INTERNAL BALLISTICS OF COMPOSITE CHARGES WITH NON-LINEAR LAW OF BURNING

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(Communicated by R. S. Varma, F.N.I.)

(*Received June 26, 1954; read March 4, 1955*)

1. Introduction

The problem of using composite charges in a gun has been discussed in the 'Theory of Internal Ballistics of Guns' by Corner (1950) and in the 'Internal Ballistics' (1951) by reducing the problem to one of an equivalent single charge. In 'Internal Ballistics' (1951) only a particular case of two charges with different shapes and web sizes of the same propellant is considered and an approximate solution is given by making use of a modified form factor. Corner (1950) considers the more general problem of two charges of different shapes, sizes and compositions. The problem is reduced to that of a single equivalent charge with adjusted parameters.

Venkatesan and myself (1953) have, however, given a direct treatment of the general problem based on the Hunt-Hinds system under the assumptions that the co-volume correction is negligible for each propellant and that γ is the same for both the charges.

But in all these works, the linear law of burning has been assumed. In the present paper, I have discussed the problem of two composite charges (of different shapes, sizes and compositions) under the non-linear law of burning and for any values of the form-coefficients θ_1 and θ_2. The treatment comes out similar to the Clemmow's system (1928; 1951) for a single charge.

2. Assumptions

The following assumptions have been made in the discussion of this paper :—

(i) That the co-volume of the gases equals the specific volume for each propellant, i.e., the co-volume correction is negligible.

(ii) That $\gamma_1 = \gamma_2 = \gamma$. This assumption is justified by the fact that γ is practically the same for most of the propellants.

(iii) That the pressure-index law is the same for both the propellants, i.e., $\alpha_1 = \alpha_2 = \alpha$.

Under the above assumptions, the theory is also applicable to a very large number of cases in which composite charges of different shapes and sizes and of the same propellant are used.

3. Basic Equations

The classical basic equations of Internal Ballistics with a single charge have been derived in 'Internal Ballistics' (1951) and in our case they become,

$$\frac{F_1 C_1 z_1 + F_2 C_2 z_2}{Al} = p\left(1+\frac{x}{l}\right) + \frac{\gamma-1}{2Al} w_1 v^2 \qquad \ldots \quad \ldots \qquad (1)$$

where $$w_1 = 1{\cdot}06w + \frac{C_1}{3} + \frac{C_2}{3} \quad \ldots \quad \ldots \quad \ldots \quad (1A)$$

and $$Al = K_0 - \left(\frac{C_1}{\delta_1} + \frac{C_2}{\delta_2}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1B)$$

$$w_1 \frac{dv}{dt} = Ap \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$z_1 = (1-f_1)(1+\theta_1 f_1) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3A)$$

$$z_2 = (1-f_2)(1+\theta_2 f_2) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3B)$$

$$D_1 \frac{df_1}{dt} = -\beta_1 p^{\alpha} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4A)$$

$$D_2 \frac{df_2}{dt} = -\beta_2 p^{\alpha} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4B)$$

where C_1, F_1, β_1, D_1, θ_1, f_1, z_1 refer to the first charge, and C_2, F_2, β_2, D_2, θ_2, f_2, z_2 to the second charge, the letters having the usual meanings.

4. Solution of the Basic Equations

In order to make the quantities dimensionless, we make the following substitutions :—

$$\left(1+\frac{x}{l}\right) = \xi \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

$$\frac{AD_1}{F_1C_1\beta_1}\left(\frac{F_1C_1}{Al}\right)^{1-\alpha} v = \eta_1 \quad \ldots \quad \ldots \quad \ldots \quad (6A)$$

$$\frac{AD_2}{F_2C_2\beta_2}\left(\frac{F_2C_2}{Al}\right)^{1-\alpha} v = \eta_2 \quad \ldots \quad \ldots \quad \ldots \quad (6B)$$

$$\frac{Al}{F_1C_1} \cdot p = \zeta_1 \quad \ldots \quad \ldots \quad \ldots \quad (6C)$$

$$\frac{Al}{F_2C_2} \cdot p = \zeta_2 \quad \ldots \quad \ldots \quad \ldots \quad (6D)$$

$$\frac{A^2D_1^2}{F_1C_1\beta_1^2 w_1} \cdot \left(\frac{F_1C_1}{Al}\right)^{2-2\alpha} = M_1 \quad \ldots \quad \ldots \quad \ldots \quad (6E)$$

$$\frac{A^2D_2^2}{F_2C_2\beta_2^2 w_1} \cdot \left(\frac{F_2C_2}{Al}\right)^{2-2\alpha} = M_2 \quad \ldots \quad \ldots \quad \ldots \quad (6F)$$

Then the equations (1), (2), (3) and (4) become

$$\left(z_2 + \frac{F_1C_1}{F_2C_2} z_1\right) = \zeta_2 \xi + \tfrac{1}{2}(\gamma - 1)\frac{\eta_2^2}{M_2} \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

$$\eta_1 \frac{d\eta_1}{d\xi} = M_1 \zeta_1 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8A)$$

$$\eta_2 \frac{d\eta_2}{d\xi} = M_2\zeta_2 \qquad \text{(8B)}$$

$$z_1 = (1-f_1)(1+\theta_1 f_1) \qquad \text{(9A)}$$

$$z_2 = (1-f_2)(1+\theta_2 f_2) \qquad \text{(9B)}$$

$$\eta_1 \frac{df_1}{d\xi} = -\zeta_1^{\alpha} \qquad \text{(10A)}$$

$$\eta_2 \frac{df_2}{d\xi} = -\zeta_2^{\alpha} \qquad \text{(10B)}$$

Differentiating (7) and using (8B), we get

$$d\left(z_2 + \frac{F_1C_1}{F_2C_2} z_1\right) = \xi d\zeta_2 + \gamma\zeta_2 d\xi$$
$$= \xi^{1-\gamma} d(\zeta_2\xi^{\gamma}) \qquad \text{(11)}$$

