ON THE RADIAL PULSATIONS OF AN INFINITE CYLINDER WITH A MAGNETIC FIELD PARALLEL TO ITS AXIS

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1. Introduction

Chandrasekhar and Fermi (1953) deduced the general equations governing the adiabatic radial pulsations of an infinitely conducting, infinite cylinder, under its own gravity, and with a constant magnetic field parallel to its axis. They obtained an integral formula for the period of the pulsations. We have, in this note, deduced the pulsation equation for an axial field varying with the distance from the axis. An expression for the amplitude of the magnetic variation (as a consequence of the radial pulsations) in terms of the characteristic amplitudes of the pulsations is also obtained. We have obtained explicit expressions for the characteristic functions and values for the particular case in which the magnetic field is assumed to be proportional to the square root of the pressure inside the cylinder (with a finite magnetic field at the surface). The numerical calculations for six different models of such a field have been carried out. The dependence of the period of pulsations on the prevalent magnetic field is investigated. It is found to decrease with the magnetic field.

2. The Pulsation Equation

Following Chandrasekhar and Fermi, we assume the gas to be compressible, of uniform density ρ, and having an infinite electrical conductivity. The effect of the last assumption is that the lines of force shall be pushed aside along with the matter in motion, which shall result in the magnetic pulsations of the same frequency as that of the radial pulsations of the cylinder.

Imagine an element of cylinder, of radius r and unit height, to be displaced through δr. From the conservation of mass, the equation of continuity can be written as

$$\frac{d}{dm}(\pi r^2) = \frac{1}{\rho} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where $m = \pi r^2 \rho$, is the mass of the element under consideration. The equation of motion can be written as

$$\frac{\partial^2 r}{\partial t^2} = -2\pi r \frac{\partial P}{\partial m} - \frac{2Gm}{r} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where the total pressure is

$$P = p + \frac{H^2}{8\pi} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

p = gas pressure, $\frac{H^2}{8\pi}$ = magnetic pressure, and H being the magnetic field at the distance r from the centre. Let δP and $\delta\rho$ denote the corresponding changes in the total pressure, density and the magnetic field at the distance r from the axis.

Chandrasekhar and Fermi gave the following equations governing the radial oscillations of small amplitude

$$\frac{\partial}{\partial m}(2\pi r\,\delta r) = -\frac{\delta\rho}{\rho^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

and

$$\frac{\partial^2}{\partial t^2}\,\delta r = -2\pi r\frac{\partial}{\partial m}\,\delta P + \frac{4Gm}{r^2}\,\delta r. \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

From equation (3)

$$\delta P = \delta p + \frac{H.\delta H}{4\pi}$$

$$= \frac{\gamma p}{\rho}\delta\rho + \frac{H.\delta H}{4\pi} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where γ is the ratio of specific heats. It is, however, assumed to be constant within the cylinder.

In a medium of infinite electric conductivity, the change δH in the magnetic field, as we follow the motion, is given by

$$\delta H = \text{curl}\,(\delta r \times H) + (\delta r.\,\text{grad})H \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

Since H is in the z-direction and δr is radial, the only non-vanishing component of δH is

$$\delta H_z = -\frac{1}{r}\frac{\partial}{\partial r}(Hr\,\delta r) + \delta r.\frac{\partial H}{\partial r}$$

$$= -\frac{H}{r}\frac{\partial}{\partial r}(r\,\delta r) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

and

$$\frac{H.\delta H}{4\pi} = -\frac{H^2}{4\pi r}\frac{\partial}{\partial r}(r\,\delta r) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

Substituting from (9) in (6), we get

$$\delta P = \frac{\gamma p}{\rho}\,\delta\rho - \frac{H^2}{4\pi r}\frac{d}{dr}(r\,\delta r) \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

which on substitution in (5) yields

$$\frac{d^2}{dt^2}\,\delta r = -\frac{1}{\rho}\frac{d}{dr}\left[\frac{\gamma p}{\rho}\delta\rho - \frac{H^2}{4\pi r}\frac{d}{dr}(r\,\delta r)\right] + \frac{4Gm}{r^2}\,\delta r.$$

