \Psi \int_{Z_N}^{Z} \left[\frac{(T')^{-\frac{1}{2}}}{\sqrt{(\theta+1)^2 - 4\theta Z}} \right] dZ \quad \ldots \quad \ldots \quad (9a)$$

Using equations (3*b*), (3*c*), (3*d*) and (3*f*) we obtain

$$\frac{d}{d\tau}(NT') = -\frac{(\bar{\gamma}-1)\sigma}{M}\eta\frac{d\eta}{d\tau} + \frac{dZ}{d\tau} - \gamma\Psi(T')^{\frac{1}{2}}\frac{1}{\sqrt{(\theta+1)^2-4\theta Z}}\frac{dZ}{d\tau} \quad \ldots \quad (10)$$

Assuming that the nozzle opens before the shot starts, the integration of equation (10) gives

$$NT' = -\frac{(\bar{\gamma}-1)\sigma}{2M}\eta^2 + Z - \gamma\Psi \int_{Z_N}^{Z} \left[\frac{(T')^{\frac{1}{2}}}{\sqrt{(\theta+1)^2-4\theta Z}} \right] dZ \quad \ldots \quad (10a)$$

Substituting for η and N from (8*a*) and (9*a*) we get

$$T'\left[Z - \Psi \int_{Z_N}^{Z} \left\{ \frac{(T')^{-\frac{1}{2}}}{\sqrt{(\theta+1)^2-4\theta Z}} \right\} dZ \right] = -\frac{(\bar{\gamma}-1)\sigma}{2M}\frac{M^2}{4\theta^2\sigma^2}\left[a - \sqrt{(\theta+1)^2-4\theta Z}\right]^2 + Z$$

$$-\gamma\Psi \int_{Z_N}^{Z} \left\{ \frac{(T')^{\frac{1}{2}}}{\sqrt{(\theta+1)^2-4\theta Z}} \right\} dZ \quad \ldots \quad (10b)$$

where $a = \sqrt{(\theta+1)^2 - 4\theta Z_0}$.

Equation (10*b*) can be written as

$$ZT' - \Psi T' \int_{Z_N}^{Z} \left\{ \frac{(T')^{-\frac{1}{2}}}{\sqrt{(\theta+1)^2-4\theta Z}} \right\} dZ + \gamma\Psi \int_{Z_N}^{Z} \left\{ \frac{(T')^{\frac{1}{2}}}{\sqrt{(\theta+1)^2-4\theta Z}} \right\} dZ = A(Z) \quad (10c)$$

where

$$A(Z) = Z - \tfrac{1}{2}\frac{M(\gamma-1)}{4\theta^2\sigma}\left[a - \sqrt{(\theta+1)^2-4\theta Z}\right]^2 \quad \ldots \quad (11)$$

If the nozzle opens after the shot starts equation (10) is valid with $\Psi = 0$, up to the instant of nozzle opening. The solution of the system of equations (3) is thus reduced to the solution of the single equation (10*c*) for T'. When T' is obtained from (10*c*) as a function of Z, we find N from (9*a*) and NT from (10*a*). To obtain ξ we eliminate ζ between (3*a*) and (3*b*). Thus

$$\eta\frac{d\eta}{d\xi} = \frac{M}{\sigma}\frac{NT'}{\xi}\left(1 + \frac{kCN}{6W}\right) = \frac{(2+\sigma)M}{3\sigma}\frac{NT'}{\xi}.$$

Integrating we get

$$\log \xi = \frac{3\sigma}{(2+\sigma)M}\int_0^{\eta}\frac{\eta\, d\eta}{NT'}.$$

Using (8*a*) and (6) we obtain

$$\log \xi = \frac{3M}{2\theta\sigma(2+\sigma)}\int_{Z_0}^{Z}\left\{\frac{a}{\sqrt{(\theta+1)^2-4\theta Z}} - 1\right\}\frac{dZ}{NT'} \quad \ldots \quad \ldots \quad (12)$$

The pressure ζ is given by (3*a*), as

$$\zeta = \frac{(2+\sigma)}{3}\frac{NT'}{\xi} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

3. The approximate solution of the equation

Introducing a new variable u defined by

$$u = \int_{Z_N}^{Z} \left\{ \frac{(T')^{\frac{1}{2}}}{\sqrt{(\theta+1)^2-4\theta Z}} \right\} dZ \quad \ldots \quad (14)$$

equation (10c) becomes

$$Z[(\theta+1)^2-4\theta Z] \left(\frac{du}{dZ}\right)^2 - A(Z) =$$
$$\Psi \left[\{(\theta+1)^2-4\theta Z\} \left(\frac{du}{dZ}\right)^2 \int_0^u \frac{1}{\{(\theta+1)^2-4\theta Z\}} \left(\frac{dZ}{du}\right)^2 du - \gamma u \right] \quad \ldots \quad (15)$$

For the solution of the above equation we substitute for u an expansion in powers of Ψ,

$$u = [u^{(0)} + \Psi u^{(1)} + \Psi u^{(2)} + \ldots.] \quad \ldots \quad (16)$$

and determine the coefficients of the successive terms in this expansion by comparison of coefficients of powers of Ψ on both sides of equation (15). Since Ψ is small in most of the practical cases, we neglect the terms involving Ψ^2 and higher powers of Ψ.