Equations (8A) and (10A) give

$$\frac{d\eta_1}{df_1} = -M_1\zeta_1^{1-\alpha} \qquad \text{(12A)}$$

and equations (8B) and (10B) give

$$\frac{d\eta_2}{df_2} = -M_2\zeta_2^{1-\alpha} \qquad \text{(12B)}$$

∴ from (12A) and (10A), we get

$$\frac{d}{df_1}\left(\zeta_1^{\alpha} \frac{d\xi}{df_1}\right) = M_1\zeta_1^{1-\alpha} \qquad \text{(13A)}$$

Similarly

$$\frac{d}{df_2}\left(\zeta_2^{\alpha} \frac{d\xi}{df_2}\right) = M_2\zeta_2^{1-\alpha} \qquad \text{(13B)}$$

Initially when the shot starts, we have

$$\left.\begin{array}{l} x = 0\,;\ v = 0\,;\ p = p_0\,;\ \xi = 1\,;\ f_1 = f_{10}\,;\ f_2 = f_{20}\,; \\ z_1 = z_{10}\,;\ z_2 = z_{20}\,;\ \eta_1 = \eta_2 = 0. \end{array}\right\} \qquad \text{(14)}$$

Let $$\frac{\beta_1}{D_1} = \beta' \text{ and } \frac{\beta_2}{D_2} = \beta'' \qquad \text{(15)}$$

Dividing (4A) by (4B) we get

$$\frac{df_1}{df_2} = \frac{\beta'}{\beta''} \qquad \text{(16)}$$

which, on integrating and using initial conditions, give

$$\beta' f_2 - \beta'' f_1 = \beta' f_{20} - \beta'' f_{10} \qquad \text{(17)}$$

Using (9A) and (9B), equation (17) becomes

$$\beta'\left(\frac{1+\theta_2}{4\theta_2}\right)\left[\sqrt{1-k_2z_2} - \sqrt{1-k_2z_{20}}\right] = \beta''\left(\frac{1+\theta_1}{4\theta_1}\right)\left[\sqrt{1-k_1z_1} - \sqrt{1-k_1z_{10}}\right] \qquad \text{(18)}$$

where

$$k_1 = \frac{4\theta_1}{(1+\theta_1)^2} \text{ and } k_2 = \frac{4\theta_2}{(1+\theta_2)^2} \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

Equations (9A) and (9B) give

$$df_1 = -\frac{1}{1+\theta_1} \cdot \frac{dz_1}{\sqrt{1-k_1 z_1}} \quad \ldots \quad \ldots \quad \ldots \quad (20A)$$

and

$$df_2 = -\frac{1}{1+\theta_2} \cdot \frac{dz_2}{\sqrt{1-k_2 z_2}} \quad \ldots \quad \ldots \quad \ldots \quad (20B)$$

From (18), we get

$$\sqrt{1-k_1 z_1} = \lambda\sqrt{1-k_2 z_2}+\mu \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

where

$$\lambda = \frac{\beta'}{\beta''} \cdot \frac{1+\theta_2}{1+\theta_1} \cdot \frac{\theta_1}{\theta_2} \quad \ldots \quad \ldots \quad \ldots \quad (22A)$$

and

$$\mu = \sqrt{1-k_1 z_{10}}-\lambda\sqrt{1-k_2 z_{20}} \quad \ldots \quad (22B)$$

Clearly when ignition takes place,

$$z_1 = z_2 = 0.$$

$\therefore$ equation (21) gives $\quad \mu = 1-\lambda \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22C)$

Hence $\quad \sqrt{1-k_1 z_{10}}-\lambda\sqrt{1-k_2 z_{20}} = 1-\lambda \quad \ldots \quad \ldots \quad (22D)$

an expression giving the relation between z_{10} and z_{20}.

We can easily get with the help of (21)

$$z_2+\frac{F_1 C_1}{F_2 C_2} z_1 \equiv \mu'+\nu' z_2-\frac{2\lambda\lambda'\mu}{k_2}\sqrt{1-k_2 z_2} \quad \ldots \quad \ldots \quad (23)$$

where

$$\left.\begin{aligned} \lambda' &= \frac{F_1 C_1}{F_2 C_2} \cdot \frac{k_2}{k_1} \\ \mu' &= \frac{1-\mu^2-\lambda^2}{k_2} \cdot \frac{F_1 C_1}{F_2 C_2} \cdot \frac{k_2}{k_1} \\ &= \frac{1-\mu^2-\lambda^2}{k_2} \lambda' \\ \text{and} \quad \nu' &= 1+\lambda^2 . \frac{F_1 C_1}{F_2 C_2} \cdot \frac{k_2}{k_1} \\ &= 1+\lambda^2 . \lambda' \end{aligned}\right\} \quad \ldots \quad \ldots \quad (24)$$

