A NOTE ON THE (RELATIVISTIC) STATISTICAL MECHANICS OF AN ASSEMBLY IN MASS-MOTION

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The present paper deals with a relativistic study of the Statistical Mechanics of an ideal gaseous assembly in mass-motion. In order to determine the distribution appropriate to the laboratory system S, we may proceed by introducing the conservation of the net linear momentum $\vec{P}$ of the assembly as an additional constraint besides the usual constraints of the total number N of the particles and the total energy E. It is shown in the first section that the distribution law so derived is formally the same as an observer in the rest system S^0 would obtain without introducing the momentum-conservation (since $\vec{P} = 0$ in S^0): in fact, the one follows from the other by applying the Lorentz transformations for the energy of the particle, the components of its momentum and the temperature of the assembly. The second section deals with the derivation of the equations connecting the macroscopic quantities referring to the two systems of co-ordinates. It is thereby shown that it is the quantities $\left[\vec{P}, \frac{i}{c}(E+pV)\right]$, and not $\left(\vec{P}, \frac{i}{c}E\right)$, that transform like the components of a four-vector.

1. The restrictive conditions controlling the distribution with respect to the laboratory system are

$$\left.\begin{aligned} \sum_j n_j &= N \\ \sum_j n_j \epsilon_j &= E \\ \sum_j n_j \vec{p}_j &= \vec{P} \end{aligned}\right\} . \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Then, the distribution law states that n_j, the number of particles possessing an energy $\epsilon_j = c\,(p_j^2+m^2c^2)^{\frac{1}{2}}$, where $\vec{p}_j$ has the components p_x, p_y and p_z as measured in S, is given by

$$n_j = \frac{g_j}{\exp\left(-\alpha + \frac{\epsilon_j}{kT} - \vec{\gamma}\cdot\vec{p}_j\right) + \beta} \qquad \ldots \quad \ldots \quad \ldots \quad (2)$$

where g_j is the number of wave functions or states of the particle with eigenvalue ϵ_j of the energy. β depends upon the type of statistics obeyed by the particles. $\vec{\gamma}$ is the Lagrange's undetermined multiplier that takes care of the conservation of net momentum $\vec{P}$ just as α is for the conservation of N. Evidently $\vec{\gamma}$, whose magnitude γ has the dimensions of $(\text{momentum})^{-1}$, is a vector parallel to the only privileged direction, i.e. that of the mass-motion. We take the z-axis of our co-ordinate system in this direction so that if $\vec{k}$ is the unit vector along the z-axis, we have

$$\vec{\gamma} = \gamma\vec{k},\ \vec{P} = P\vec{k} \text{ and } \vec{\gamma}\,.\,\vec{p}_j = \gamma p_z.$$

We can, therefore, put the distribution law in the form

$$n(\vec{p})d^3p = \frac{V}{h^3}\,\frac{d^3p}{\exp\left(-\alpha+\frac{\epsilon}{kT}-\gamma p_z\right)+\beta}. \qquad (3)$$

In order to determine γ as a function of the macroscopic properties of the assembly, we calculate the value of v, the velocity of mass-motion. This will obviously be equal to $\bar{u}_z$, the z-component of the particle velocity averaged over the whole assembly in its equilibrium state.

Now

$$u_z = \frac{\partial\epsilon}{\partial p_z} = \frac{c^2\,.\,p_z}{\epsilon}. \qquad (4)$$

Hence

$$v = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{d^3p}{\exp\left(-\alpha+\frac{\epsilon}{kT}-\gamma p_z\right)+\beta}\,.\,\frac{c^2p_z}{\epsilon}}{\displaystyle\int_{-\infty}^{\infty} \frac{d^3p}{\exp\left(-\alpha+\frac{\epsilon}{kT}-\gamma p_z\right)+\beta}}. \qquad (5)$$

