ON THE RADIAL ADIABATIC PULSATIONS OF AN INFINITE CYLINDER IN THE PRESENCE OF MAGNETIC FIELD PARALLEL TO ITS AXIS—II

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1. INTRODUCTION

S. Chandrasekhar and E. Fermi (1953) deduced the general equations governing the radial adiabatic pulsations of an infinite cylinder of compressible medium of infinite conductivity under its own gravity, with a constant magnetic field parallel to its axis. They obtained an integral formula for the period of pulsations, viz.

$$\omega^2 = \int_0^M r^2\,dm = 4\{(\gamma-1)+\mathbf{M}\} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where

ω = frequency of pulsations.
$\gamma = C_p/C_v$ (the ratio of the specific heats at constant pressure and at constant volume).
r = radial distance from the axis.
$m = \pi r^2 \rho$ (mass per unit length of the cylinder, interior to r).
M = Mass per unit length of the cylinder.
$\mathbf{M}$ = total magnetic energy.
$\mathbf{U}$ = heat energy.

E. Lytkens (1954) obtained the explicit solutions of the pulsation equation for the case where there is prevalent a magnetic field proportional to the square root of pressure, and vanishing at the surface. K. P. Chopra and S. P. Talwar (1955) have shown in a recent note, hereinafter called Paper I, that the amplitude equation, viz.

$$A\psi = -\frac{d}{dx}\left\{(1-x^2+f)\frac{1}{x}\frac{d}{dx}(x\psi)\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where $\psi = \delta r/R\,;\quad x = r/R$

and $$f = \frac{H^2}{8\pi^2 G\rho^2 R^2}\left(\frac{2}{\gamma}-1\right)$$

holds in general for any field. The corresponding equation in terms of the characteristic functions ϕ $\left(\text{defined by } \psi = \frac{d\phi}{dx}\right)$ is

$$A\phi = -(1-x^2+f)\frac{1}{x}\frac{d}{dx}\left(x\frac{d\phi}{dx}\right) \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

They further obtained the explicit solutions of the pulsation equation in the special case where the variation of the magnetic field with distance from the axis is given by

$$H^2 = (H_0^2-H_s^2)(1-x^2)+H_s^2 \qquad \text{(4)}$$

where H_s = field at the surface of the cylinder (i.e. at $x = 1$),
H_0 = field along the axis of the cylinder (i.e. at $x = 0$).

The mechanical and magnetic pulsations occur in phase (on account of infinite conductivity) with a frequency ω given by

$$A = \frac{1}{\gamma}\left(\frac{\omega^2}{\pi G\rho}+4\right) \qquad \text{(5)}$$

The frequency parameter A can take up the values

$$A = 4j^2(1+B)\,; \quad j = 1, 2, 3, \ldots \qquad \text{(6)}$$

where

$$B = \frac{H_0^2-H_s^2}{8\pi^2 G\rho^2 R^2}. \qquad \text{(7)}$$

The characteristic functions ϕ's and their amplitudes ψ's satisfy the normality and orthogonality relations.

$$\int_0^1 \psi_j\psi_k\, x\, dx = 4j^2(1+B)\int_0^1 \frac{\phi_j\phi_k}{1-x^2+f}\, x\, dx = \delta_{jk} \qquad \text{(8)}$$

where δ_{jk} is the Kronecker delta, satisfying the properties

$$\left.\begin{aligned} \delta_{jk} &= 1 \quad \text{for } j = k \\ \delta_{jk} &= 0 \quad \text{for } j \neq k. \end{aligned}\right\} \qquad \text{(9)}$$

The characteristic functions normalized in this way are:

$$\left.\begin{aligned} \phi_1 &= \frac{1-Lx^2}{\sqrt{L(2-L)}} \\ \phi_2 &= \frac{(1-Lx^2)(1-3Lx^2)}{\sqrt{2L(4-14L+20L^2-9L^3)}} \\ \phi_3 &= \frac{(1-Lx^2)(1-8Lx^2+10L^2x^4)}{\sqrt{3L(6-51L+200L^2-366L^3+312L^4-100L^5)}} \end{aligned}\right\} \qquad \text{(10)}$$

with their characteristic amplitudes,

$$\left.\begin{aligned} \psi_1 &= \frac{-2Lx}{\sqrt{L(2-L)}} \\ \psi_2 &= \frac{-8Lx+12L^2x^3}{\sqrt{2L(4-14L+20L^2-9L^3)}} \\ \psi_3 &= \frac{-18Lx+72L^2x^3-60L^3x^5}{\sqrt{3L(6-51L+200L^2-366L^3+312L^4-100L^5)}} \end{aligned}\right\} \qquad \text{(11)}$$

where

$$L = \frac{1+B}{1+B+C} \qquad \text{(12)}$$

2. Formulation of the Problem

In this note, called Paper II, I will make use of the well-known Rietz method of evaluating Eigen-values in calculating the higher approximations to the characteristic function A, and the contributory influence of the other modes corresponding to the values of $j = 1, 2, 3,$

