FERMI'S THEORY OF MULTIPLE PION PRODUCTION AND THE ZERO-POINT ENERGY

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INTRODUCTION

I. Fermi's statistical theory of multiple pion production in high energy nucleon-nucleon collisions (1950) has received considerable attention in recent years. The earlier calculations have been improved upon (Fermi, 1953) by taking into account the conservation of charge and isotopic spin (this has been done up to a maximum number of pions produced $\leqslant 3$). As has been stressed by Fermi himself, and others (Bhabha, 1953), the theory is not free from 'conceptional' difficulties and, because of this and other simplifying assumptions introduced in the treatment, the theory can at best provide a rough picture of the phenomena concerned. Nevertheless, the results are on the whole in fair agreement with experiment, though this may be partly due to accident (this view is supported by considerations of section III).

In the next section we extend (following Cocconni, 1954) Fermi's theory to the case of a nucleon-nucleus collision. The incident high energy nucleon is supposed to interact with the nucleons, say n in number, which lie in its path as it 'tunnels' through the target nucleus (for heavy nuclei, $A \sim 100$, a reasonable value for n is 5). Further, we assume that for a pion produced in the collision process, the lowest permissible energy, ϵ_0 say, is equal to the ground state energy of a free particle of mass μ (the pion rest mass) enclosed in a volume equal to the volume of the 'reaction-space'. In the case of a nucleon-nucleon collision, as Auluck and Kothari (1953) have pointed out, because of the large contraction of the reaction-space in the centre of mass system (henceforth called C-system), the zero-point energy becomes comparable with the total available energy, resulting in a substantial decrease in the number of pions produced. For very high primary energies, the number of particles (pions) produced becomes independent of energy. In the case of a nucleon-nucleus collision, because of the length of the 'tunnel' being n times the transverse direction, for relatively small bombarding energies ($\sim$ a few Bev), there is no appreciable increase in the zero-point energy due to Lorentz contraction, but it is still fairly effective in cutting down the number of pions produced. For very high energies the contraction becomes large, and as in the case of the nucleon-nucleon collision, we obtain an upper limit for the number of pions produced. The introduction of the zero-point energy implies a serious departure from Fermi's original theory and it is consequently of considerable importance to determine which of the two versions of the theory is in reasonable accord with observation.

In section III, we return to Fermi's original thermodynamic theory and show that his estimate of the total number of pions generated is in error by a factor of about 2 (Kothari, 1954). It is seen that as an indirect consequence of the contraction of the reaction-space, one obtains, in addition to the number of pions estimated by Fermi, an approximately equal number N_0, which, in the C-system, have all the same energy μc^2. The existence of this number N_0 is inherent in Fermi's theory.

II. We consider a collision between a high energy nucleon and a nucleus at rest in the laboratory system. We assume that the nucleon in going through the nucleus hits n nucleons, the collision being regarded as taking place between the primary nucleon and the 'lump' of these n nucleons. For a heavy nucleus ($A \sim 100$), $n \simeq 5$. The velocity of the centre of mass of the total system composed of the primary nucleon and the lump of n nucleons is (in units of c)

$$\beta_c = \frac{(\gamma^2-1)^{\frac{1}{2}}}{(\gamma+n)}, \quad \ldots \quad (1)$$

where γMc^2 is the energy of the primary nucleon in the laboratory system. The total available energy in the C-system is

$$W = Mc^2(2n\gamma+n^2+1)^{\frac{1}{2}}, \quad \ldots \quad (2)$$

and the Lorentz contraction factor is

$$\frac{1}{\gamma_c} = \frac{(2n\gamma+n^2+1)^{\frac{1}{2}}}{(\gamma+n)}. \quad \ldots \quad (3)$$

The volume (in the C-system) in which the pions are produced may be taken as

$$\Omega = \frac{n\Omega_0}{\gamma_c}, \quad \ldots \quad (4)$$

where $$\Omega_0 = \frac{4\pi}{3}\left(\frac{\xi\hbar}{\mu c}\right)^3, \quad \xi \sim 1.$$

If the final state consists of N particles, the probability $P(N)$ of this state being realized according to Fermi's theory is

$$P(N) = k\left(\frac{\Omega}{h^3}\right)^{N-1}\frac{dQ_{N-1}}{dW}, \quad \ldots \quad (5)$$

where k is a constant and $Q_{N-1}(W)$ is the volume of the momentum space containing all those states for which the energy is less than or equal to W. Assuming that the final state consists of N relativistic pions and S non-relativistic nucleons we write

$$W = \frac{1}{2M}\left[\sum_{i=1}^{S-1} P_i^2 + \left(-\sum_{i=1}^{S-1} P_i\right)^2\right] + SMc^2 + \sum_{i=1}^{N} cp_i \quad \ldots \quad (6)$$