With the help of equations (3) and (4), it reduces to

$$\left(\frac{d^2}{dt^2} - \frac{4Gm}{r^2}\right)\delta r = \frac{1}{\rho}\frac{d}{dr}\left[\frac{1}{r}\left\{\gamma P + \frac{H^2}{8\pi}(2-\gamma)\right\}\frac{d}{dr}(r\,\delta r)\right]$$

or since $m = \pi r^2\rho$

$$\left(\frac{d^2}{dt^2} - 4\pi G\rho\right)\delta r = \frac{1}{\rho}\frac{d}{dr}\left[\frac{1}{r}\left\{\gamma P + \frac{H^2}{8\pi}(2-\gamma)\right\}\frac{d}{dr}(r\,\delta r)\right] \quad \ldots \quad \ldots \quad (11)$$

The gravitational pressure, at a distance r from the axis, is given by

$$P = -\int_r^R \rho\,.\frac{2Gm}{r}\,dr = \pi G\rho^2(R^2 - r^2) \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

If we assume that all the physical variables vary with time as $e^{i\omega t}$, then equation (11), with the help of equation (12), will reduce to

$$(\omega^2+4\pi G\rho)\delta r = -\frac{1}{\rho}\frac{d}{dr}\left[\left\{\gamma\pi G\rho^2(R^2-r^2)+\frac{H^2}{8\pi}(2-\gamma)\right\}\frac{1}{r}\frac{d}{dr}(r\,\delta r)\right] \quad .. \quad (13)$$

For convenience, we put

$$A = \frac{1}{\gamma}\left(\frac{\omega^2}{\pi G\rho}+4\right) \text{ and } f = \left(\frac{2}{\gamma}-1\right)\cdot\frac{H^2}{8\pi^2R^2\rho^2G} \quad .. \quad .. \quad (14)$$

and introduce new variables

$$x = \frac{r}{R}, \Psi = \frac{\delta r}{R} \quad .. \quad .. \quad .. \quad .. \quad (15)$$

The differential equation (13) takes the form

$$A\Psi = -\frac{d}{dx}\left\{(1-x^2+f)\frac{1}{x}\frac{d}{dx}(x\Psi)\right\} \quad .. \quad .. \quad .. \quad (16)$$

Let us define a function Φ such that

$$\frac{d\Phi}{dx} = \Psi \quad .. \quad .. \quad .. \quad .. \quad .. \quad (17)$$

then the equation (16), on integration, gives

$$A\Phi = -(1-x^2+f)\frac{1}{x}\frac{d}{dx}\left(x\frac{d\Phi}{dx}\right) \quad .. \quad .. \quad .. \quad (18)$$

The constant of integration is zero, because

$$\text{at } x = 0, \frac{d\Phi}{dx} = 0 \quad .. \quad .. \quad .. \quad .. \quad (19)$$

The equation (18) is independent of the manner of variation of H.

3. Expression of δH in terms of x and Ψ

Using the substitutions (15), the equation (8) can be written as:—

$$\delta H_z = -\frac{H}{x}\frac{d}{dx}(x\Psi) \quad .. \quad .. \quad .. \quad .. \quad (20)$$

4. Characteristic Values for the Magnetic Field proportional to the square root of Pressure

We assume that the magnetic field is proportional to the square root of pressure, such that

$$H^2 = H_s^2+(H_0^2-H_s^2)(1-x^2) \quad .. \quad .. \quad .. \quad (21)$$

The values of the magnetic field at the centre and surface being given by

$$H = H_0 \text{ at } x = 0; \quad H = H_s \text{ at } x = 1.$$

Then

$$f = \frac{\left(\frac{2}{\gamma}-1\right)}{8\pi^2 G\rho^2 R^2}\left\{H_s^2+(H_0^2-H_s^2)(1-x^2)\right\}$$

$$\left.\begin{aligned} &= C+B(1-x^2) \\ \text{where} \quad C &= \frac{H_s^2\left(\frac{2}{\gamma}-1\right)}{8\pi^2 G\rho^2 R^2} \\ \text{and} \quad B &= \frac{(H_0^2-H_s^2)\left(\frac{2}{\gamma}-1\right)}{8\pi^2 G\rho^2 R^2}\end{aligned}\right\} \quad \ldots \quad (22)$$