In general, it is reasonable to put

$$\Phi = a_1\phi_1 + a_2\phi_2 + a_3\phi_3 + \ldots + a_n\phi_n \qquad (13)$$

where $a_1, a_2, a_3, \ldots, a_n$ are constants, to be chosen suitably. The expression (13) gives a very good approximation to the characteristic functions. The corresponding expression for the amplitudes is

$$\Psi = a_1\psi_1 + a_2\psi_2 + a_3\psi_3 + \ldots + a_n\psi_n \qquad (14)$$

Let $$x\Psi = \chi. \qquad (15)$$

Then we can write (2) in the form:

$$\frac{A}{x}\chi = -\frac{d}{dx}\left\{(1-x^2+f)\frac{1}{x}\frac{d\chi}{dx}\right\} \qquad (16)$$

while the characteristic functions of this equation are given by

$$\chi = a_1 x\psi_1 + a_2 x\psi_2 + a_3 x\psi_3 + \ldots + a_n x\psi_n. \qquad (17)$$

3. The Characteristic Function A

Let us form the quotient,

$$A = -\frac{\int_0^1 \chi \frac{d}{dx}\left\{(1-x^2+f)\frac{1}{x}\frac{d\chi}{dx}\right\} dx}{\int_0^1 \frac{1}{x}\chi^2\, dx} \qquad (18)$$

Substituting for χ from (17) and differentiating with respect to a_j we get

$$\frac{dA}{da_j} = \frac{\left[C\sum_{k=1}^{n} a_k\left\{\psi_j\frac{d}{dx}(x\psi_k)+\psi_k\frac{d}{dx}(x\psi_j)\right\}\Big|_0^1 - 2\sum_{k=1}^{n} a_k\int_0^1 (1-x^2+f)\frac{1}{x}\times \times\frac{d}{dx}(x\psi_j)\cdot\frac{d}{dx}(x\psi_k)\,dx - 2A\sum_{k=1}^{n} a_k\int_0^1 \psi_k\psi_j x\,dx\right]}{\sum_{j=1}^{n}\sum_{k=1}^{n}\int_0^1 a_j a_k\psi_j\psi_k x\,dx} \qquad (19)$$

If A is to be minimum, then this expression should vanish.

Since ψ's are normalized and orthogonal, equation (19) reduces to

$$Aa_j + \tfrac{1}{2}C\sum_{k=1}^{n} a_k\left\{\psi_j\frac{d}{dx}(x\psi_k)+\psi_k\frac{d}{dx}(x\psi_j)\right\}\Big|_0^1 - \sum_{k=1}^{n} a_k\int_0^1 (1-x^2+f)\frac{1}{x}\frac{d}{dx}(x\psi_j)\frac{d}{dx}(x\psi_k)\,dx = 0. \qquad (20)$$

On replacing ψ by the derivative of ϕ we have

$$Aa_j+\tfrac{1}{2}C\sum_{k=1}^{n}a_k\left\{\psi_j\frac{d}{dx}\left(x\frac{d\phi_k}{dx}\right)+\psi_k\frac{d}{dx}\left(x\frac{d\phi_j}{dx}\right)\right\}\Bigg|_0^1$$

$$-\sum_{k=1}^{n}a_k\int_0^1(1-x^2+f)\frac{1}{x}\frac{d}{dx}\left(x\frac{d\phi_j}{dx}\right)\frac{d}{dx}\left(x\frac{d\phi_k}{dx}\right)dx=0. \quad .. \quad (21)$$

With the help of the equation (3), the second and the third terms in (21) can be reduced to:

$$\tfrac{1}{2}C\sum_{k=1}^{n}a_k\left\{\psi_j\frac{d}{dx}\left(x\frac{d\phi_k}{dx}\right)+\psi_k\frac{d}{dx}\left(x\frac{d\phi_j}{dx}\right)\right\}\Bigg|_{x=0}^{1}$$

$$=-2C\sum_{k=1}^{n}a_k\left[k^2\frac{\psi_j\phi_k x}{1-x^2}+j^2\psi_k\phi_j\frac{x}{1-x^2}\right]\Bigg|^{x=1}$$