Now let

$$z_2+\frac{F_1 C_1}{F_2 C_2} z_1 = \mu'+\nu' z_2-\frac{2\lambda\lambda'\mu}{k_2} . \sqrt{1-k_2 z_2}$$
$$= Z_2 \text{, say} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

and

$$\zeta_2 \xi^\gamma = Y_2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

Then equation (11) becomes

$$d(Z_2) = \xi^{1-\gamma} dY_2 \quad \ldots \quad (27)$$

From (13B), (20B) and (25), eliminating f_2 we get

$$(1+\theta_2)^2 \left[\lambda'\lambda\mu+\nu'\sqrt{1-k_2z_2}\right] \cdot \frac{d}{dZ_2}\left[\zeta_2^{\alpha}\left\{\lambda'\lambda\mu+\nu'\sqrt{1-k_2z_2}\right\}\frac{d\xi}{dZ_2}\right]$$

$$= M_2\zeta_2^{1-\alpha} \quad \ldots \quad (28)$$

From (25), we have

$$\lambda'\lambda\mu+\nu'\sqrt{1-k_2z_2} = a^{\frac{1}{2}}\sqrt{1-bZ_2} \quad \ldots \quad (29)$$

where

$$a = \lambda'^2\lambda^2\mu^2+\nu'^2+k_2\mu'\nu' \quad \ldots \quad (30A)$$

and

$$b = \frac{k_2}{a}\nu' \quad \ldots \quad (30B)$$

Let

$$bZ_2 = Z \quad \ldots \quad (31A)$$

and

$$bY_2 = b\zeta_2\xi^{\gamma} = Y \quad \ldots \quad (31B)$$

Then equation (27) gives

$$dZ = \xi^{1-\gamma}\, dY \quad \ldots \quad (32)$$

and equations (28), (29) and (31) give

$$(1-Z)\left[\xi''+\alpha\xi'\cdot\frac{\zeta'_2}{\zeta_2}\right] - \tfrac{1}{2}\xi' = \frac{M_2}{ab^2(1+\theta_2)^2}\cdot\xi_2^{1-2\alpha} \quad \ldots \quad (33)$$

where the dashes ′ and ″ denote first and second derivatives with respect to Z.

From (32), we have

$$(\gamma-1)\frac{\xi'}{\xi} = \frac{Y''}{Y'} \quad \ldots \quad (34A)$$

and

$$\frac{\xi''}{\xi'} = \frac{Y'''}{Y''} + \frac{2-\gamma}{\gamma-1}\cdot\frac{Y''}{Y'} \quad \ldots \quad (34B)$$

From equation (31), we have

$$\frac{\zeta'_2}{\zeta_2} = \frac{Y'}{Y} - \frac{\gamma\xi'}{\xi} = \frac{Y'}{Y} - \frac{\gamma}{\gamma-1}\cdot\frac{Y''}{Y'} \quad \ldots \quad (34C)$$

Eliminating ξ, ξ', ξ'', ζ_1 and ζ'_1 from equation (33) with the help of (31), (32), (34A), (34B) and (34C), we get

$$(1-Z)\left[\frac{Y'''}{Y''} + (n-2)\frac{Y''}{Y'} + \alpha\frac{Y'}{Y}\right] - \frac{1}{2} = Q\cdot\frac{(Y')^{2-2n}}{Y''Y^{2\alpha-1}} \quad \ldots \quad (35)$$

where

$$n = \frac{\gamma(1-\alpha)}{\gamma-1} \text{ and } Q = \frac{M_2(\gamma-1)}{(1+\theta_2)^2ab^{3-2\alpha}} \quad \ldots \quad (36)$$

—an equation similar to that obtained by Clemmow (1951; p. 123) for a single charge.

The equation (35) can be integrated numerically to give a series of values of Y, Y', and Y'' in terms of Z for any specific given propellant (i.e. γ and α given) and a given value of Q, representing the loading conditions.

The initial conditions are :—

(i) when a finite shot-start pressure is assumed :

$$\text{then} \quad \left.\begin{aligned} Z_0 &= bZ_{20} = b\left(z_{20} + \frac{F_1C_1}{F_2C_2} z_{10}\right) \\ Y_0 &= bY_{20} = b\zeta_{20} = b \cdot \frac{Al}{F_2C_2} p_0 \\ (Y')_0 &= 1 \text{ and } (Y'')_0 = 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

and (ii) when the shot-start pressure is taken to be zero :—

$$\text{then} \quad \left.\begin{aligned} Z_0 = 0; \quad Y_0 = 0 \\ (Y')_0 = 1 \text{ and } (Y'')_0 = 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

The shot-travel is obtained from (32) as

$$\xi^{\gamma-1} = \frac{dY}{dZ} \quad \ldots \quad \text{(37A)}$$

$$\text{i.e.,} \quad x = l\left[(Y')^{\frac{1}{\gamma-1}} - 1\right] \quad \ldots \quad \text{(37B)}$$

and the pressure from (31B) as

$$\zeta_2 = \frac{Y}{b(Y')^{\frac{\gamma}{\gamma-1}}} \quad \ldots \quad \text{(38A)}$$

$$\text{i.e.,} \quad p = \frac{F_2C_2}{Alb} \cdot \frac{Y}{(Y')^{\frac{\gamma}{\gamma-1}}} \quad \ldots \quad \text{(38B)}$$

Then equation (7) gives

$$\eta_2^2 = \frac{2M_2}{(\gamma-1)b}\left[Z - \frac{Y}{Y'}\right] \quad \ldots \quad \text{(39A)}$$

$$\text{or} \quad v^2 = \frac{2F_2C_2}{(\gamma-1)bw_1}\left[Z - \frac{Y}{Y'}\right] \quad \ldots \quad \text{(39B)}$$

The equations (37A), (38A) and (39A), (or (37B), (38B) and (39B) repectively) giving the shot-travel, pressure and velocity in terms of Z, Y and the derivative Y' are valid so long as both the propellants are burning.

5. Different Cases of All-burnt

Following Venkatesan and Patni (1953), we see from (17) that if

(i) $\beta' f_{20} < \beta'' f_{10}$, then f_2 cannot become zero before f_1. Hence charge C_1 must be burnt out first.