Let us assume that the solution of the equation (20) is of the series form

$$\Phi = \sum_{k=0}^{\infty} a_k \, . \, x^{k+\alpha} \qquad \ldots \quad (23)$$

The indicial equation

$$(C+1+B)x^2 a_0 = 0$$

gives $\alpha = 0$, so that

$$\Phi = \sum_{k=0}^{\infty} a_k x^k$$

is the solution of equation (20). We obtain the following recurrence formula for the coefficients:—

$$\begin{aligned}\frac{a_{k+2}}{a_k} &= \frac{k^2}{(k+2)^2}\cdot\frac{1+B}{1+B+C}\left[1-\frac{A}{(1+B)k^2}\right] \\ &= \frac{k^2}{(k+2)^2}\cdot L\left[1-\frac{A}{(1+B)k^2}\right] \qquad \ldots \quad (24)\end{aligned}$$

where

$$L = \frac{1+B}{1+B+C}$$

The coefficients a_1, a_3, a_5, a_7, etc., are all zero.
It is clear from the recurrence relation (24), that the series terminates for

$$\begin{aligned} A &= (1+B)k^2; \quad k = 0, \text{ or even} \\ &= 4j^2(1+B), \quad j = \text{integer or zero.} \qquad \ldots \quad (25)\end{aligned}$$

Normalization of the characteristic values and functions

If the n-th characteristic function is denoted by ϕ_n, then the corresponding amplitude ψ_n is given by

$$\psi_n = \frac{d\phi_n}{dx} \qquad \ldots \quad (26)$$

subject to the orthogonality relation

$$\int_0^1 \psi_j \psi_k x \, dx = 0 \qquad \ldots \quad (27)$$

Partially integrating and remembering that $A = 4j^2(1+B)$ we obtain

$$\int_0^1 \psi_j \psi_k x\, dx = 4j^2 L \int_0^1 \frac{\phi_j \phi_k}{1-Lx^2} x\, dx \quad \ldots \quad (28)$$

since this formula must be valid even if j and k are interchanged, the integrals vanish if $j \neq k$. We choose the arbitrary factor in the characteristic functions so that we have

$$\left.\begin{aligned} &\int_0^1 \psi_j \psi_k x\, dx = 4j^2 L \int_0^1 \frac{\phi_j \phi_k}{1-Lx^2} x\, dx = \delta_{jk} \\ \text{where} \quad &\delta_{jk} = 1 \quad \text{for} \quad j = k \\ &\quad\;\; = 0 \quad \text{for} \quad j \neq k \end{aligned}\right\} \quad \ldots \quad (29)$$

Characteristic functions normalized in this fashion are :

$$\left.\begin{aligned} \phi_1 &= \frac{1}{\sqrt{L(2-L)}}(1-Lx^2) \\ \phi_2 &= \frac{1-4Lx^2+3L^2x^4}{\sqrt{2L(4-14L+20L^2-9L^3}} = \frac{(1-3Lx^2)(1-Lx^2)}{\sqrt{2L(4-14L+20L^2-9L^3}} \\ \phi_3 &= \frac{1-9Lx^2+18L^2x^4-10L^3x^6}{\sqrt{3L(6-51L+200L^2-366L^3+312L^4-100L^5}} \end{aligned}\right\} \quad \ldots \quad (30)$$

. . . .

while the corresponding amplitudes are:

$$\left.\begin{aligned} \psi_1 &= \frac{1}{\sqrt{L(2-L)}}(-2Lx) \\ \psi_2 &= \frac{1}{\sqrt{2L(4-14L+20L^2-9L^3}}(-8Lx+12L^2x^3) \\ \psi_3 &= \frac{(-18Lx+72L^2x^3-60L^3x^5)}{\sqrt{3L(6-51L+200L^2-366L^3+312L^4-100L^5)}} \end{aligned}\right\} \quad \ldots \quad (31)$$

5. Discussion

(a) Dependence of period of pulsations on the magnetic field.

