FLOW OF A COMPRESSIBLE FLUID AROUND A CORNER

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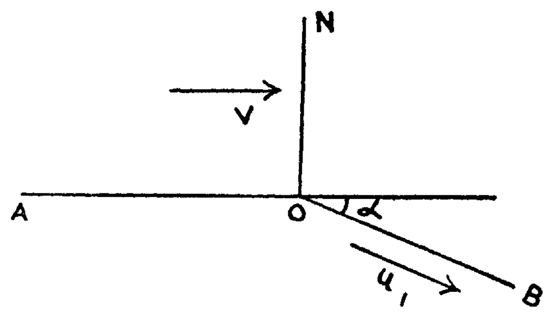
1. INTRODUCTION

Prandtl's problem of the expansion of a uniform two-dimensional stream of gas flowing around a corner at a supersonic speed had been studied (Durand, 1935) under adiabatic conditions and it had been shown that the velocity perpendicular to the radius vector from the corner is always equal to the speed of sound at the local conditions of pressure and density. Bernoulli's equation had been used to find the components of velocity in terms of the angle of deflection and also to find the relation between the pressure and the same angle.

In this paper the same problem has been studied starting with non-adiabatic conditions, conductivity and viscosity taken into account but it is found that the equations lead to adiabatic conditions and that the transverse velocity is equal to the velocity of sound as before. The equations can be straight integrated giving the velocity in terms of deflection. Expressions for the components of velocity and for pressure are analytically different from those obtained by the other method and are of some interest.

2. EQUATIONS OF MOTION

A uniform stream of gas flows parallel to a rigid boundary AO. At O the boundary makes an angle α with AO produced and then takes a straight course OB. It is proposed to investigate the flow around the corner O in the angle between ON and OB where ON is perpendicular to AO at O.



Take O as origin, (u, v) be components of velocity along and perpendicular to the radius vector from O.

Components of strain are

$$e_{rr} = 2\frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{2}{r}\frac{\partial v}{\partial \theta} + \frac{2u}{r}, \quad e_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}. \quad \cdots \quad (1)$$

The divergence is

$$\Delta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}. \quad \cdots \quad (2)$$

The stress components are

$$\left.\begin{aligned} p_{rr} &= -p+\lambda\Delta+\mu e_{rr}, \quad p_{\theta\theta} = -p+\lambda\Delta+\mu e_{\theta\theta} \\ \text{and} \quad p_{r\theta} &= \mu e_{r\theta} \end{aligned}\right\} \quad \ldots \quad (3)$$

where p is the pressure.

The equations of motions are

$$\left.\begin{aligned} \rho\left(u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}-\frac{v^2}{r}\right) &= \frac{\partial}{\partial r}p_{rr}+\frac{1}{r}\frac{\partial}{\partial \theta}p_{r\theta}+\frac{1}{r}(p_{rr}-p_{\theta\theta}), \\ \text{and } \rho\left(u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+\frac{uv}{r}\right) &= \frac{\partial}{\partial r}p_{r\theta}+\frac{1}{r}\frac{\partial}{\partial \theta}p_{\theta\theta}-\frac{2}{r}p_{r\theta}, \end{aligned}\right\} \quad \ldots \quad (4)$$

where ρ is the density.

The equation of continuity is

$$\frac{\partial}{\partial r}(r\rho u)+\frac{\partial}{\partial \theta}(\rho v) = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

The equation of energy is

$$\rho\left(u\frac{\partial i}{\partial r}+\frac{v}{r}\frac{\partial i}{\partial \theta}\right)-\left(u\frac{\partial p}{\partial r}+\frac{v}{r}\frac{\partial p}{\partial \theta}\right)$$

$$=\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu}{\sigma}\cdot r\frac{\partial i}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{\mu}{\sigma}\cdot\frac{\partial i}{\partial \theta}\right)+\Phi \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where i, called the enthalpy, is the heat content, σ the Prandtl number, and Φ, dissipation function given by

$$4\Phi = \lambda(e_{rr}+e_{\theta\theta})^2+2\mu(e_{rr}^2+e_{\theta\theta}^2+2e_{r\theta}^2). \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

Also the equation of state for a perfect gas is

$$i\rho = \frac{\gamma}{\gamma-1}p \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

where γ is the ratio of specific heats at constant pressure and at constant volume.

