SOME PROPERTIES OF GENERALISED LAPLACE TRANSFORM

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1. The Laplace transform is defined by the equation

$$f(p) = p\int_0^\infty e^{-px}\, h(x)\, dx \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

when the integral is convergent and $R(p) > 0$.

R. S. Varma (1951) generalised (1) by the relation

$$\phi(p) = p\int_0^\infty e^{-\frac{1}{2}px}(px)^{m-\frac{1}{2}}\, W_{k,\,m}(px)h(x)\, dx \quad \ldots \quad \ldots \quad (2)$$

which reduces to (1) when $k = -m+\frac{1}{2}$ by virtue of the identity

$$W_{-m+\frac{1}{2},\,m}(z) \equiv z^{-m+\frac{1}{2}}e^{-\frac{1}{2}z} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

We shall denote (2) by

$$\phi(p) \underset{k,\,m}{\overset{v}{=}} h(x) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

and as usual, (1) will be represented by

$$f(p) \doteqdot h(x).$$

In this paper a few theorems involving (1) and (2) are proved and some interesting results deduced.

2. In this section we collect together the generalised transforms of some functions. These have been obtained by taking suitable functions for $h(x)$ in (2) and evaluating the integrals with the help of known results (Erdelyi, 1939; Goldstein, 1932; Meijer, 1936; Pasricha, 1943; Rathie, 1953, 1954). Some of these will be required in our investigations later on.

$$x^{\nu-1} \underset{k,\,m}{\overset{v}{=}} \frac{\Gamma(\nu)\Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})}\, p^{1-\nu}, \quad R(\nu) > 0,\ R(\nu+2m) > 0. \quad \ldots \quad (5)$$

$$e^{-ax}x^{\nu-1} \underset{k,\,m}{\overset{v}{=}} \frac{\Gamma(\nu)\Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})}\, p^{1-\nu}\, {}_2F_1\left(\nu,\ \nu+2m;\ \nu+m-k+\tfrac{1}{2};\ -\frac{a}{p}\right), \quad (6)$$

$R(\nu) > 0,\ R(\nu+2m) > 0,\ R(p) > R(a) > 0.$

$$x^{-\lambda-m-\frac{1}{2}}(1+x)^{k+\lambda-1} \underset{k,\,m}{\overset{v}{=}} \frac{\Gamma(\frac{1}{2}-\lambda+m)\Gamma(\frac{1}{2}-\lambda-m)}{\Gamma(1-\lambda-k)}\, p^{m+\frac{1}{2}}e^{\frac{1}{2}p}\, W_{\lambda,\,m}(p), \quad (7)$$

$R(\frac{1}{2}-\lambda\pm m) > 0.$

$$x^{-\lambda-\mu-2m-\frac{1}{2}}\,{}_2F_1(\tfrac{1}{2}-\lambda+\mu,\ 1-\lambda-\mu-k-m;\ \tfrac{1}{2}-\lambda-\mu-2m;\ -x)$$
$$\underset{k,\,m}{\overset{v}{=}} \frac{\Gamma(\frac{1}{2}-\lambda-\mu)\Gamma(\frac{1}{2}-\lambda-\mu-2m)}{\Gamma(1-\lambda-\mu-k-m)}\,p^{2m+\mu+\frac{1}{2}}e^{\frac{1}{2}p}W_{\lambda,\,\mu}(p), \quad \ldots \quad (8)$$

$R(\frac{1}{2}-\lambda-\mu)>0$, $R(\frac{1}{2}-\lambda-\mu-2m)>0$.

$$x^{c-2m-1}\,{}_3F_2(\tfrac{1}{2}-\lambda+\mu,\ \tfrac{1}{2}-\lambda-\mu,\ \tfrac{1}{2}+c-k-m;\ c,\ c-2m;\ -x)$$
$$\underset{k,\,m}{\overset{v}{=}} \frac{\Gamma(c)\Gamma(c-2m)}{\Gamma(\frac{1}{2}+c-k-m)}\,p^{2m-c-\lambda+1}e^{\frac{1}{2}p}W_{\lambda,\,\mu}(p), \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

$R(c)>0$, $R(c-2m)>0$.

$$x^{\nu-1}\,{}_rF_s(\alpha_1,\ \ldots,\ \alpha_r;\ \beta_1,\ \ldots,\ \beta_s;\ \pm x)$$
$$\underset{k,\,m}{\overset{v}{=}} \frac{\Gamma(\nu)\Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})}\,p^{1-\nu}\,{}_{r+2}F_{s+1}\left\{\begin{matrix}\alpha_1,\ \ldots,\ \alpha_r,\ \nu,\ \nu+2m; \\ \beta_1,\ \ldots,\ \beta_s,\ \nu+m-k+\frac{1}{2};\end{matrix}\ \pm\frac{1}{p}\right\}, \quad \ldots \quad (10)$$