(provided $C < B$)

and $$-\sum_{k=1}^{n}a_k\int_0^1(1-x^2+f)\frac{1}{x}\frac{d}{dx}\left(x\frac{d\phi_j}{dx}\right)\frac{d}{dx}\left(x\frac{d\phi_k}{dx}\right)=-4j^2(1+B)\,a_j$$

so that (21) can be written as:

$$\{A-4j^2(1+B)\}\,a_j-2C\sum_{k=1}^{n}a_k\left\{k^2\frac{\psi_j\phi_k}{1-x^2}x+j^2\frac{\psi_k\phi_j x}{1-x^2}\right\}\Bigg|_{x=1}=0. \quad .. \quad (22)$$

In Table I are given the values of the two terms within the parenthesis of the second term in (22)

TABLE I

j	k	$k^2\psi_j\phi_k\dfrac{x}{1-x^2}\Big\vert_{x=1}$	$j^2\psi_k\phi_j\dfrac{x}{1-x^2}\Big\vert_{x=1}$
1	1	-2	-2
1	2	$8\sqrt{2}$	$2\sqrt{2}$
1	3	$-18\sqrt{3}$	$-2\sqrt{3}$
2	1	$2\sqrt{2}$	$8\sqrt{2}$
2	2	-16	-16
2	3	$18\sqrt{6}$	$8\sqrt{6}$
3	1	$-2\sqrt{3}$	$-18\sqrt{3}$
3	2	$8\sqrt{6}$	$18\sqrt{6}$
3	3	-54	-54

Substituting the corresponding values from Table I (for $j = 1, 2, 3$; $k = 1, 2, 3$) in the equation (21) we get the following linear homogeneous equations for the coefficients a_1, a_2, a_3:—

For $j = 1$

$$\{A-4(1+B-2C)\}a_1-20\sqrt{2}Ca_2+40\sqrt{3}Ca_3 = 0$$

For $j = 2$

$$-20\sqrt{2}Ca_1+\{A-16(1+B-4C)\}a_2-52\sqrt{6}Ca_3 = 0 \quad \ldots \quad (23)$$

For $j = 3$

$$40\sqrt{3}Ca_1-52\sqrt{6}Ca_2+\{A-36(1+B-6C)\}a_3 = 0$$

If only the first characteristic function was retained

$$A_1 = 4(1+B-2C)$$

If only the second characteristic function was retained

$$A_2 = 16(1+B-4C). \quad \ldots \quad \ldots \quad (24)$$

If only the third characteristic function was retained

$$A_3 = 36(1+B-6C).$$

The exact values of A_1, A_2, A_3 (when the influence of the other two is also taken into account) are obtained by solving the determinant:

$$\begin{vmatrix} A-4(1+B-2C) & -20\sqrt{2}C & 40\sqrt{3}C \\ -20\sqrt{2}C & A-16(1+B-4C) & -52\sqrt{6}C \\ 40\sqrt{3}C & -52\sqrt{6}C & A-36(1+B-6C) \end{vmatrix} = 0. \quad \ldots \quad (25)$$

On expansion the determinant yields a cubic in A

$$A^3-(\alpha+\beta+\gamma)A^2+\{(\alpha\beta+\beta\gamma+\gamma\alpha)-21824C^2\}A-$$
$$-\{\alpha\beta\gamma-32(507\alpha+150\beta+25\gamma+15600C)C^2\} = 0 \quad \ldots \quad (26)$$

where

$$\left.\begin{aligned} \alpha &= 4(1+B-2C) \\ \beta &= 16(1+B-4C) \\ \gamma &= 36(1+B-6C) \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

Let the roots of (26) be α', β', γ', such that

$$\left.\begin{aligned} \alpha' &= \alpha+\epsilon_1 \\ \beta' &= \beta+\epsilon_2 \\ \gamma' &= \gamma+\epsilon_3 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

where ϵ_1, ϵ_2, ϵ_3 remain to be determined. But a cubic whose roots are α', β', γ' is

$$A^3-(\alpha'+\beta'+\gamma')A^2+(\alpha'\beta'+\beta'\gamma'+\gamma'\alpha')A-\alpha'\beta'\gamma' = 0. \quad \ldots \quad \ldots \quad (29)$$

Comparison of (26) and (29) yields, with the help of (23) and (24),

$$\left.\begin{aligned} \epsilon_1 &= -217\frac{C^2}{1+B} \\ \epsilon_2 &= -744\frac{C^2}{1+B} \\ \epsilon_3 &= 961\frac{C^2}{1+B} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

Thus the exact values of the characteristic functions are:

$$\left.\begin{aligned} A_1 &= 4\left(1+B-2C-54\,\frac{C^2}{1+B}\right) \\ A_2 &= 16\left(1+B-4C-46{\cdot}5\,\frac{C^2}{1+B}\right) \\ A_3 &= 36\left(1+B-6C+27\,\frac{C^2}{1+B}\right) \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (31)$$

In Table II are given the values of A_1, A_2 and A_3 for two different models: (1) $C/B = 10^{-2}$; (2) $C/B = 10^{-4}$.