We further assume that μ varies as some power of absolute temperature, and therefore as i^n,

$$\text{i.e.} \quad \frac{\mu}{i^n} = \text{const.} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

where n is usually positive and less than unity.

These are all the equations. To solve these we make some simplifications.

3. Simplifications of the Equations

In the present problem we assume that the velocity, density and pressure are constant along a radius, hence the above equations reduce to the following :—

$$\left.\begin{aligned} \frac{\rho v}{r}\left(\frac{du}{d\theta}-v\right) &= -\frac{\lambda+2\mu}{r^2}\left(u+\frac{dv}{d\theta}\right)+\frac{1}{r^2}\frac{d}{d\theta}\left\{\mu\left(\frac{du}{d\theta}-v\right)\right\}, \\ \frac{\rho v}{r}\left(\frac{dv}{d\theta}+u\right) &= -\frac{3\mu}{r^2}\left(\frac{du}{d\theta}-v\right)-\frac{1}{r}\frac{dp}{d\theta}+\frac{1}{r^2}\frac{d}{d\theta}\left\{(\lambda+2\mu)\left(u+\frac{dv}{d\theta}\right)\right\} \end{aligned}\right\} \quad (4a)$$

$$\rho u+\frac{d}{d\theta}(\rho v)=0, \qquad \text{(5a)}$$

$$\Phi=\frac{\lambda+2\mu}{r^2}\left(u+\frac{dv}{d\theta}\right)^2+\frac{\mu}{r^2}\left(\frac{du}{d\theta}-v\right)^2, \qquad \text{(7a)}$$

$$\frac{v}{r}\left(\rho\frac{di}{d\theta}-\frac{dp}{d\theta}\right)=\frac{1}{r^2}\frac{d}{d\theta}\left(\frac{\mu}{\sigma}\frac{di}{d\theta}\right)+\Phi. \qquad \text{(6a)}$$

Again the coefficients of r in various equations must all be zero. Therefore we get

$$\frac{du}{d\theta}-v=0, \qquad \text{(10)}$$

$$\rho v\left(u+\frac{dv}{d\theta}\right)=-\frac{dp}{d\theta}, \qquad \text{(11)}$$

$$\lambda+2\mu=0,$$

$$\Phi=0,$$

$$\frac{d}{d\theta}\left(\frac{\mu}{\sigma}\frac{di}{d\theta}\right)=0, \qquad \text{(12)}$$

$$\rho\frac{di}{d\theta}=\frac{dp}{d\theta}. \qquad \text{(13)}$$

It is to be noted that the minimum value of $\lambda+2\mu$ is also zero.

4. Solutions of the Equations

From (12), taking σ as constant, we have with the help of (9),

$$i^n\frac{di}{d\theta}=-c, \qquad \text{(14)}$$

where c is a positive constant, so that from (13),

$$\frac{dp}{d\theta}=-\frac{c\rho}{i^n}. \qquad \text{(15)}$$

From (8), we have by differentiation with respect to θ,

$$i\frac{d\rho}{d\theta}+\rho\frac{di}{d\theta}=\frac{\gamma}{\gamma-1}\frac{dp}{d\theta}=-\frac{\gamma}{\gamma-1}\frac{c\rho}{i^n}, \qquad \text{from (15)}$$

or

$$i^{n+1}\frac{d\rho}{d\theta}+\rho i^n\frac{di}{d\theta}=-\frac{\gamma}{\gamma-1}c\rho$$

or

$$i^{n+1}\frac{d\rho}{d\theta}=-\frac{1}{\gamma-1}c\rho, \qquad \text{from (14)}$$

or

$$(\gamma-1)\frac{d\rho}{\rho}=-\frac{cd\theta}{i^{n+1}}=\frac{di}{i}, \qquad \text{from (14)}$$

Hence $i = \alpha\rho^{\gamma-1}$ where α is a constant,

so that
$$\frac{\gamma}{\gamma-1}p = i\rho = \alpha\rho^{\gamma}$$

or
$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \qquad (16)$$

where suffix 0 denotes value when at rest.

This is the adiabatic law.

