ON A PROBLEM OF G. KUREPA

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The purpose of this note is to give a characterisation of ramified tables in which there exists no maximal chain intersecting every maximal antichain. Also a minimal ramified set possessing this property is constructed, and a question communicated to me by G. Kurepa is answered in the affirmative.

A chain (antichain) of a partially ordered set P is any subset S of P, every pair (no pair) of elements of which is comparable. If further S is such that $(S \cup p)$ is a chain (antichain) for no element $p \in P-S$, then S is a maximal chain (antichain) of P.

It may be noted that the antichain A is a maximal antichain of P if and only if every element of P is comparable with some element of A.

A ramified partially ordered set is a partially ordered set in which the set of all elements $<$ any given element is a chain. A ramified table is a ramified set in which every descending chain is finite.

A subset Q of the p.o. set P is cofinal with P if every element of P is $\leqslant$ some corresponding $q \in Q$. G. Kurepa (1952) has proposed the following questions:—

(1) Does there exist a partially ordered set in which there is no maximal antichain which intersects every maximal chain?

(2) Does there exist a partially ordered set in which there is no maximal chain which intersects every maximal antichain?

Both these questions have been answered in the affirmative by J. C. Sheperdson (1954), who has constructed a ramified set possessing both these properties. These properties (of a p.o. set, referred to in questions (1) and (2) respectively) will be referred to as properties (1) and (2) respectively in the sequel. Necessary and sufficient conditions for a ramified set to possess property (2) are established in 1·1 and 2·2. In 2·1 a minimal ramified set possessing property (2) is constructed by means of partitions of ordertypes (i.e. of completely ordered sets). It is pointed out that every ramified table possessing property (2) has a cofinal subset which can be represented by means of segments of an ordertype, the segments being ordered by set inclusion. Re-ordering of these segments according to their place in the completely ordered set leads to the answer (2·4) of the following question communicated to me by G. Kurepa:

(3) Does there exist a partially ordered set S so that for each maximal antichain $A \subset S$, each maximal chain of $S-A$ is one such in S, too?

1·1. A necessary and sufficient condition for the ramified set P to possess property (2) is that for every element $p \in P$ there exists at least one pair p_1, p_2 of mutually incomparable elements of P, each $> p$.

(A) To prove that the condition is necessary, consider any ramified set P, not satisfying this condition. There exists an element $p \in P$ for which the set C_p of all elements $\geqslant p$ is a chain. Also the set C^p of all elements $< p$ is a chain, by the definition of a ramified set. Therefore, since every element of C^p is $<$ every element of C_p, $C^p + C_p = C$, the set of all elements, comparable with p, is a chain. Also no element $x \in C_p$ is comparable with any element $y \bar{\epsilon} C$, for $y > x$ would imply that $y > x \geqslant p$, while $y < x$ would imply that $y \in C^x$, the chain of all elements $< x$, and

$C^x \supset p$ if $x \neq p$. In either case $y \in C$. Now, if A were a maximal antichain of P, which does not intersect C, every element of P, and hence every element of C_p is comparable with some element of A. But this has been shown above to be impossible. Therefore, since C_p is nonnull, it follows that C is a chain which intersects every maximal antichain, i.e. P does not possess property (2), which shows that the condition is necessary.

To prove that the condition is sufficient, it is evidently enough to show that

(B) If P is a ramified set satisfying the condition of 1.1, then for every maximal chain C of P, every maximal antichain A of $P-C$ is a maximal antichain of P also.

For, corresponding to every maximal chain C of P, there will be a maximal antichain A of $P-C$, and $A \cap C$ is evidently null. Hence (assuming that $P \not\subseteq C$, which condition can be easily seen to be true when P satisfies the condition of 1.1), to prove that P possesses property (2), it is sufficient to show that every such maximal antichain A of $P-C$ is a maximal antichain of P.

Consider any ramified set P satisfying the condition of 1.1. Let C be any maximal chain of P and A any maximal antichain of $(P-C)$. Then every element of $(P-C)$ is comparable with some corresponding element of A, and hence to prove that A is a maximal antichain of P, it is sufficient to show that every element p of C is comparable with some corresponding element of A. By hypothesis there exist two mutually incomparable elements p_1, p_2 of P, each $> p$, and at most one of these can belong to C. Let $p_1 \in (P-C)$. Then, since A is a maximal antichain of $(P-C)$, A contains an element a comparable with p_1. a is comparable with p also, since $a \geqslant p_1$ implies that $a \geqslant p_1 > p$, while $a < p_1$ implies that $a \in C^{p_1}$, the chain of all elements $< p_1$, which also contains p. Hence corresponding to any element $p \in C$, there exists $a \in A$ such that a is comparable with p, which proves (B). Hence the condition of 1.1 is sufficient for P to possess property (2).