(ii) $\beta' f_{20} < \beta'' f_{10}$, then f_1 cannot become zero before f_2. Hence charge C_2 must be burnt out first.

(iii) $\beta' f_{20} = \beta'' f_{10}$, then both the charges will have to be burnt out simultaneously.

Hence we see that two different cases arise, viz.—

I. The two propellants burn out at different times.

II. Both the propellants burn out simultaneously.

For the sake of definiteness, let us call that propellant which will burn first as C_1.

Case I :—We have to consider this in two parts :—

(i) When only C_2 is burning (and C_1 has been burnt out).

(ii) When C_2 is also burnt out.

Case I(i) :—In this case, equation (7) becomes

$$\left(z_2+\frac{F_1C_1}{F_2C_2}\right) = \zeta_2\xi+\tfrac{1}{2}(\gamma-1)\frac{\eta_2^2}{M_2} \quad \ldots \quad (40)$$

which on differentiation and using (8B) gives

$$d\left(\frac{F_1C_1}{F_2C_2}+z_2\right) = \xi^{1-\gamma}d(\zeta_2\xi^{\gamma}) \quad \ldots \quad (41)$$

Let the suffix (2, 1) denote the position when the charge C_1 is just burnt out. Then the initial conditions for this case are :—

$$\left.\begin{array}{l} x = x_{2,1}\,; \quad v = v_{2,1}\,; \quad z_1 = 1\,; \quad z_2 = z_{2;\,2,1} \\ Z_2 = Z_{2;\,2,1}\,; \quad \xi = \xi_{2,1}\,; \quad \eta_2 = \eta_{2;\,2,1} \text{ and } \zeta_2 = \zeta_{2;\,2,1} \end{array}\right\} \quad \ldots \quad (42)$$

The equation (21) gives the value of z_2 at this position (2, 1) as

$$z_{2;\,2,1} = \frac{1}{k_2}\left[1-\left\{\frac{\sqrt{1-k_1}-\mu}{\lambda}\right\}^2\right] \quad \ldots \quad (43)$$

Also from (31) and (25), we get

$$Z_{2,1} = b\left(\frac{F_1C_1}{F_2C_2}+z_{2;\,2,1}\right) \quad \ldots \quad (44)$$

Then $\xi_{2,1}$, $\zeta_{2;\,2,1}$ and $\eta_{2;\,2,1}$ are given by (37A) (38A), and (39A).

Let

$$\frac{F_1C_1}{F_2C_2}+z_2 = z_{\Sigma} \quad \ldots \quad (45)$$

and

$$\zeta_2\xi^{\gamma} = Y_2, \text{ as before} \quad \ldots \quad (46)$$

Then (42) gives

$$d(z_{\Sigma}) = \xi^{1-\gamma}dY_2 \quad \ldots \quad (47)$$

Proceeding, as for equations (32) and (33), we shall get here

$$dZ = \xi^{1-\gamma}dY \quad \ldots \quad (48)$$

and

$$(1-Z)\left[\xi''+\alpha\xi'\,\frac{\zeta_2'}{\zeta_2}\right]-\tfrac{1}{2}\xi' = \frac{M_2}{a'(b')^2(1+\theta_2)^2}\cdot\zeta_2^{1-2\alpha} \quad \ldots \quad (49)$$

where

$$a' = \frac{F_2C_2+F_1C_1k_2}{F_2C_2} \quad \ldots \quad (50A)$$

$$b' = \frac{k_2}{a'} \quad \ldots \quad (50B)$$

$$Z = b'z_{\overline{2}} = b'\left(\frac{F_1C_1}{F_2C_2}+z_2\right) \quad \ldots \quad (50C)$$

and

$$Y = b'Y_2 = b'\zeta_2\xi^{\gamma} \quad \ldots \quad (50D)$$

Hence the corresponding equation (35) becomes in this case

$$(1-Z)\left[\frac{Y'''}{Y''}+(n-2)\frac{Y''}{Y'}+\alpha\frac{Y'}{Y}\right]-\frac{1}{2} = Q'\cdot\frac{(Y')^{2-2n}}{Y''Y^{2\alpha-1}} \quad \ldots \quad (51)$$

where

$$n = \frac{\gamma(1-\alpha)}{\gamma-1} \quad \text{and} \quad Q' = \frac{M_2(\gamma-1)}{a'(b')^{3-2\alpha}(1+\theta_2)^2} \quad \ldots \quad (52)$$

with initial conditions

$$Z = Z_{2,1} = b'\left(\frac{F_1C_1}{F_2C_2}+z_{2;\,2,1}\right) \quad \ldots \quad (53A)$$

$$Y = Y_{2,1} = b'\zeta_{2;\,2,1}\,\xi_{2,1}^{\gamma} \quad \ldots \quad (53B)$$

$$Y' = (Y')_{2,1} = \xi_{2,1}^{\gamma-1} \quad \ldots \quad (53C)$$

and

$$Y'' = (Y'')_{2,1} \quad \ldots \quad (54D)$$

—the quantities $Y_{2,1}$, $(Y')_{2,1}$, $(Y'')_{2,1}$ being obtained from the numerical integration of equation (35) when $Z = Z_{2,1}$.