$R(\nu)>0$, $R(\nu+2m)>0$, $R(p)>0$ $(r<s)$; $R(p)>1$ $(r=s)$.

$$x^{\nu-1}\,{}_rF_s(\alpha_1,\ \ldots,\ \alpha_r;\ \beta_1,\ \ldots,\ \beta_s;\ \pm x^2)$$
$$\underset{k,\,m}{\overset{v}{=}} \frac{\Gamma(\nu)\Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})}\,p^{1-\nu}\,{}_{r+4}F_{s+2}\left\{\begin{matrix}\alpha_1,\ \ldots,\ \alpha_r,\ \frac{1}{2}\nu,\ \frac{1}{2}(\nu+1),\ \frac{1}{2}(\nu+2m),\ \frac{1}{2}(\nu+2m+1); \\ \beta_1,\ \ldots,\ \beta_s,\ \frac{1}{2}(\nu+m-k+\frac{1}{2}),\ \frac{1}{2}(\nu+m-k+\frac{3}{2});\end{matrix}\ \pm\frac{4}{p^2}\right\}, \quad (11)$$

$R(\nu)>0$, $R(\nu+2m)>0$; $R(p)>0\,(s>r+1)$; $R(p)>2\,(s=r+1)$.

$$e^{-ax}x^{\nu-1}\,{}_rF_s(\alpha_1,\ \ldots,\ \alpha_r;\ \beta_1,\ \ldots,\ \beta_s;\ \pm x^{\lambda})$$
$$\underset{k,\,m}{\overset{v}{=}} p^{1-\nu}\sum_{n=0}^{\infty}\left\{\frac{(\alpha_1)_n\ldots(\alpha_r)_n\Gamma(\nu+\lambda n)\Gamma(\nu+2m+\lambda n)}{(\beta_1)_n\ldots(\beta_s)_n\Gamma(\nu+m-k+\frac{1}{2}+\lambda n)}(\pm p)^{-\lambda n}\right.$$
$$\left.\times\,{}_2F_1\left(\nu+\lambda n,\ \nu+2m+\lambda n;\ \nu+m-k+\tfrac{1}{2}+\lambda n;\ -\frac{a}{p}\right)\right\}, \quad \ldots \quad (12)$$

$R(\nu)>0$, $R(\nu+2m)>0$; $R(a+p)>0$ and $\left|\frac{a}{p}\right|<1$ when $\lambda<s-r+1$;

$R(a+p)>s-r+1$ and $\left|\frac{a}{p}\right|<1$ when $\lambda=s-r+1$; λ being real and positive.

$$e^{-\frac{a}{x}}x^{k-m-\frac{3}{2}} \underset{k,\,m}{\overset{v}{=}} 2a^{k-\frac{1}{2}}p^{m+1}K_{2m}(2\sqrt{ap}) \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

$$e^{-\frac{1}{2x}}x^{\frac{1}{2}(\mu+k-m-\frac{3}{2})}\,W_{-\frac{1}{2}(3m+\mu+k-\frac{3}{2}),\ \frac{1}{2}(m+\mu-k+\frac{1}{2})}\left(\frac{1}{x}\right) \underset{k,\,m}{\overset{v}{=}} 2p^{1-\frac{1}{2}\mu}K_{\mu}(2\sqrt{p}) \quad (14)$$

$$x^{2k-m-\frac{3}{2}}e^{-\frac{1}{2x}}W_{\frac{1}{2}-k,\,m}\left(\frac{1}{x}\right) \underset{k,\,m}{\overset{v}{=}} \sqrt{\frac{2}{\pi}}\,p^{m-k+\frac{3}{2}}K_{2k-1}(\sqrt{p})K_{2m}(\sqrt{p}) \quad \ldots \quad (15)$$

$$x^{-m-1}e^{-\frac{1}{2x}}D_{-2k}\sqrt{\frac{2}{x}} \underset{k,\,m}{\overset{v}{=}} \sqrt{\pi}\,2^{-k}p^{m+\frac{1}{2}}W_{k,\,m}(2\sqrt{p})W_{-k,\,m}(2\sqrt{p}) \quad \ldots \quad (16)$$