In writing down this expression we have assumed momentum conservation for the heavy particles. Let ϵ_0 be the least permissible energy for a pion produced in the collision. Then the volume of the momentum space bounded by the surface defined by (6) and

$$cp_i > \epsilon_0, \qquad (i = 1, 2, \ldots . N)$$

is given by

$$Q_{N-1}(W) = \frac{(4\pi)^N(2\pi M)^{\frac{3(S-1)}{2}}(W-SMc^2-N\epsilon_0)^{3N+\frac{3}{2}(S-1)}}{c^{3N}S^{\frac{3}{2}}}\left(\sum_{r=0}^{2N}\frac{a_r\alpha^{2N-r}}{\Gamma\left(r+N+\frac{3S}{2}-\frac{1}{2}\right)}\right), \quad \ldots \quad (7)$$

where $$\alpha = \frac{\epsilon_0}{W-SMc^2-N\epsilon_0},$$

and the coefficients a_r are given by

$$(2+2k+k^2)^N = \sum_{r=0}^{2N} a_r k^{2N-r} \quad \ldots \quad (8)$$

Substituting in (5) for Ω and $Q_{N-1}(W)$ from (4) and (7), we obtain after some simple calculation

$$P(S,N) = kS^{-3/2}(\pi/2)^{(S-1)/2}\left(\frac{4\xi^3}{3\pi}\frac{M^3}{\mu^3}\right)^{N+S-1}(Mc^2)^{-3N-\frac{3}{2}(S-1)}\frac{n^2w}{(n^2+w^2-1)^{N+S-1}}\times$$
$$\times\sum_{r=0}^{2N}\left[\frac{a_r\alpha^{2N-r}W_0^{3N+\frac{3S}{2}-\frac{5}{2}}}{\Gamma\left(r+N+\frac{3S}{2}-\frac{1}{2}\right)}\left\{3\left(N+\frac{S-1}{2}\right)\frac{W_0}{\epsilon_0}\frac{d\epsilon_0}{dW}+\left(N+r+\frac{3S}{2}-\frac{3}{2}\right)\times\right.\right.$$
$$\left.\left.\times\left(1-N\frac{d\epsilon_0}{dW}-\frac{W_0}{\epsilon_0}\frac{d\epsilon_0}{dW}\right)\right\}\right]. \quad \ldots \quad (9)$$

Here
$$w = \frac{W}{Mc^2},$$
and
$$W_0 = (W-SMc^2-N\epsilon_0). \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

If the 'tunnel' be considered as a cylinder with area of cross-section πr_0^2 and length $\dfrac{2nr_0}{\gamma_c}\left(r_0 = \dfrac{\eta\hbar}{\mu c},\ \eta \sim 1\right)$, the zero-point energy of a pion will be given by

$$\epsilon_0 = \frac{c\hbar}{r_0}\left[5{\cdot}784+\eta^2+\frac{\pi^2\gamma_c^2}{4n^2}\right]^{\frac{1}{2}}$$
$$= \frac{c\hbar}{r_0}\left[5{\cdot}784+\eta^2+\frac{\pi^2}{4n^2}\frac{(\gamma+n)^2}{(2n\gamma+n^2+1)}\right]^{\frac{1}{2}} \quad \ldots \quad \ldots \quad (11)$$

The maximum number, $N_{\max.}$, of pions produced is

$$N_{\max.} \leqslant \frac{W-SMc^2}{\epsilon_0}$$
$$= \frac{Mc^2r_0}{c\hbar}\left[1-\frac{S}{(2n\gamma+n^2+1)^{\frac{1}{2}}}\right]\times\left[\frac{5{\cdot}784+\eta^2}{(2n\gamma+n^2+1)}+\frac{\pi^2}{4n^2}\frac{(\gamma+n)^2}{(2n\gamma+n^2+1)^2}\right]^{-\frac{1}{2}} \ldots \quad (12)$$

For very high primary energies

$$N_{\max.} \leqslant \frac{4}{\pi}\frac{M}{\mu}\eta n^2 = 8{\cdot}49\eta n^2, \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

taking
$$\frac{\mu}{M} = 0{\cdot}15.$$

Explicit calculations of the relatives probabilities have been made for $n = 1$ and $n = 5$. $n = 1$ corresponds to the case of a collision between two nucleons, while $n = 5$ corresponds to the collision between a nucleon and a heavy nucleus.