Then the shot-travel, pressure and velocity at any point between the position when C_1 is just burnt out and the position when C_2 is also burnt out, are given in terms of Z, Y and Y' by the equations

$$\xi^{\gamma-1} = Y' \quad \ldots \quad (54A)$$

i.e.

$$x = l[(Y')^{\frac{1}{\gamma-1}}-1] \quad \ldots \quad (54B)$$

$$\zeta_2 = \frac{1}{b'}\cdot\frac{Y}{[Y']^{\frac{\gamma}{\gamma-1}}} \quad \ldots \quad (55A)$$

i.e.

$$p = \frac{F_2C_2}{Alb'}\cdot\frac{Y}{[Y']^{\frac{\gamma}{\gamma-1}}} \quad \ldots \quad (55B)$$

and

$$\eta_2^2 = \frac{2M_2}{b'(\gamma-1)}\left[Z-\frac{Y}{Y'}\right] \quad \ldots \quad (56A)$$

i.e.

$$v^2 = \frac{2F_2C_2}{(\gamma-1)w_1b'}\left[Z-\frac{Y}{Y'}\right] \quad \ldots \quad (56B)$$

The quantities x_2, p_2 and v_2 corresponding to the all-burnt position are obtained by putting

$$Z = (Z)_2 = b'\left(\frac{F_1C_1}{F_2C_2}+1\right)$$

and the corresponding values $(Y)_2$ and $(Y')_2$ of Y and Y' as obtained from the numerical integration of equation (51) in equations (54B), (55B) and (56B).

Case I (ii) :—In this case, the equation (7) becomes

$$1+\frac{F_1C_1}{F_2C_2} = \zeta_2\xi+\tfrac{1}{2}(\gamma-1)\frac{\eta_2^2}{M_2} \quad \ldots \quad \ldots \quad \ldots \quad (57)$$

Differentiating (57) and using (8B), we get

$$\xi^{1-\gamma}d(\zeta_2\xi^{\gamma}) = 0$$

$$\therefore\ \zeta_2\xi^{\gamma} = \text{constant} = \zeta_{2;2}\,\xi_2^{\gamma}$$

i.e.
$$\zeta_2 = \frac{\zeta_{2;2}\,\xi_2^{\gamma}}{\xi^{\gamma}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (58A)$$

or
$$p = \frac{p_2\xi_2^{\gamma}}{\xi^{\gamma}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (58B)$$

where ξ_2, $\zeta_{2;2}$ and $\eta_{2;2}$ are the values of ξ, ζ_2 and η_2 at all-burnt position, obtained from equations (54A), (55A) and (56A) as

$$\xi_2^{\gamma-1} = (Y')_2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (59A)$$

$$\zeta_{2;2} = \frac{(Y)_2}{b'[(Y')_2]^{\frac{\gamma}{\gamma-1}}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (59B)$$

and

$$\eta_{2;2}^2 = \frac{2M_2}{(\gamma-1)b'}\left[(Z)_2-\frac{(Y)_2}{(Y')_2}\right] \quad \ldots \quad \ldots \quad \ldots \quad (59C)$$

From equations (57) and (58A), we get

$$\eta_2^2 = \frac{2M_2}{\gamma-1}\left[\frac{F_2C_2+F_1C_1}{F_2C_2}-\frac{\zeta_{2;2}\,\xi_2^{\gamma}}{\xi^{\gamma-1}}\right] \quad \ldots \quad \ldots \quad \ldots \quad (60A)$$

or
$$v^2 = \frac{2F_2C_2}{(\gamma-1)w_1}\left[\frac{F_2C_2+F_1C_1}{F_2C_2}-\frac{\xi_2^{\gamma}\zeta_{2;2}}{\xi^{\gamma-1}}\right] \quad \ldots \quad \ldots \quad (60B)$$

The equations (58B) and (60B) give the values of pressure and velocity at any point after all-burnt position in terms of ξ and the various elements at the complete all-burnt position.

Case II :—Both the propellants are burnt out simultaneously,

i.e. $\beta' f_{20} = \beta'' f_{10}$.

At complete 'all-burnt', we shall have from equations (31) and (25)

$$(Z)_2 = bZ_{2;2} = b\left(\frac{F_2C_2+F_1C_1}{F_2C_2}\right).$$

Then from equation (35), $(Y)_2$ and $(Y')_2$, the corresponding values of Y and Y' are calculated. Then equations (37), (38) and (39) give ξ_2, $\zeta_{2;2}$ and $\eta_{2;2}$, the values of ξ, ζ_2 and η_2 at the complete 'all-burnt' as

$$\xi_2^{\gamma-1} = (Y')_2 \qquad \ldots \quad (61)$$

$$\zeta_{2;2} = \frac{1}{b}\cdot\frac{(Y)_2}{[(Y')_2]^{\frac{\gamma}{\gamma-1}}} \qquad \ldots \quad (62)$$

and

$$\eta^2_{2;2} = \frac{2M_2}{b(\gamma-1)}\left[(Z)_2 - \frac{(Y)_2}{(Y')_2}\right] \qquad \ldots \quad (63)$$

After all-burnt, the values of ζ_2 (giving pressure) and η_2 (giving velocity) at any point are given by equations (58A) and (60A).

6. Muzzle Velocity

Case I.—(When the charges burn out separately) :—

The muzzle velocity is given from equation (60B) as

$$v_3^2 = \frac{2Al}{(\gamma-1)w_1}\left[\frac{F_2C_2+F_1C_1}{Al} - \frac{p_2\xi_2^{\gamma}}{\xi_3^{\gamma-1}}\right] \qquad \ldots \quad (64)$$

where p_2 and ξ_2 are obtained from equations (59B) and (59A).

Case II.—(When the charges burn out simultaneously) :—

In this case the muzzle velocity v_3 is given by the same equation (64) but p_2 and ξ_2 are obtained from equations (61) and (62).

7. Maximum Pressure

The following cases may arise, viz., that the maximum pressure occurs when—

(*a*) both the charges are burning;
(*b*) C_1 is burnt out and C_2 is still burning; or
(*c*) at the position of simultaneous 'all-burnt' of both the propellants.