From equations (14) and (15), we have

$$\frac{1}{\gamma}\left(\frac{\omega^2}{\pi G\rho}+4\right) = A_n = 4n^2(1+B),$$

$$B = \frac{(H_0^2-H_s^2)\left(\dfrac{2}{\gamma}-1\right)}{8\pi^2 G\rho^2 R^2}$$

which clearly shows that cylinder is stable for radial oscillations.

If T denotes the period in the presence of the magnetic field, and T_0 without the magnetic field, then it follows that

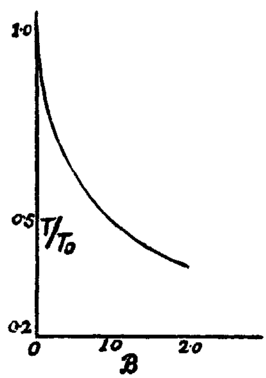
$$\frac{T}{T_0} = \sqrt{\frac{\gamma-1}{\gamma(1+B)-1}} \quad \ldots \quad (32)$$

This relation shows that the time period decreases with magnetic field. The Table I and figure 1 exhibit the variation of $\frac{T}{T_0}$ with B for $\gamma = 1{\cdot}5$.

TABLE I

The Dependence of Period on Magnetic Field

B	0·0	0·5	1·0	1·5	2·0
T/T_0	1·0	0·6325	0·5000	0·4372	0·3780



TEXT-FIG. 1.

(b) The amplitudes of radial pulsations

In Table III are listed the calculated values of the amplitudes ψ, corresponding to the following six different models of the magnetic field:—

Model I. $H_0 = 0$, $H_s = 0$; which corresponds to the case of zero magnetic field.

Model II. $H_s = 0$, $H^2 = H_0^2\,(1-x^2)$; which corresponds to the case 'magnetic field proportional to the square root of pressure inside the cylinder, but zero at the surface'.

Model III. $H_s = H_0$; which corresponds to the case of uniform field.

Model IV, Model V, Model VI: $H^2 = H_s^2+(H_0^2-H_s^2)(1-x^2)$; with $H_s = 10^{-1}\,H_0$, $H_s = 10^{-2}\,H_0$, $H_s = 10^{-3}\,H_0$ respectively.

The Models IV, V, VI correspond to the case 'magnetic field proportional to the square root of pressure within the cylinder but finite at the surface'.

The values of ψ change linearly with x (as shown in figure 2).

TABLE II
Values of L

b \ H_s/H_0	1·0	0·1	0·01	0·001
0·0	1·000	1·0000	1·000000	1·00000000
1·0	0·500	0·9950	0·999950	0·99999950
1·5	0·400	0·9940	0·999940	0·99999940
3·0	0·250	0·9925	0·999925	0·99999925

TABLE III
The Values of Characteristic Amplitudes
Mode I ($n = 1$)

Model	$H = 0$	$H_s = 0$	$H_s/H_0 = 1$		
x \ L	1	1	0·5	0·4	0·25
0·00	0·00	0·00	0·00	0·00	0·00
0·20	−0·40	−0·40	−0·2309	−0·20	−0·1511
0·40	−0·80	−0·80	−0·4619	−0·40	−0·3023
0·60	−1·20	−1·20	−0·6928	−0·60	−0·4535
0·80	−1·60	−1·60	−0·9238	−0·80	−0·6047
1·0	−2·00	−2·00	−1·1547	−1·00	−0·7559

TABLE III—(*contd.*)

Model	$H_s/H_0 = 10^{-1}$			$H_s/H_0 = 10^{-2}$		
x \ L	0·995	0·994	0·9925	0·999950	0·999940	0·9999925
0·00	0·0	0·0	0·0	0·0	0·0	0·0
0·20	−0·3980	−0·3976	−0·3970	−0·3999	−0·3999	−0·3999
0·40	−0·7960	−0·7952	−0·7940	−0·7999	−0·7999	−0·7999
0·60	−1·1940	−1·1928	−1·1910	−1·1999	−1·1999	−1·1999
0·80	−1·5920	−1·5904	−1·5880	−1·5999	−1·5999	−1·5998
1·00	−1·9900	−1·9880	−1·9850	−1·9999	−1·9998	−1·9998