(i) *Nucleon-Nucleon Collision:*

In this case

$$\frac{\epsilon_0}{Mc^2} = \frac{\mu}{M}\frac{1}{\eta}\left(5{\cdot}784+\eta^2+\frac{\pi^2w^2}{16}\right)^{\frac{1}{2}} \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

Calculating $\left(\frac{d\epsilon_0}{dW}\right)$ and substituting in (9), we obtain (neglecting nucleon pair creation)

$$P(2, N) = \frac{k}{Mc^2}\left(\frac{4\xi^3}{3\pi}\frac{M^3}{\mu^3}\right)^{N+1} w^{-(N+1)}\left(\frac{\epsilon_0}{Mc^2}\right)^{3N+\frac{1}{2}} \times$$
$$\times\left[\frac{3\pi^2}{16}\left(\frac{\mu}{\eta M}\right)^2 \frac{wMc^2}{\epsilon_0}(N+\tfrac{1}{2})\left(\sum_{r=0}^{2N}\frac{a_r z^{N+r+\frac{3}{2}}}{\Gamma(N+r+\frac{5}{2})}\right)+\right.$$
$$\left.+\left\{1-\frac{\pi^2}{16}\left(\frac{\mu}{\eta M}\right)^2\frac{w(w-2)}{(\epsilon_0/Mc^2)^2}\right\}\left\{\sum_{r=0}^{2N}\frac{a_r z^{N+r+\frac{1}{2}}}{\Gamma(N+r+\frac{3}{2})}\right\}\right], \quad \ldots \quad (15)$$

where
$$z = \frac{W_0}{\epsilon_0} = \frac{W-2Mc^2-N\epsilon_0}{\epsilon_0} = \frac{w-2}{(\epsilon_0/Mc^2)} - N,$$

and
$$w = \frac{W}{Mc^2} = (2\gamma+2)^{\frac{1}{2}}.$$

γMc^2 denotes the energy of the bombarding nucleon in the laboratory system. Table I shows the relative probabilities for the production of various numbers N of pions calculated from equation (15) for $\xi = 1$ and $w = 3$, 4 and 5. $\bar{N}$ is the average number of pions produced. For comparison we reproduce in Table II Fermi's results, obtained without introducing the zero-point energy.

TABLE I

γ	w	ϵ_0/Mc^2	$N = 0$	1	2	3	$\bar{N}$	$N_{max.}$
3·5	3	0·597	20	80			0·80	1·67
7·0	4	0·696	2	55	43		1·42	2·88
11·5	5	0·805	0	12	78	10	1·98	3·73

Relative probabilities for the production of N pions in a *nucleon-nucleon* collision when the zero-point energy is taken into account. $N_{max.}$ is the maximum number of pions produced, while $\bar{N}$ is the average number of pions produced during the collision.

TABLE II

γ	w	$N = 0$	1	2	3	4	5	6	7	$\bar{N}$
3·5	3	9	59	30	2					1·2
7·0	4		13	40	33	11	2			2·5
11·5	5		2	15	34	31	14	3	1	3·5

Relative probabilities for the production of N pions in a *nucleon-nucleon* collision, disregarding the zero-point energy.

(ii) *Nucleon-Nucleus Collision:*

In this case we take $n = 5$. ϵ_0 then changes inappreciably in the range $\gamma \leqslant 10$, and hence in differentiating Q_{N-1} we may regard it as constant. Carrying out the calculation, we finally obtain for the probability of a state consisting of six nucleus and N pions,

$$P(6, N) = \frac{k}{Mc^2}\left(\frac{\epsilon_0}{Mc^2}\right)^{3N+\frac{13}{2}}\left(\frac{4\xi^3}{3\pi}\frac{M^3}{\mu^3}\right)^{N+5}\left(\frac{25w}{24+w^2}\right)^{N+5}\left(\sum_{r=0}^{2N}\frac{a_r z^{N+r+\frac{13}{2}}}{\Gamma(N+r+\frac{15}{2})}\right), \qquad (16)$$

where $z = \dfrac{w-6}{(\epsilon_0/Mc^2)} - N.$

$\left(\dfrac{\epsilon_0}{Mc^2}\right)$ may be taken as

$$\frac{\epsilon_0}{Mc^2} = \frac{ch}{2(\xi\hbar/\mu c)}\frac{1}{Mc^2} = \frac{\pi}{\xi}\frac{\mu}{M} = \frac{0{\cdot}471}{\xi} \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

Table III shows the relative probabilities for the production of various numbers N of pions calculated from eqn. (16) for $w = 8$ and $w = 10$ and $\xi = 1$. In Table IV are given the relative probabilities obtained by letting $\epsilon_0 \to 0$. The last column in each table gives the average number of pions emitted.