To simplify the integrals involved in this equation, we make the substitutions

$$\left.\begin{aligned} p_x = p_x^0,\quad p_y = p_y^0,\quad p_z &= \frac{p_z^0+\frac{\gamma kT}{c^2}\,.\,\epsilon^0}{\left(1-\frac{\gamma^2k^2T^2}{c^2}\right)^{\frac{1}{2}}} \\ \epsilon &= \frac{\epsilon^0+\gamma kT\,.\,p_z^0}{\left(1-\frac{\gamma^2k^2T^2}{c^2}\right)^{\frac{1}{2}}} \end{aligned}\right\}. \qquad (6)$$

so that

$$c^2\left(p_x^2+p_y^2+p_z^2\right)-\epsilon^2 = c^2\left(p_x^{0^2}+p_y^{0^2}+p_z^{0^2}\right)-\epsilon^{0^2} = -m^2c^4. \qquad (7)$$

$$\begin{aligned} d^3p &= d^3p^0\,\frac{\partial(p_x, p_y, p_z)}{\partial(p_x^0, p_y^0, p_z^0)} \\ &= d^3p^0\,\frac{1+\gamma kT\,.\,p_z^0/\epsilon^0}{\left(1-\frac{\gamma^2k^2T^2}{c^2}\right)^{\frac{1}{2}}} \end{aligned} \qquad (8)$$

and $$\frac{\epsilon}{kT}-\gamma p_z=\frac{\epsilon^0}{kT}\left(1-\frac{\gamma^2k^2T^2}{c^2}\right)^{\frac{1}{2}}. \qquad (9)$$

The equation (5) thus becomes

$$v=\frac{\displaystyle\int_{-\infty}^{\infty}\frac{d^3p^0\,.\,(c^2p_z^0/\epsilon^0+\gamma kT)}{\exp\left[-\alpha+\frac{\epsilon^0}{kT}\left(1-\frac{\gamma^2k^2T^2}{c^2}\right)^{\frac{1}{2}}\right]+\beta}\cdot\left(1-\frac{\gamma^2k^2T^2}{c^2}\right)^{-\frac{1}{2}}}{\displaystyle\int_{-\infty}^{\infty}\frac{d^3p^0(1+\gamma kT\,.\,p_z^0/\epsilon^0)}{\exp\left[-\alpha+\frac{\epsilon^0}{kT}\left(1-\frac{\gamma^2k^2T^2}{c^2}\right)^{\frac{1}{2}}\right]+\beta}\cdot\left(1-\frac{\gamma^2k^2T^2}{c^2}\right)^{-\frac{1}{2}}}. \qquad (10)$$

Remembering the equation (7) whereby ϵ^0 is an even function of p_z^0, the quotient on the right hand side of (10) immediately turns out to be γkT. This is, therefore, the velocity of the rest system S^0 with respect to the laboratory system S. Consequently the microscopic quantities with the superscript 0 occurring in (6) and (7) are appropriate to S^0. We thus obtain the relation

$$\vec{\gamma}=\frac{\vec{v}}{kT}. \qquad (11)$$

Now the distribution law as stated in (2) becomes

$$n_j=\frac{g_j}{\exp\left(-\alpha+\frac{\epsilon_j-vp_z}{kT}\right)+\beta}. \qquad (12)$$

For an observer in the rest system, $v=0$ and hence the distribution law in S^0 would be

$$n_j^0=\frac{g_j^0}{\exp\left(-\alpha^0+\frac{\epsilon_j^0}{kT_0}\right)+\beta}. \qquad (13)$$

The same is the distribution law arrived at by conserving N and E_0 in the rest system.

Further, in order that the laws of thermodynamics be form-invariant under a Lorentz transformation, we have for the transformation of temperature (Tolman, 1934)

$$T=T_0\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}. \qquad (14)$$

This relation, coupled with the special Lorentz transformations (6), immediately gives

$$\frac{\epsilon-vp_z}{kT}=\frac{\epsilon^0}{kT_0}. \qquad (15)$$

Thus we see that the distribution function derived in S by introducing momentum-conservation is formally the same as that derived in S^0 without conserving the momentum, except for the appropriate Lorentz transformations.