Thus

$$Z[(\theta+1)^2-4\theta Z] \left(\frac{du^{(0)}}{dZ}\right)^2 = A(Z) \quad \ldots \quad (17)$$

so that

$$u^{(0)} = \int_{Z_N}^{Z} \left[\frac{A(Z)}{Z\{(\theta+1)^2-4\theta Z\}} \right]^{\frac{1}{2}} dZ \quad \ldots \quad (18)$$

$$Z\{(\theta+1)^2-4\theta Z\} \frac{du^{(1)}}{dZ} = \tfrac{1}{2} \left[\left[\frac{\{(\theta+1)^2-4\theta Z\} A(Z)}{Z} \right]^{\frac{1}{2}} \int_{Z_N}^{Z} \left[\frac{Z}{A(Z)\{(\theta+1)^2-4\theta Z\}} \right]^{\frac{1}{2}} dZ \right.$$
$$\left. - \gamma \left[\frac{Z\{(\theta+1)^2-4\theta Z\}}{A(Z)} \right]^{\frac{1}{2}} \int_{Z_N}^{Z} \left[\frac{A(Z)}{Z\{(\theta+1)^2-4\theta Z\}} \right]^{\frac{1}{2}} dZ \right] \quad \ldots \quad (19)$$

To the same order we have from (9a) and (10a)

$$N = Z - \Psi \int_{Z_N}^{Z} \left\{ \frac{(T')^{\frac{1}{2}}}{\sqrt{(\theta+1)^2-4\theta Z}} \right\} dZ$$
$$= Z - \Psi \int_{Z_N}^{Z} \left[\frac{Z}{A(Z)\{(\theta+1)^2-4\theta Z\}} \right]^{\frac{1}{2}} dZ \quad \ldots \quad (20)$$

$$NT' = A(Z) - \gamma\Psi \int_{Z_N}^{Z} \left\{ \frac{(T')^{\frac{1}{2}}}{\sqrt{(\theta+1)^2 - 4\theta Z}} \right\} dZ$$

$$= A(Z) - \gamma\Psi u^{(0)}$$

$$= A(Z) - \gamma\Psi \int_{Z_N}^{Z} \left[\frac{A(Z)}{Z\{(\theta+1)^2 - 4\theta Z\}} \right]^{\frac{1}{2}} dZ \quad \ldots \quad \ldots \quad (21)$$

Equations (20) and (21) give N and NT' as explicit functions of Z, from which the variation of T' with Z can be derived. Substituting the value of NT' from equation (21) in equation (12) we get ξ as a function of Z and thence ζ is given by (13). Thus all the ballistic parameters can be determined by evaluating a few integrals. These integrals can be easily evaluated by the use of any of the standard formulae for approximate quadrature, e.g. Simpson's Rule. Assuming Z_0 and Z_N to be small (which is the case), the integrands of these integrals can be expanded in powers of Z_0 and Z_N and we need retain only the terms up to the first order in Z_0 and Z_N, then the resulting integrals can be evaluated. Since this latter procedure is very laborious, it is convenient to evaluate the integrals by numerical quadrature.

4. Maximum Pressure

The maximum pressure occurs when $\frac{d\zeta}{dZ} = 0$. Differentiating (13) with respect to Z we get

$$\frac{d\zeta}{dZ} = \frac{(2+\sigma)}{3} \left[\frac{1}{\xi} \frac{d}{dZ}(NT') - NT' \frac{1}{\xi^2} \frac{d\xi}{dZ} \right] = 0$$

or

$$\frac{d}{dZ}(NT') = NT' \frac{d}{dZ}(\log \xi) = NT' \frac{d}{dZ}(\log \xi).$$

Substituting for $\log \xi$ and NT' from equations (12) and (21) and denoting the value of Z for maximum pressure by Z_1 we get the following equation for the determination of Z_1

$$A'(Z_1) - \gamma\Psi \left[\frac{A(Z_1)}{Z_1\{(\theta+1)^2 - 4\theta Z_1\}} \right]^{\frac{1}{2}} = \frac{3M}{2\theta\sigma(2+\sigma)} \left[\frac{a}{\sqrt{(\theta+1)^2 - 4\theta Z_1}} - 1 \right] \quad (22)$$

The solution of (22) may be obtained by the method of successive approximations. Neglecting the term in Ψ and solving the equation we determine Z_1 to the first approximation. If we substitute the value of Z_1 in the coefficient of Ψ in (22) and again solve for Z_1, we obtain Z_1 to the second approximation. In this way Z_1 can be calculated to any desired accuracy.

The peak pressure ζ_1 is then calculated from (21), (12) and (13),

$$\zeta_1 = \frac{(2+\sigma)}{3} \frac{(NT')_1}{\xi_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

5. Nozzle-start Pressure and Shot-start Pressure

Now we consider the relation between maximum pressure and nozzle-start pressure for a given value of shot-start pressure. If the shot starts after the nozzle opening, $(Z_N < Z_0)$, it is clear that the instant of the shot-start will depend on that of the nozzle-start for a given shot-start pressure. From (13) we obtain for $Z = Z_0$,

$$x = 0, \quad \xi = 1, \quad \zeta_0 = \frac{(2+\sigma)}{3}(NT')_0 ,$$

and hence from (21) and (11) we get

$$\zeta_0 = \frac{(2+\sigma)}{3}\left[Z_0 + \gamma\Psi\{\sqrt{(\theta+1)^2-4\theta Z_0} - \sqrt{(\theta+1)^2-4\theta Z_N}\}\right] \quad .. \quad (24)$$

The variation of Z_N with Z_0 for a particular value of ζ_0 is expressed by the above equation. It may be noted that the relation is not linear, except when $\theta = 0$, i.e. for tubular propellants.

Abstract

In this paper the author has extended the results obtained by Thiruvenkatachar and Venkatesan (1953) for the tubular propellants to the general case of a propellant of any shape, i.e. for any value of θ.

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