Case (*a*) :— From equation (38A),

$$\zeta_2 = \frac{Y}{b[Y']^{\frac{\gamma}{\gamma-1}}} \qquad \ldots \quad (38A)$$

$$\text{or } p = \frac{F_2C_2}{Alb}\cdot\frac{Y}{[Y']^{\frac{\gamma}{\gamma-1}}} \qquad \ldots \quad (38B)$$

The pressure is maximum when

$$\frac{Y'}{Y}=\frac{\gamma}{\gamma-1}\cdot\frac{Y''}{Y'} \quad \text{.. .. ..} \quad (65)$$

The equation (65) can be solved numerically for Z from the tabulated values of Z, Y, Y' and Y'' as obtained from (35) and then the value and the position of maximum pressure (i.e. p_1 and ξ_1) and the velocity v_1 at that instant can be determined from equations (38B), (37A) and (39B).

Conditions :—The conditions for the occurrence of maximum pressure in this position are

$$\left.\begin{aligned} & f_{1\,;\,1}>0 \text{ and } f_{2\,;\,1}>0 \\ \text{i.e.}\quad & z_{1\,;\,1}<1 \text{ and } z_{2\,;\,1}<1 \end{aligned}\right\} \quad \text{.. .. ..} \quad (66)$$

where $z_{1;\,1}$ and $z_{2;\,1}$ are the values of z_1 and z_2 at the position of maximum pressure.

With the help of equations (29), (31) and (21), the conditions (66) reduce to

$$\left[a^*-\sqrt{1-(Z)_1}\right]\left[b^*+\sqrt{1-(Z)_1}\right]<\frac{\nu'^2\,k_2}{a} \quad \text{.. ..} \quad (67\text{A})$$

$$\left[c^*-\sqrt{1-(Z)_1}\right]\left[d^*+\sqrt{1-(Z)_1}\right]<\frac{\nu'^2\,k_1}{\lambda^2 a} \quad \text{.. ..} \quad (67\text{B})$$

where $(Z)_1 =$ the value of Z obtained as the solution of equation (65), and

$$a^*=\frac{\nu'+\lambda\lambda'\mu}{\sqrt{a}} \quad \text{.. .. .. ..} \quad (68\text{A})$$

$$b^*=\frac{\nu'-\lambda\lambda'\mu}{\sqrt{a}} \quad \text{.. .. .. ..} \quad (68\text{B})$$

$$c^*=\frac{\nu'-\mu}{\lambda\sqrt{a}} \quad \text{.. .. .. ..} \quad (68\text{C})$$

$$d^*=\frac{\nu'+\mu}{\lambda\sqrt{a}} \quad \text{.. .. .. ..} \quad (68\text{D})$$

Case (*b*) :—Here

$$\beta' f_{20}>\beta'' f_{10}.$$

Equation (55B) gives

$$p=\frac{F_2C_2}{Alb'}\cdot\frac{Y}{[Y']^{\frac{\gamma}{\gamma-1}}} \quad \text{.. .. .. ..} \quad (55\text{B})$$

∴ pressure is maximum, if

$$\frac{Y'}{Y}=\frac{\gamma}{\gamma-1}\,\frac{Y''}{Y'} \quad \text{.. .. .. ..} \quad (69)$$

This equation can be solved numerically for Z from the tabulated values of Z, Y, Y' and Y'' obtained from equation (51). Then the value of maximum pressure can be found out from equation (55B).

Conditions :—The conditions for the occurrence of maximum pressure in this position are

$$\left.\begin{array}{l} f_{1;\,1} = 0 \text{ and } f_{2;\,1} \geqslant 0 \\ \text{i.e., } z_{1;\,1} = 1 \text{ and } z_{2;\,1} < 1 \end{array}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (70)$$

But in view of the condition $\beta' f_{20} > \beta'' f_{10}$ and equation (17), $f_{2;\,1}$ cannot be equal to zero.

Then with the help of (21) and (50C), the conditions reduce to

$$\beta' f_{20} > \beta'' f_{10} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (71A)$$

$$\mu + \lambda\sqrt{1-k_2} < \sqrt{1-k_1} \qquad \ldots \quad \ldots \quad \ldots \quad (71B)$$

and

$$\frac{F_2C_2(Z)_1 - F_1C_1}{b'F_2C_2} < 1 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (71C)$$

where $(Z)_1$ is the value of Z obtained as the solution of equation (69).

Case (*c*) :—Here

$$\beta' f_{20} = \beta'' f_{10}.$$

In this case the maximum pressure p_1 is given by the equations (65) and (38B).

Conditions :—The conditions for the occurrence of maximum pressure in this position are

$$\left.\begin{array}{ll} & \beta' f_{20} = \beta'' f_{10} \\ \text{and} & f_{1;\,1} = f_{2;\,1} = 0 \\ \text{or} & z_{1;\,1} = z_{2;\,1} = 1 \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (72)$$

These conditions are reduced to

$$\left[a^* - \sqrt{1-(Z)_1}\right]\left[b^* + \sqrt{1-(Z)_1}\right] = \frac{\nu'^2 k_2}{a} \qquad \ldots \quad \ldots \quad (73A)$$

and

$$\left[c^* - \sqrt{1-(Z)_1}\right]\left[d^* + \sqrt{1-(Z)_1}\right] = \frac{\nu'^2 k_1}{\lambda^2 a} \qquad \ldots \quad \ldots \quad (73B)$$

where a^*, b^*, c^* and d^* are given by (68A), (68B), (68C) and (68D) and $(Z)_1$ has the same value as in (67A) and (67B).