2. Let us now define certain macroscopic quantities referring to S^0. With the help of the distribution function (13), we have

$$N = \frac{V_0}{h^3}\int_{-\infty}^{\infty} \frac{d^3p^0}{\exp\left(-\alpha^0+\frac{\epsilon^0}{kT_0}\right)+\beta}. \quad \ldots \quad (16)$$

$$E_0 = \frac{V_0}{h^3}\int_{-\infty}^{\infty} \frac{d^3p^0}{\exp\left(-\alpha^0+\frac{\epsilon^0}{kT_0}\right)+\beta}\,\epsilon^0. \quad \ldots \quad (17)$$

Moreover if u^0 and p^0 represent the velocity and the momentum of a particle respectively, then by the usual arguments of the kinetic theory, the pressure p_0 of the assembly will be given by (Kothari and Singh, 1942)

$$p_0 = \tfrac{1}{3}\int_{mc^2}^{\infty} \frac{u^0\,.\,p^0\,.\,n(\epsilon^0)\,d\epsilon^0}{V_0}$$

$$\therefore \quad p_0V_0 = \frac{V_0}{h^3}\int_{-\infty}^{\infty} \frac{c^2 p_z^{0\,2}}{\epsilon^0}\,\frac{d^3p^0}{\exp\left(-\alpha^0+\frac{\epsilon^0}{kT_0}\right)+\beta}. \quad \ldots \quad (18)$$

Having set up the integrals for the 'proper' macroscopic quantities N, E_0 and p_0V_0, we now take up the evaluation of the macroscopic quantities referring to S in terms of the 'proper' ones.

From the distribution function (12) and the restrictive conditions (1), one gets

$$N = \frac{V}{h^3}\int_{-\infty}^{\infty} \frac{d^3p}{\exp\left[-\alpha+\frac{\epsilon-vp_z}{kT}\right]+\beta} \quad \ldots \quad (19)$$

$$P = \frac{V}{h^3}\int_{-\infty}^{\infty} \frac{d^3p}{\exp\left[-\alpha+\frac{\epsilon-vp_z}{kT}\right]+\beta}\,.\,p_z \quad \ldots \quad (20)$$

$$E = \frac{V}{h^3}\int_{-\infty}^{\infty} \frac{d^3p}{\exp\left[-\alpha+\frac{\epsilon-vp_z}{kT}\right]+\beta}\,.\,\epsilon. \quad \ldots \quad (21)$$

Rewriting the substitutions (6) and (8) with the help of (11), we have

$$\left.\begin{aligned} p_x = p_x^0, \quad p_y = p_y^0, \quad p_z &= \frac{p_z^0+\frac{v}{c^2}\epsilon^0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} \\ \epsilon &= \frac{\epsilon^0+vp_z^0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} \end{aligned}\right\} \quad \ldots \quad (6')$$

and

$$d^3p = d^3p^0 \frac{1+vp_z^0/\epsilon^0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}. \qquad (8')$$

Also as a consequence of the Lorentz contraction

$$V = V_0\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}. \qquad (22)$$

Using these relations in collaboration with (15), the equations (19), (20) and (21) become

$$N = \frac{V_0}{h^3}\int_{-\infty}^{\infty} \frac{d^3p^0(1+vp_z^0/\epsilon^0)}{\exp\left(-\alpha+\frac{\epsilon^0}{kT_0}\right)+\beta} \qquad (23)$$

$$P = \frac{V_0}{h^3}\int_{-\infty}^{\infty} \frac{d^3p^0(1+vp_z^0/\epsilon^0)}{\exp\left(-\alpha+\frac{\epsilon^0}{kT_0}\right)+\beta}\cdot\frac{p_z^0+\frac{v}{c^2}\epsilon^0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}. \qquad (24)$$

$$E = \frac{V_0}{h^3}\int_{-\infty}^{\infty} \frac{d^3p^0(1+vp_z^0/\epsilon^0)}{\exp\left(-\alpha+\frac{\epsilon^0}{kT_0}\right)+\beta}\cdot\frac{\epsilon^0+vp_z^0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}. \qquad (25)$$

The second part in the integral (23) vanishes and the remaining integral, when compared with its equal (16), yields the result

$$\alpha = \alpha^0.$$

This means that the Lagrange's multiplier for the conservation of N is an invariant under a Lorentz transformation provided that we include the rest energy of a particle into ϵ_j. Remembering (17) and (18) along with this result, it follows from (24) and (25) that

$$P = \frac{E_0+p_0V_0}{c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\cdot v \qquad (26)$$

$$E = \frac{E_0+\frac{v^2}{c^2}p_0V_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} \qquad (27)$$

Further, the transformation equation for pressure, i.e.

$$p = p_0, \qquad (28)$$

coupled with (22) and (27) gives

$$E+pV = \frac{E_0+p_0V_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}. \qquad (29)$$