TABLE III—(*contd.*)

Model	$H_s/H_0 = 10^{-3}$		
x \ L	0·99999950	0·99999940	0·99999925
0·00	0·0	0·0	0·0
0·20	−0·3999	−0·3999	−0·3999
0·40	−0·7999	−0·7999	−0·7999
0·60	−1·1999	−1·1999	−1·1999
0·80	−1·5999	−1·5999	−1·5999
1·00	−1·9999	−1·9999	−1·9999

TABLE III—(*contd.*)

Mode II ($n = 2$)

Model	$H = 0$	$H_s = 0$	$H_s/H_0 = 1$		
L / x	1	1	0·5	0·4	0·25
0·00	0·0	0·0	0·0	0·0	0·0
0·20	−1·0634	−1·0634	−2·6233	−2·1824	−1·3889
0·40	−1·7197	−1·7970	−4·7598	−4·0428	−2·6509
0·60	−1·5613	−1·5613	−5·9228	−5·2592	−3·6591
0·80	−0·1810	−0·1810	−5·6253	−5·5096	−4·2866
1·00	−2·8284	+2·8284	−3·3806	−4·4721	−4·4065

TABLE III—(*contd.*)

Model	$H_s/H_0 = 10^{-1}$			$H_s/H_0 = 10^{-2}$		
L / x	0·995	0·994	0·9925	0·999950	0·999940	0·999925
0·00	0·0	0·0	0·0	0·0	0·0	0·0
0·20	−1·0586	−1·0576	−1·0563	−1·0634	−1·0634	−1·0634
0·40	−1·7139	−1·7128	−1·7113	−1·7196	−1·7196	−1·7196
0·60	−1·5627	−1·5630	−1·5636	−1·5613	−1·5611	−1·5613
0·80	−0·2017	−0·2058	−0·2120	−0·1812	−0·1812	−0·1813
1·00	+2·7723	+2·7612	+2·7448	+2·8278	+2·8277	+2·8276

TABLE III—(*contd.*)

Model	$H_s/H_0 = 10^{-3}$		
L / x	0·99999950	0·99999940	0·99999925
0·00	0·0	0·0	0·0
0·20	−1·0634	−1·0634	−1·0634
0·40	−1·7196	−1·7196	−1·7196
0·60	−1·5612	−1·5612	−1·5612
0·80	−0·1810	−0·1810	−0·1810
1·00	+2·8284	+2·8284	+2·8283

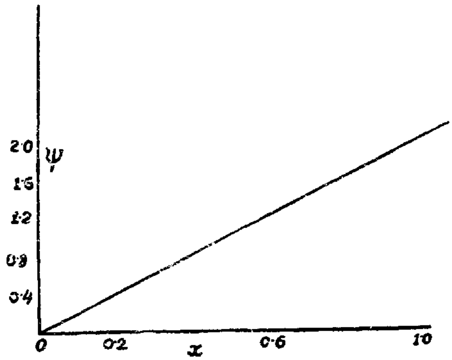
TABLE IV

Values of $\delta H/H$ at the surface ($x = 1$)

Model	$H = 0$	$H_s = 0$	$H_s/H_0 = 1$			$H_s/H_0 = 10^{-1}$		
$L \rightarrow$ / $\downarrow n$	1	1	0·50	0·40	0·25	0·9950	0·9940	0·9925
1	4·00	4·00	2·3094	0·20	0·1512	3·9800	3·9760	3·9701
2	−22·6276	−22·6276	−16·0750	−4·4721	+3·5252	−22·3477	−22·2923	−22·2115

TABLE—IV (*contd.*)

Model	$H_s/H_0 = 10^{-2}$			$H_s/H_0 = 10^{-3}$		
$L \rightarrow$ / $\downarrow n$	0·999950	0·999940	0·999925	0·99999950	0·99999940	0·99999925
1	3·9998	3·9997	3·9997	3·999998	3·999998	3·999997
2	−22·6248	−22·6242	−22·6234	−22·62758	−22·62756	−22·62755



TEXT-FIG. 2.