8. A Special Case. ($\theta_1 = \theta_2 = 0$)

When both of θ_1 and θ_2 or any one of them be zero, some of the equations derived above present some difficulty as they may take the undetermined form. Accordingly those equations are to be modified. For a long tubular propellant, we generally take $\theta = 0$; but as is theoretically determined, this $\theta = \dfrac{R_0 - r_0}{H_0}$; so even if H_0 (i.e. the length of the grain) be 200–500 times the annulus $(R_0 - r_0)$, θ is greater than ·002 and we may not neglect it. With this device, the theory developed in the previous sections is equally applicable to tubular propellants also. However, we give here principal equations which differ from those given above when θ_1 and θ_2 both are taken exactly zero.

Since

$$\theta_1 = \theta_2 = 0, \quad k_1 = k_2 = 0.$$

The equations (18), (21), (25), (32) and (33) become here

$$\beta'' z_1 - \beta' z_2 = \beta'' z_{10} - \beta' z_{20} \qquad (74)$$

$$z_1 = \frac{\beta'}{\beta''} z_2 + \frac{\beta'' z_{10} - \beta' z_{20}}{\beta''} \qquad (75)$$

$$z_2 + \frac{F_1 C_1}{F_2 C_2} z_1 = \lambda^* z_2 + \mu^* = Z \qquad (76)$$

$$dZ = \xi^{1-\gamma} dY \qquad (77)$$

$$\xi'' + \alpha \xi' \frac{\zeta_2'}{\zeta_2} = \frac{M_2}{\lambda^2} \zeta_2^{1-2\alpha} \qquad (78)$$

where

$$\lambda^* = 1 + \frac{F_1 C_1}{F_2 C_2} \frac{\beta'}{\beta''} \qquad (79)$$

$$\mu^* = \frac{F_1 C_1}{F_2 C_2} \left[z_{10} - \frac{\beta'}{\beta''} z_{20} \right] \qquad (80)$$

and

$$Y = \zeta_2 \xi^{\gamma} \qquad (81)$$

Accordingly the fundamental equation (35) becomes here

$$\frac{Y'''}{Y''} + (n-2)\frac{Y''}{Y'} + \alpha \frac{Y'}{Y} = Q^* \cdot \frac{(Y')^{2-2n}}{Y'' Y^{2\alpha-1}} \qquad (82)$$

where

$$n = \frac{\gamma(1-\alpha)}{\gamma-1} \text{ and } Q^* = \frac{M_2(\gamma-1)}{(\lambda^*)^2} \qquad (83)$$

and the independent variable is $Z = \lambda^* z_2 + \mu^*$.

The equation (82) does not contain the independent variable Z explicitly; it can, therefore, be solved numerically by treating Y' as a function of Y.

With zero shot-start pressure, the series solution of (82) takes the form as given by Clemmow (1951, p. 124),

$$Y = Z\left[1 + \frac{Z^*}{4-2\alpha} + \frac{\{3-2\alpha-n(2-\alpha)\}(Z^*)^2}{(5-3\alpha)(4-2\alpha)} + \cdots\cdots\right] \qquad (84)$$

where

$$Z^* = \frac{Q^* Z^{3-2\alpha}}{(3-2\alpha)(2-\alpha)} \qquad (85)$$

The quantities ξ, ζ_2 and η_2 giving the shot-travel (x), pressure (p) and velocity (v) when both the charges are burning, are given by

$$\xi^{\gamma-1} = \frac{dY}{dZ} \qquad (86)$$

$$\zeta_2 = \frac{Y}{[Y']^{\frac{\gamma}{\gamma-1}}} \qquad (87)$$

and

$$\eta_2^2 = \frac{2M_2}{\gamma-1}\cdot\left[Z-\frac{Y}{Y'}\right] \quad .. \quad .. \quad .. \quad (88)$$

where Z, Y and Y' have the values obtained from (82). Clearly the equations (86), (87) and (88) correspond to equations (37A), (38A) and (39A) of the general case.

9. Different Cases of All Burnt

Case I (*i*).—(When C_1 is burnt out and C_2 is still burning.)

The initial conditions for this case are

$$z_{2;\,2,1} = \frac{\beta''(1-z_{10})+\beta' z_{20}}{\beta'} \quad .. \quad .. \quad .. \quad (89)$$

$$Z_{2,1} = \lambda^* z_{2;\,2,1}+\mu^* \quad .. \quad .. \quad .. \quad .. \quad (90)$$

and $\xi_{2,1}$, $\zeta_{2;\,2,1}$ and $\eta_{2;\,2,1}$ are given by (86), (87) and (88) for the value

$$Z = Z_{2,1}.$$

Now let

$$\left.\begin{aligned} z_2+\frac{F_1C_1}{F_2C_2} &= Z \\ \text{and} \qquad \zeta_2\xi^\gamma &= Y \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad (91)$$

Then here we shall have

$$dZ = \xi^{1-\gamma}dY \quad .. \quad .. \quad .. \quad .. \quad (92)$$

$$\xi''+\alpha\xi'\frac{\zeta_2'}{\zeta_2} = M_2\zeta_2^{1-2\alpha} \quad .. \quad .. \quad .. \quad (93)$$

and the equation corresponding to (51) is

$$\frac{Y'''}{Y''}+(n-2)\frac{Y''}{Y'}+\alpha\frac{Y'}{Y} = Q'^*\cdot\frac{(Y')^{2-2n}}{Y''Y^{2\alpha-1}}. \quad .. \quad .. \quad (94)$$

where

$$n = \frac{\gamma(1-\alpha)}{\gamma-1} \quad \text{and} \quad Q'^* = M_2(\gamma-1) \quad .. \quad .. \quad (95)$$

The equation (94) is similar to (82) with Q'^* for Q^* and with initial conditions

$$\left.\begin{aligned} Z &= Z_{2,1} \\ Y &= \zeta_{2;\,2,1}\,\xi_{2,1}^\gamma \\ Y' &= \xi_{2,1}^{\gamma-1} \\ \text{and} \qquad Y'' &= (Y'')_{2,1} \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad (96)$$

$=$ the value of Y'' corresponding to $Z = Z_{2,1}$.