From the relations (26) and (27) it follows that the total momentum $\vec{P}$ and the total energy E of the assembly do not transform like the components of a four-vector, rather the combination of (26) and (29) suggests that the quantities $\left[\vec{P}, \frac{i}{c}(E+pV)\right]$ form a four-vector. This is due to the fact that the assembly under consideration is not a closed system in that it consists of a thermodynamic fluid contained in a vessel under the influence of the 'external' pressure from the walls of the vessel. In fact, the very result that $\left(\vec{P}, \frac{i}{c}E\right)$ is not a four-vector may, in general, be taken as a proof that the system considered is non-closed (Möller, 1952). The transformations obtained above from the statistical considerations of the microscopic properties of the assembly are well-known transformations in relativistic thermodynamics and can also be established macroscopically by treating a perfect fluid in motion according to the principles of special relativity (Tolman, 1934, and Möller, 1952):

It may be mentioned in passing that the treatment of the text involves the following limiting cases of general interest:

(i) $kT_0 \gg mc^2$, so that the single-particle energy spectrum may be taken to be

$$\epsilon^0 = p^0 . c.$$

Evidently a moving 'Hohlraum', being an assembly of particles with zero rest-mass, suggests itself as an example of this case.

(ii) $kT_0 \ll mc^2$, so that the energy-momentum relation becomes

$$\epsilon^0 = mc^2 + \frac{1}{2m} p^{0 2}.$$

An ideal gas for which the particle velocities in S^0 are non-relativistic is an example of this case. Further, the velocity of mass-motion may also be either relativistic or non-relativistic.

The application of the general results to these limiting cases is quite straightforward.

The Statistical Mechanics of a gaseous assembly in rotation is being studied by introducing the angular-momentum conservation and will be discussed in subsequent papers.

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Summary

Linear momentum-conservation is applied to an ideal relativistic gaseous assembly in mass-motion. It is shown that the distribution function thus obtained is formally the same as one would obtain in the rest system without conserving the momentum, except for the appropriate Lorentz transformations. The four-vector character of the quantities $\left[\vec{P}, \frac{i}{c}(E+pV)\right]$ is thereby established.

References

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Möller, C. (1952). The Theory of Relativity (Oxford: Clarendon Press).

Tolman, R. C. (1934). Relativity, Thermodynamics and Cosmology (Oxford: Clarendon Press).

Note added in proof: Expressed in terms of inverse hyperbolic functions, the equations of transformation, from one inertial system to another, of the macroscopic quantities pertaining to the assembly can be shown to take an interesting form.* As is customary in the special relativity, we have two inertial systems S and S' such that the velocity of S' with respect to S is v_1 in the direction of one of the common axes, say the z-axis. Let the rest system S^0 of the gaseous assembly have a parallel velocity v_2 in S'. Its velocity v in S is then given by

$$\tanh^{-1}\frac{v}{c}=\tanh^{-1}\frac{v_1}{c}+\tanh^{-1}\frac{v_2}{c}.$$

From (26) and (29), however, one readily obtains

$$\tanh^{-1}\frac{v}{c}=\sinh^{-1}\frac{Pc}{E_0+p_0V_0}=\cosh^{-1}\frac{E+pV}{E_0+p_0V_0}.$$

We can, therefore, write

$$\sinh^{-1}\frac{Pc}{E_0+p_0V_0}=\sinh^{-1}\frac{P_1c}{E_0+p_0V_0}+\sinh^{-1}\frac{P_2c}{E_0+p_0V_0}$$

and

$$\cosh^{-1}\frac{E+pV}{E_0+p_0V_0}=\cosh^{-1}\frac{E_1+p_1V_1}{E_0+p_0V_0}+\cosh^{-1}\frac{E_2+p_2V_2}{E_0+p_0V_0}.$$

In these equations v, P and $(E+pV)$ are the macroscopic quantities of the assembly, as observed in S; v_2, P_2 and $(E_2+p_2V_2)$ are those observed in S' while v_1, P_1 and $(E_1+p_1V_1)$ are those observed in S for an identical assembly at rest in S', i.e. they appear by virtue of the relative motion of the two systems.

* The corresponding treatment for a single particle will be appearing in an early issue of the *American Journal of Physics*.

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