1.2 Corollary :—From (B) and 1.1 it follows immediately that

(C) If P is any ramified set possessing property (2), and C any maximal chain of P, then every maximal antichain of $(P-C)$ is a maximal antichain of P also.

§2. It has been pointed out by G. Kurepa (1935, p. 112) that every set of successive partitions of a completely ordered set corresponds to a ramified table whose elements are the segments of the partitions, the segments being ordered by the relation of set inclusion reversed. Using this notion a minimal ramified set possessing property (2) can be constructed as follows:—

2.1. The ramified table corresponding to the partition Q of the real number segment $O \leqslant x \leqslant 1$, in which $O \leqslant x \leqslant 1$ is itself a segment, and which contains with every segment, the two subsets formed by bisecting it, each segment of Q being of length $1/2^n$, $n \geqslant O$, is a minimal ramified set possessing property (2) (i.e. Q is imbeddable in every ramified set possessing property (2)).

It may be noted that every minimal ramified table possessing property (2) is also a minimal ramified set possessing property (2), on account of the following

2.2 Lemma. A ramified set P possesses property (2) if and only if there exists a ramified table cofinal with P, possessing property (2).

Since it is known (1950, p. 125) that for every ramified set there exists a cofinal ramified table, it is sufficient to show that P possesses property (2) if and only if every cofinal subset of P possesses this property. Let R be any subset of P, cofinal with P.

Assuming that P possesses property (2), consider any element $r \in R$. Then, since $r \in P$, by 1.1, there exist two mutually incomparable elements p_1, p_2 of P, both $> r$. Since R is cofinal with P, there exist elements r_1, r_2 of R such that $r_1 \geqslant p_1$, $r_2 \geqslant p_2$. Therefore, since P is a ramified set and p_1, p_2 are incomparable, r_1, r_2 are incomparable. (If $r_1 > r_2$, p_1, p_2 would both belong to the set of all elements $\leq r_1$, which is a chain.) Also $r_1 > r$, $r_2 > r$. Hence corresponding to any $r \in R$, there exist two mutually incomparable elements r_1, r_2 of R each $> r$, and hence by 1.1 R

possesses property (2). It follows that every cofinal subset of the ramified set P possessing property (2) possesses property (2).

Conversely, if R possesses property (2), and p is any element of P, there exists an element $r \in R$ such that $r \geqslant p$ (R being cofinal with P), and also two mutually incomparable elements r_1, r_2 of R each $> r$ (since R possesses property (2)). Since r_1, r_2 also belong to P, and each $> p$, it follows that P possesses property (2) if it has a cofinal subset possessing this property. This proves 2.2.

Proof of 2.1 :—The partition Q defined in the hypothesis of 2.1 can be easily seen to be minimal in the class of partitions P of any ordertype E, where P possesses the following property:—

(D) Every segment I of the ordertype E, which belongs to P, contains more than one subsegment belonging to P.

For, let P be any partition of an ordertype E, satisfying the condition (D). Consider the following mapping ϕ of the partition Q (defined in 2.1) into P: Let the lowest element of Q, namely the segment $0 \leqslant x \leqslant 1$, be mapped on to any (arbitrarily chosen) element of P. Suppose that the element $\phi(I)$ of P has been defined for all elements I of Q (i.e. segments of $o \leqslant x \leqslant 1$ belonging to Q) of length $\geqslant 1/2^n$, in such a way that order is preserved, i.e. $\phi(I_1) \subset \phi(I_2)$ in E if and only if $I_1 \subset I_2$. Then by (D), every segment $\phi(I)$ of E contains at least two disjoint subsegments p_1, p_2 both belonging to P, while every $I \in Q$ of length $1/2^n$ contains only two subsegments q_1, q_2 of length $1/2^{n+1}$ each, belonging to Q. Let $\phi(q_1)$, $\phi(q_2)$ be defined as p_1, p_2 respectively. It is evident that order is preserved. Also this defines $\phi(I)$ for all $I \in Q$ of length $\geqslant 1/2^{n+1}$. Hence from the definition of $\phi(I)$ for $I \equiv 0 \leqslant x \leqslant 1$, $\phi(I)$ can be defined by induction for all $I \in Q$, so that order is preserved, which proves that Q is imbeddable in P.