$$x^{-m}e^{-\frac{1}{2x}}W_{k,\,m}\left(\frac{1}{x}\right) \underset{k,\,m}{\overset{v}{=}} \sqrt{\pi}\,2^{\frac{1}{2}-2k}p^{m+\frac{1}{2}}e^{-\sqrt{p}}\,W_{2k-\frac{1}{2},\,2m}(2\sqrt{p}) \quad \ldots \quad (17)$$

$$x^{-m-1}\, e^{\frac{1}{2x}} D_{2k-1}\left(\sqrt{\frac{2}{x}}\right) \underset{k,\,m}{\overset{v}{=}} \frac{\Gamma(\frac{1}{2}+m-k)\,\Gamma(\frac{1}{2}-m-k)}{\Gamma(1-2k)} \times 2^{-k-\frac{1}{2}}\, p^{m+\frac{1}{2}}\, W_{k,m}(2i\sqrt{p})\, W_{k,m}(-2i\sqrt{p}), \quad \ldots \quad (18)$$

$R(\frac{1}{2}-k\pm m)>0.$

$$e^{\frac{1}{2}x} x^{k-\lambda-m-\frac{3}{2}} W_{\lambda,\,\mu}(x) \underset{k,\,m}{\overset{v}{=}} \frac{\Gamma(k-m-\lambda+\mu)\,\Gamma(k-m-\lambda-\mu)\,\Gamma(k+m-\lambda+\mu)\,\Gamma(k+m-\lambda-\mu)}{\Gamma(\frac{1}{2}-\lambda-\mu)\,\Gamma(\frac{1}{2}-\lambda+\mu)\,\Gamma(2k-2\lambda)} \times p\, {}_2F_1(k-\lambda-m+\mu,\, k-\lambda-m-\mu;\, 2k-2\lambda;\, 1-p), \quad \ldots \quad (19)$$

$R(k-\lambda\pm m\pm\mu)>0.$

$$x^{\gamma-1}(1+x)^{-\beta} \underset{k,\,m}{\overset{v}{=}} \frac{p^{1-\gamma}}{\Gamma(\beta)} E(\beta,\,\gamma,\,\gamma+2m:\,\gamma+m-k+\tfrac{1}{2}:\,p), \quad \ldots \quad (20)$$

$R(\gamma)>0,\ R(\gamma+2m)>0.$

$$x^{\gamma-1}\,{}_2F_1(\beta,\,\gamma+m-k+\tfrac{1}{2};\,\delta;\,-x) \underset{k,\,m}{\overset{v}{=}} \frac{\Gamma(\delta)}{\Gamma(\beta)\,\Gamma(\gamma+m-k+\frac{1}{2})}\, p^{1-\gamma} E(\beta,\,\gamma,\,\gamma+2m:\,\delta:\,p), \quad \ldots \quad \ldots \quad (21)$$

$R(\gamma)>0,\ R(\gamma+2m)>0.$

$$x^{\gamma-1}\,{}_3F_2(\alpha,\,\beta,\,\gamma+m-k+\tfrac{1}{2};\,\delta,\,\gamma+2m;\,-x) \underset{k,\,m}{\overset{v}{=}} \frac{\Gamma(\delta)\,\Gamma(\gamma+2m)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma+m-k+\frac{1}{2})}\, p^{1-\gamma} E(\alpha,\,\beta,\,\gamma:\,\delta:\,p), \quad \ldots \quad \ldots \quad (22)$$

$R(\gamma)>0,\ R(\gamma+2m)>0.$

$$x^{\lambda-1}\,{}_4F_3(\alpha,\,\beta,\,\gamma,\,\lambda+m-k+\tfrac{1}{2};\,\delta,\,\lambda,\,\lambda+2m;\,-x) \underset{k,\,m}{\overset{v}{=}} \frac{\Gamma(\delta)\,\Gamma(\lambda)\,\Gamma(\lambda+2m)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma+m-k+\frac{1}{2})}\, p^{1-\lambda} E(\alpha,\,\beta,\,\gamma:\,\delta:\,p), \quad \ldots \quad \ldots \quad (23)$$

$R(\lambda)>0,\ R(\lambda+2m)>0.$

$$x^{\gamma-1} E\left(\alpha_1,\,\ldots,\,\alpha_r:\,\beta_1,\,\ldots,\,\beta_s:\,\frac{1}{x}\right) \underset{k,\,m}{\overset{v}{=}} p^{1-\gamma}\, E(\alpha_1,\,\ldots,\,\alpha_r,\,\gamma,\,\gamma+2m:\,\beta_1,\,\ldots,\,\beta_s,\,\gamma+m-k+\tfrac{1}{2}:\,p), \quad \ldots \quad (24)$$