From equations (11) and (15), we have
$$\rho v\left(u+\frac{dv}{d\theta}\right) = \frac{c\rho}{i^n}, \qquad (17)$$

which with the help of (5*a*) gives
$$v\left\{-\frac{d}{d\theta}(v\rho)+\rho\frac{dv}{d\theta}\right\} = \frac{c\rho}{i^n}$$

or
$$v^2\frac{d\rho}{d\theta} = -\frac{c\rho}{i^n}.$$

But we have already seen that
$$\frac{d\rho}{d\theta} = -\frac{c\rho}{(\gamma-1)i^{n+1}},$$

hence
$$v^2 = (\gamma-1)i = \gamma\frac{p}{\rho} = a^2, \qquad (18)$$

where a is the local velocity of sound.

Again from (17), with the help of (14), we get
$$v\left(u+\frac{dv}{d\theta}\right) = -\frac{di}{d\theta}$$

or
$$u\frac{du}{d\theta}+v\frac{dv}{d\theta} = -\frac{di}{d\theta},$$

giving,
$$u^2+v^2 = c_1-2i, \qquad (19)$$

where c_1 is a constant.

Also from (14) we have
$$i^{n+1} = i_0^{n+1}-(n+1)c\theta$$

therefore (19) gives
$$q^2 = u^2+v^2 = c_1-2i_0\left\{1-\frac{(n+1)c\theta}{i_0^{n+1}}\right\}^{\frac{1}{n+1}}. \qquad (20)$$

Measuring θ from the line ON at which
$$u = 0,\ v = V$$

we have
$$V^2 = c_1-2i_0$$

so that
$$c_1 = V^2+2i_0 \qquad (21)$$

where
$$i_0 = \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0}.$$

The flow will be purely radial and uniform along the boundary OB if when

$$\theta = \frac{\pi}{2}+\alpha,\ v = 0,\ u = u_1,\ \text{say}$$

then

$$u_1^2 = V^2+2i_0-2i_0\left\{1-\frac{(n+1)c\left(\frac{\pi}{2}+\alpha\right)}{i_0^{n+1}}\right\}^{\frac{1}{n+1}} \quad \ldots \quad \ldots \quad (22)$$

Also

$$u^2 = q^2-v^2 = V^2-v^2+2i_0-2i_0\left\{1-\frac{(n+1)c\theta}{i_0^{n+1}}\right\}^{\frac{1}{n+1}}$$

$$= V^2-\gamma\frac{p}{\rho}+\frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}\left[1-\left\{1-\frac{(n+1)c\theta}{i_0^{n+1}}\right\}^{\frac{1}{n+1}}\right]. \quad \ldots \quad \ldots \quad (23)$$

Evidently $p = 0$ on the boundary OB. OB may be regarded as the free surface of the stream.

In the particular case when μ is constant, so that $n = 0$, we get

$$\left.\begin{aligned} q^2 &= V^2+2c\theta \\ u_1^2 &= V^2+2c\left(\frac{\pi}{2}+\alpha\right) \\ \therefore\ q^2 &= V^2+\frac{u_1^2-V^2}{\frac{\pi}{2}+\alpha}\cdot\theta \\ \text{and} \qquad u^2 &= V^2-\gamma\frac{p}{\rho}+\frac{u_1^2-V^2}{\frac{\pi}{2}+\alpha}\cdot\theta. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

In this particular case, we have from (15)

$$\frac{dp}{d\theta} = -c\rho$$

$$= -c\rho_0\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{from (16)}$$

which gives on integration

$$\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = \frac{c\rho_0}{p_0}\cdot\frac{\gamma-1}{\gamma}\left(\frac{\pi}{2}+\alpha-\theta\right), \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

as $p = 0$ when $\theta = \frac{\pi}{2}+\alpha$, i.e. on OB.

This gives the relation between the pressure and θ.

Thus when

$$\theta = 0,$$

$$\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = \frac{c\rho_0}{p_0}\frac{\gamma-1}{\gamma}\left(\frac{\pi}{2}+\alpha\right)$$

$$= \frac{\gamma-1}{2\gamma}\frac{\rho_0}{p_0}(u_1^2 - V^2).$$

This may be taken as the pressure of the uniform stream parallel to AO.

Reference

Durand, W. F. (1935). Aerodynamic Theory, **3**, 243.

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