To prove 2.1 it remains to show that every ramified table possessing property (2) has a subset which can be represented by a partition of an ordertype E satisfying condition (D). This latter result follows from the representation of ramified tables by complexes given by Kurepa (1935, pp. 84–89). Every ramified table P can be represented by a set of complexes $X = (x_0, \ldots x_\beta, \ldots)$ $\beta < \xi = \xi(X)$, where each $\xi(X) \leqslant \gamma = \gamma(P)$, the rank of P, the complexes being ordered so that $X < Y$ if and only if $\xi(X) < \xi(Y)$ and $x_\beta = y_\beta$ for $\beta < \xi(X)$. In this representation it can be assumed without loss of generality (as can be easily seen from the proof given in (1935, pp. 84–89) that for every $X \in P$ and every nonlimiting ordinal $\beta + 1 < \xi(X)$, the complex $(X)^{\beta+1} = (x_0, \ldots x_\beta)$ consisting of the first $\beta + 1$ co-ordinates of X, also belongs to P. Let $\overline{P}$ denote the set of all complexes $Y = (y_0, \ldots y_\beta, \ldots)$ $\beta < \eta = \eta(Y)$ such that for every nonlimiting ordinal $\beta + 1 \leqslant \eta(Y)$, the set $(Y)^{\beta+1}$ of first $\beta + 1$ co-ordinates of Y belongs to P, and there exists no complex $X \in P$ such that $\xi(X) > \eta(Y)$ and $x_\beta = y_\beta$ for $\beta < \eta(Y)$. For each $Y \in \overline{P}$ and each $\beta < \eta(Y)$ let $L(Y)^\beta$ denote the set of values of the co-ordinate y_β corresponding to the fixed set of first β co-ordinates $(Y)^\beta$. By Zermelo's axiom this set $L(Y)^\beta$ can be completely ordered. Let $M(Y)^\beta$ be any complete ordering of $L(Y)^\beta$. Let $\overline{P}$ be ordered lexicographically, i.e. $Y_1 = (y_0^1, \ldots y_\beta^1, \ldots) < Y_2 = (y_0^2, \ldots y_\beta^2, \ldots)$ if and only if there exists $\lambda <$ both $\eta(Y_1)$ and $\eta(Y_2)$ such that $y_\beta^1 = y_\beta^2$ for $\beta < \lambda$, i.e. $(Y_1)^\lambda = (Y_2)^\lambda$, and $y_\lambda^1 < y_\lambda^2$ in the ordertype $M(Y_1)^\lambda = M(Y_2)^\lambda$. From the definition of $\overline{P}$ it is clear that λ exists for every pair of distinct complexes Y_1, Y_2 of $\overline{P}$ [since otherwise $\eta(Y_1) < \eta(Y_2)$ would imply that $Y_1 = (Y_2)^{\eta(Y_1)}$ and hence there exists $X \in P$ such that $\xi(X) > \eta(Y_1)$ and $Y_1 = (X)^{\eta(Y_1)}$]. Hence this is a complete ordering of $\bar{P}$. It is clear that $\bar{P}$ is a 'product with variable factors' defined by Hausdorff (1908, p. 470).