TABLE II

Characteristic Functions

B	$C/B = 10^{-2}$			$C/B = 10^{-4}$		
	A_1	A_2	A_3	A_1	A_2	A_3
0·20	4·7832	19·0695	42·7712	4·7991	19·1962	43·1956
0·40	5·5655	22·1355	49·5470	5·5972	22·3889	50·3913
0·60	6·3471	25·1992	56·3258	6·3946	25·5794	57·5870
0·80	7·1282	28·2615	63·1065	7·1916	28·7684	64·7827
1·00	7·9091	31·3228	69·8886	7·9883	31·9564	71·9784
1·20	8·7898	34·3833	74·0796	8·7848	35·1436	79·1740
1·40	9·4702	37·4432	83·4553	9·4821	38·3302	86·3697
1·60	10·2506	40·5027	90·2397	10·2783	41·5265	93·5654
1·80	11·0308	43·5617	97·0246	11·0744	44·7122	100·7251
2·00	11·8110	46·6208	103·8096	11·8704	47·8880	107·9568
2·20	12·5908	49·6784	110·2996	12·3680	51·0723	115·1524
2·40	13·4712	52·7379	117·3805	13·4622	54·2585	122·3481
2·60	14·1512	55·7963	124·1665	14·1551	57·4536	129·5438
2·80	14·9312	59·8545	130·9525	14·9549	60·6285	136·7395
3·00	15·7111	61·9126	137·7387	15·7507	63·8134	143·9352

Substituting the values of A from (31) in (23) we obtain, for C, sufficiently less than B, the values of ratios of coefficients:—

For the first characteristic function:

$$\frac{a_2}{a_1} = -2{\cdot}357\,\frac{C}{1+B}; \quad \frac{a_3}{a_1} = 2{\cdot}165\,\frac{C}{1+B}.$$

For the second characteristic function:

$$\frac{a_1}{a_2} = 2{\cdot}357\,\frac{C}{1+B}, \quad \frac{a_3}{a_2} = -6{\cdot}212\,\frac{C}{1+B}.$$

For the third characteristic function:

$$\frac{a_2}{a_3} = 6{\cdot}368\,\frac{C}{1+B}, \quad \frac{a_1}{a_3} = -2{\cdot}165\,\frac{C}{1+B}.$$

In Table III the calculated values of the ratios of coefficients for the three characteristic functions are given.

TABLE III

B	First Characteristic Function		Second Characteristic Function		Third Characteristic Function	
	a_2/a_1	a_3/a_1	a_1/a_2	a_3/a_2	a_1/a_3	a_2/a_3
0·20	−0·0039	0·0036	0·0039	−0·0103	0·0106	−0·0039
0·40	−0·0067	0·0061	0·0067	−0·0177	0·0181	−0·0067
0·60	−0·0088	0·0081	0·0088	−0·0232	0·0238	−0·0088
0·80	−0·0104	0·0096	0·0104	−0·0276	0·0282	−0·0104
1·00	−0·0117	0·0108	0·0117	−0·0310	0·0318	−0·0117
1·20	−0·0128	0·0118	0·0128	−0·0338	0·0347	−0·0128
1·40	−0·0137	0·0126	0·0137	−0·0362	0·0371	−0·0137
1·60	−0·0145	0·0133	0·0145	−0·0382	0·0391	−0·0145
1·80	−0·0151	0·0139	0·0151	−0·0399	0·0409	−0·0151
2·00	−0·0157	0·0144	0·0157	−0·0414	0·0424	−0·0157
2·20	−0·0163	0·0150	0·0163	−0·0430	0·0441	−0·0163
2·40	−0·0166	0·0152	0·0166	−0·0438	0·0449	−0·0166
2·60	−0·0170	0·0156	0·0170	−0·0448	0·0459	−0·0170
2·80	−0·0173	0·0159	0·0173	−0·0457	0·0469	−0·0173
3·00	−0·0176	0·0162	0·0176	−0·0465	0·0477	−0·0176

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ABSTRACT

The radial adiabatic pulsations of an infinite cylinder with a prevalent magnetic field had already been considered in a previous paper. In this note the higher approximations to the characteristic functions and the contributions of other modes are dealt with.

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