TABLE III

γ	w	$N = 0$	1	2	3	4	5	6	$\bar{N}$
3·8	8	12	54	33	1				1·22
7·4	10			5	26	46	21	2	3·89

Relative probabilities for the production of N pions in a *nucleon-nucleus* collision, taking into account the zero-point energy; γMc^2 is the energy of the bombarding nucleon and $\bar{N}$ is the average number of pions produced.

TABLE IV

γ	w	$N = 0$	1	2	3	4	5	6	7	8	9	10	11	12	$\bar{N}$
3·5	7·8	2	12	27	30	19	8	2							2·8
7	9·8					2	6	13	20	22	18	11	5	2	8·0

Relative probabilities for the production of N pions in a *nucleon-nucleus* collision, disregarding zero-point energy.

III. In this section we return to Fermi's thermodynamic theory and show that his estimate of the total number of pions produced has appreciable error (Kothari, 1954). According to Bose-Einstein statistics, the number of particles n_i in the energy state ϵ_i is

$$n_i = \frac{1}{\frac{1}{A}e^{\epsilon_i/kT}-1},$$

where A is the degeneracy parameter. Fermi assumes that the number of pions produced in an extremely high energy collision is given by

$$N = \sum_i n_i = \sum_i \frac{1}{e^{\epsilon_i/kT}-1}, \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

where T is the temperature of the pion assembly. He replaces the sum by the integral

$$\frac{4\pi g\Omega}{(ch)^3}\int_0^\infty \frac{\epsilon^2 d\epsilon}{e^{\epsilon/kT}-1} = \frac{4\pi g\Omega}{(2\pi c\hbar)^3}\cdot 2!\ \zeta(3)(kT)^3 = N_1, \text{ say} \quad \ldots \quad \ldots \quad (19)$$

g is the weight factor of the pions (equal to 3) and Ω is the volume in which the pions are assumed to be generated. It is given by

$$\Omega = \frac{4\pi}{3}\left(\frac{\hbar}{\mu c}\right)^3 \frac{2Mc^2}{W} = \Omega_0 \frac{2Mc^2}{W}, \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

W being the total available energy in the C-system. If we write

$$N_0 = N - N_1,$$

then

$$N_0 = \alpha g(kT/\mu c^2), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

α being a numerical factor of the order of unity. $(kT/\mu c^2)$ is the first term of the sum (18) for $(\mu c^2/kT)$ small. The temperature of the pion gas is given by Stefan's Law, viz.,

$$\frac{W-2Mc^2}{\Omega} = \frac{g}{2}\frac{6\zeta(4)}{\pi^2 c^3 \hbar^3}(kT)^4 \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

Substituting for kT from this equation, we obtain from (19) and (21)

$$\frac{N_0}{N_1} = \alpha\frac{[9\pi\zeta(4)]^{\frac{1}{4}}}{2\zeta(3)}\left(\frac{g\mu}{M}\right)^{\frac{1}{4}}(wf)^{-\frac{1}{4}}, \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

where $w = \dfrac{W}{Mc^2}$, and $f = \dfrac{\Omega}{\Omega_0}$.

When $f \sim 1$, $\dfrac{N_0}{N_1} \to 0$ as $w \to \infty$.

However, the actual value of Ω assumed by Fermi is (20); hence we have

$$\frac{N_0}{N_1} = \alpha\frac{[9\pi\zeta(4)]^{\frac{1}{4}}}{2\zeta(3)}\left(\frac{g\mu}{2M}\right)^{\frac{1}{4}}$$

$$\simeq 0{\cdot}54 \text{ (for } \alpha = \tfrac{1}{2}).$$

Thus we see that the number N_0 is of the same order as N_1 and hence Fermi's expression (equation 19) seriously underestimates the total number of pions produced. In the laboratory system, the N_0 pions will form a narrow jet moving along the direction of the incident nucleon, each pion having energy $\dfrac{w}{2}\mu c^2$.

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Summary

Fermi's statistical theory of multiple pion production is extended to the case of a nucleon-nucleus collision, taking into account the zero-point energy of the pions. Numerical calculations of the relative probabilities are made. It is seen that the introduction of the zero-point energy substantially reduces the number of pions produced.

Fermi's estimate of the number of pions produced on the basis of statistical thermodynamics is shown to have appreciable error.

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