(c) *The amplitude of the magnetic pulsations*

The equation (20) can be rewritten as

$$\frac{\delta H}{H} = -\frac{1}{x}\frac{d}{dx}(x\psi) = \left(-\frac{\psi}{x}+\frac{d\psi}{dx}\right)$$

For $n = 1$

$$\frac{\delta H}{H} = \frac{4L}{\sqrt{L(2-L)}}$$

which is independent of x.

For $n = 2$

$$\frac{\delta H}{H} = -2\left[\psi+\frac{24L^2x^2}{\sqrt{2L(4-14L+20L^2-9L^3)}}\right]$$

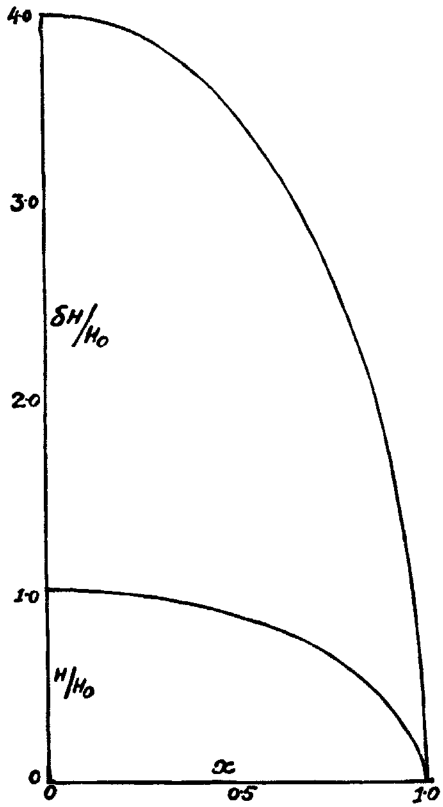
which depends both on x and L.

In Table IV, we give the evaluated values of $\frac{\delta H}{H}$ for $x = 1$ for all the models, whereas in Table V, we calculate H/H_0 and $\delta H/H_0$ for several values. of x (only, for $n = 1$). Figures (3) and (4) demonstrate the respective nature of variations.

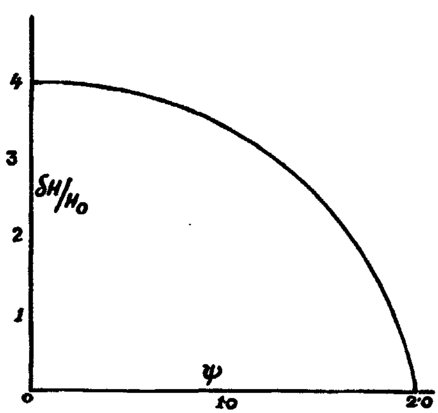
TABLE V

The Amplitude of Magnetic Pulsations

	x	ψ	H/H_0	$\delta H/H_0$
1	0·2	−0·4000	0·980	3·919
2	0·4	−0·8000	0·916	3·666
3	0·6	−1·1999	0·800	3·200
4	0·8	−1·5999	0·600	2·400
5	0·9	−1·7999	0·436	1·744
6	1·0	−1·9999	0·010	0·040



TEXT-FIG. 3.



TEXT-FIG. 4.

Abstract

The expression for the amplitudes of the radial adiabatic pulsations and consequent magnetic variations of an infinitely conducting, infinite cylinder, subject to a variable axial field, are obtained. The numerical calculations of the characteristic amplitudes are carried out for six different models of the particular case in which the magnetic field is assumed to be proportional to the square root of pressure. Further, the period of pulsation is found to decrease with the magnetic field.

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Reference

Chandrasekhar, S., and Fermi, E. (1953). Problems of gravitational stability in the presence of a magnetic field. *Ap. J.*, **118**, 116.

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