The quantities ξ, ζ_2 and η_2 are given by (86), (87) and (88) but now the quantities Z, Y and Y' are given by (91) and (94).

Case I (ii).—(When C_1 and C_2 both are burnt out.)

In this case ζ_2 and η_2 (giving pressure p and velocity v respectively) at any point after all-burnt are given in terms of ξ by equations (58A) and (60A) where ξ_2, $\zeta_{2;\,2}$ and $\eta_{2;\,2}$ (the values of ξ, ζ_2 and η_2 respectively at the complete all-burnt position) are given by equations (86), (87) and (88) for

$$Z = (Z)_2 = \frac{F_1C_1+F_2C_2}{F_2C_2},$$

and for Y and Y' obtained from equation (94) with the above value of Z.

Case II.—Both the propellants are burnt out simultaneously, i.e.

$$\beta' f_{20} = \beta'' f_{10},$$

or

$$\beta'' z_{10} - \beta' z_{20} = \beta'' - \beta'.$$

At 'all-burnt', we have from equations (86), (87) and (88)

$$\left.\begin{aligned} \xi_2^{\gamma-1} &= (Y')_2 \\ \zeta_{2;\,2} &= \frac{(Y)_2}{[(Y')_2]^{\gamma/\gamma-1}} \\ \text{and} \quad \eta^2_{2;\,2} &= \frac{2M_2}{\gamma-1}\left[(Z)_2 - \frac{(Y)_2}{(Y')_2}\right] \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (97)$$

where

$$(Z)_2 = \frac{F_1C_1+F_2C_2}{F_2C_2} \quad \text{and } (Y)_2 \text{ and } (Y')_2 \text{ are}$$

obtained from equation (82) for this value of $Z = (Z)_2$.

After 'all-burnt' position, ζ_2 and η_2 (giving pressure p and velocity v) at any point are given in terms of ξ by (58A) and (60A) with ξ_2, $\zeta_{2;\,2}$ and $\eta_{2;\,2}$ as given by (97).

10. Maximum Pressure

Case (a) :—When both the charges are burning, the maximum pressure p_1 is given from equation (87) as

$$p_1 = \frac{F_2C_2}{Al} \cdot \frac{Y}{(Y')^{\frac{\gamma}{\gamma-1}}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (98)$$

subject to the condition

$$\frac{Y'}{Y} = \frac{\gamma}{\gamma-1} \cdot \frac{Y''}{Y'} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (99)$$

which should be solved numerically from the tabulated values of Z, Y, Y' and Y'' as obtained from equation (82).

Conditions :—The conditions of equation (66) for the occurrence of maximum pressure in this position are simplified with the help of equations (75) and (76) as follows :—

$$\left.\begin{aligned} (Z)_1 &< \lambda^* + \mu^* \\ (Z)_1 &< \frac{\beta''\lambda^*}{\beta'} - \frac{\mu^*}{\lambda^*-1} \end{aligned}\right\} \quad \ldots \quad . \quad (100)$$

i.e. $(Z)_1$, the value of Z obtained as the solution of equation (99), should be less than the smaller of

$$\lambda^*+\mu^*$$

and

$$\left(\frac{\beta''\lambda^*}{\beta'}-\frac{\mu^*}{\lambda^*-1}\right).$$

Case (*b*) :—Here the maximum pressure p_1 is given by the same equations (98) and (99) but the latter equation should be solved from the tabulated values of Z, Y, Y' and Y'' as obtained from equation (94).

Conditions :—The conditions of equation (70) for the occurrence of maximum pressure in this position are simplified to

$$\left.\begin{aligned} &\beta' f_{20} > \beta'' f_{10} \\ \text{and} \quad &(Z)_1 < 1+\frac{F_1C_1}{F_2C_2}. \end{aligned}\right\} \quad \ldots \quad (101)$$

Case (*c*) :—Here both the charges are burnt out simultaneously, i.e. $\beta' f_{20} = \beta'' f_{10}$ and the maximum pressure occurs at 'all-burnt' position.

In this case the maximum pressure is given by the same equations (98) and (99).

Conditions :—The conditions of equation (72) for the occurrence of maximum pressure in this position are simplified in this case to

$$\left.\begin{aligned} &\beta' f_{20} = \beta'' f_{10} \\ \text{and} \quad (Z)_1 &= \lambda^*+\mu^* \\ &= \frac{\beta''\lambda^*}{\beta'}-\frac{\mu^*}{\lambda^*-1} \end{aligned}\right\} \quad \ldots \quad (102)$$

where λ^* and μ^* are given by equations (79) and (80) and $(Z)_1$ has the same value as found from equation (99).

11. Summary

In this paper the problem of using composite charges in conventional guns has been discussed. A solution of the basic equations of Internal Ballistics when non-linear power law for rate of burning is assumed, is derived for any values of the form-coefficients under certain conditions, viz., the value of γ—the ratio of specific heats—is the same for both the charges and the power-index law also is the same for both the propellants.

12. Acknowledgement

The author is highly thankful to Dr. R. S. Varma for encouragement and guidance in the preparation of this paper.

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Issued August 23, 1955.