The ramified table P has a cofinal subset P_1 which is similar to a partition of the ordertype $\bar{P}$. For each $Y \epsilon \bar{P}$ and each $\beta < \eta(Y)$, $(Y)^\beta$ represents a segment of $\bar{P}$, namely the set of all $Z \epsilon \bar{P}$ such that $\eta(Z) > \beta$ and $(Z)^\beta = (Y)^\beta$. For any $Z \epsilon \bar{P}$ and any $\beta \geqslant \eta(Z)$ let $(Z)^\beta$ be defined as $(Z)^{\eta(Z)} = Z$. Let $\mu = \operatorname{Max}_{Y \epsilon \bar{P}} \eta(Y)$. Then for each $\beta < \mu$, every element $Z \epsilon \bar{P}$ belongs to the corresponding segment $(Z)^\beta$ of $\bar{P}$. Also for $(Y)^\beta \neq (Z)^\beta$ the segments $(Y)^\beta$, $(Z)^\beta$ are disjoint. Hence for each $\beta < \mu$, the set of all distinct segments $\{(Y)^\beta\}$, $Y \epsilon \bar{P}$, forms a partition of $\bar{P}$. Hence the set of all distinct segments $\{(Y)^\beta\}$, $Y \epsilon \bar{P}$, where β assumes all values $< \mu$, is a set of successive partitions T' of $\bar{P}$. Every element $X \epsilon P$ is, by definition, $=$ some $(Y)^\beta$, $\beta < \mu$, and hence corresponds to an element of the partition T'. Let T denote the subset of T' consisting of all such elements which correspond to elements of P. Then T is also a partition of $\bar{P}$. Also if t_1, t_2 are any two distinct elements of T, i.e. $t_1 = (Y)^\beta \neq t_2 = (Z)^\delta$, $t_1 \leqslant t_2$ in T, i.e. the segment $(Y)^\beta \supset (Z)^\delta$ if and only if $\beta < \delta$ and $(Z)^\beta = (Y)^\beta$, i.e. if and only if the element $(Y)^\beta \leqslant (Z)^\delta$ in P. Hence the correspondence of the elements of P with those of T is an order-preserving mapping. But this mapping is not necessarily one-one, since the distinct elements $(Y)^\beta$, $(Y)^{\beta+1}$ of P correspond to the same segment of $\bar{P}$ if the range of variation $L(Y)^\beta$ of y_β corresponding to $(Y)^\beta$ consists of a single element, i.e. there exists only one element of P next higher than $(Y)^\beta$. For every element $t \epsilon T$ let $p(t)$ denote the lowest element of P which corresponds to the segment t of $\bar{P}$, [$p(t)$ exists since P satisfies the descending chain condition] and let $P_1 = \sum_{t \epsilon T} p(t)$. Then from the property (2) of P it will follow, as shown below, that P_1 is cofinal with P, and that T satisfies condition (D).

Consider any element $X \epsilon P$. Since P has property (2), by 1.1, there exist two mutually incomparable elements X_1, X_2 of P both $> X$. The elements t_1, t_2 of T corresponding to X_1, X_2 are disjoint segments of $\bar{P}$ both contained in t, the element of T corresponding to X, since the correspondence between P and T is order-preserving. Therefore the elements $p(t_1)$, $p(t_2)$ of P_1 are incomparable and both $> x$, i.e. for every $x \epsilon P$ there exists an element of $P_1 > x$, which proves that P_1 is cofinal with P. Hence by 2.2 P_1 possesses property (2). Hence by 1.1, for every element $p \epsilon P_1$, there exist two mutually incomparable elements p_1, p_2 of P_1 both $> p$. Therefore, since T is similar to P_1 (being in one-one order-preserving correspondence with the latter set), it follows that for every element $t \epsilon T$ there are two disjoint segments of $\bar{P}$ contained in t, belonging to T. This proves condition (D) for the partition T of $\bar{P}$. Hence any ramified set P possessing property (2) has a subset P_1 which can be represented by a partition T of $\bar{P}$ satisfying (D). This proves 2.1. It also follows that

2.3. A ramified table (and hence a ramified set) possessing property (2) has a cofinal subset which can be represented by a partition of an ordertype.

2.4. The above arguments also prove that a ramified table P possessing property (2) can be represented by the segments of a partition T of $\bar{P}$ (the correspondence being order-preserving but not necessarily one-one). This result leads to the answer of question (3) of Kurepa. Consider the mapping of the ramified table P on to the partition T of $\bar{P}$ defined in 2.1 above. Let R be a reordering of the aggregate of P, in which the elements X_1, X_2 of P are connected by the relation $X_1 < X_2$ if and only if the corresponding elements t_1, t_2 of T are such that every element of t_1 is $<$

every element of t_2 in the ordertype $\bar{P}$. This new relation is evidently transitive and assymmetric, so that by postulating also the relation $X < X$, R is a partial ordering of the aggregate of P. Also it is clear that any two elements of the aggregate of P are comparable in P if and only if they are incomparable in R. Hence every maximal chain (antichain) of P corresponds to a maximal antichain (chain) of R and conversely. But by 1·2, P has the following property, which is a consequence of property (2):—

If C is any maximal chain of P, every maximal antichain of $P-C$ is a maximal antichain of P also.

Hence R has the property:

If A is any maximal antichain of R, every maximal chain of $R-A$ is a maximal chain of R also.

This is the property referred to in question (3) of Kurepa. Hence every ramified table possessing property (2) can be reordered so that every pair of comparable (incomparable) elements of the old ordering becomes incomparable (comparable) in the new ordering, and the new ordering has the property mentioned in question (3).

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