$R(\gamma)>0,\ R(\gamma+2m)>0.$

$$\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}\, x^{k-m-\frac{3}{2}}\,{}_2F_1\left(a,\,b;\,c;\,-\frac{1}{x}\right) \underset{k,\,m}{\overset{v}{=}} \sum_{m,\,-m}\left\{\frac{\Gamma(-2m)\,\Gamma(a+k+m-\frac{1}{2})\,\Gamma(b+k+m-\frac{1}{2})}{\Gamma(c+k+m-\frac{1}{2})}\, p^{m} \times {}_2F_2(a+k+m-\tfrac{1}{2},\,b+k+m-\tfrac{1}{2};\,c+k+m-\tfrac{1}{2},\,1+2m;\,p)\right\}, \quad \ldots \quad (25)$$

$R(a+k\pm m-\frac{1}{2})>0,\ R(b+k\pm m-\frac{1}{2})>0.$

3. **Theorem I.** If

$$\phi(p) \underset{k,\, m}{\overset{v}{=}} h(x)$$

and

$$f(p) \underset{\sigma,\, \mu}{\overset{v}{=}} x^{2m-2\mu} h(x)$$

then

$$\phi(p) = \frac{p^{m+k-\sigma-\mu+1}}{\Gamma(\sigma-k+m-\mu)} \int_0^\infty t^{\sigma-k+m-\mu-1} (p+t)^{-1} f(p+t)$$

$$\times {}_2F_1\left(\tfrac{1}{2}-k+m,\ \sigma+\mu-k-m;\ \sigma-\mu-k+m;\ -\frac{t}{p}\right) dt, \quad .. \quad (26)$$

provided that $R(\sigma-k+m-\mu) > 0$ and the integral is convergent.

Proof. We have

$$\phi(p) = p \int_0^\infty e^{-\frac{1}{2}px} (px)^{m-\frac{1}{2}} W_{k,m}(px)\, h(x)\, dx.$$

But Erdelyi (1939) has shown that

$$W_{k,m}(z) = \frac{z^{k-\sigma}}{\Gamma(\sigma-k+m-\mu)} \int_0^\infty e^{-\frac{1}{2}u} u^{\sigma-k+m-\mu-1} W_{\sigma,\mu}(u+z)$$

$$\times \left(1+\frac{u}{z}\right)^{\mu-\frac{1}{2}} {}_2F_1\left(\tfrac{1}{2}-k+m,\ \sigma+\mu-k-m;\ \sigma-\mu-k+m;\ -\frac{u}{z}\right) du,$$

$R(\sigma-k+m-\mu) > 0.$

If we put $z = px$ and $u = xt$, this takes the form

$$W_{k,m}(px) = \frac{(px)^{k-\sigma}}{\Gamma(\sigma-k+m-\mu)} \int_0^\infty e^{-\frac{1}{2}xt} (xt)^{\sigma-k+m-\mu-1} W_{\sigma,\mu}\{(p+t)x\}$$

$$\times \left(1+\frac{t}{p}\right)^{\mu-\frac{1}{2}} {}_2F_1\left(\tfrac{1}{2}-k+m,\ \sigma+\mu-k-m;\ \sigma-\mu-k+m;\ -\frac{t}{p}\right) x\, dt.$$

Hence

$$\phi(p) = \frac{p}{\Gamma(\sigma-k+m-\mu)} \int_0^\infty e^{-\frac{1}{2}px} (px)^{k-\sigma+m-\frac{1}{2}} h(x)\, dx \int_0^\infty e^{-\frac{1}{2}xt} (xt)^{\sigma-k+m-\mu-1}$$

$$\times W_{\sigma,\mu}\{(p+t)x\} \left(1+\frac{t}{p}\right)^{\mu-\frac{1}{2}} {}_2F_1\left(\tfrac{1}{2}-k+m,\ \sigma+\mu-k-m;\ \sigma-\mu-k+m;\ -\frac{t}{p}\right) x\, dt$$

$$= \frac{p^{k-\sigma+m-\mu+1}}{\Gamma(\sigma-k+m-\mu)} \int_0^\infty t^{\sigma-k+m-\mu-1} {}_2F_1\left(\tfrac{1}{2}-k+m,\ \sigma+\mu-k-m;\ \sigma-\mu-k+m;\ -\frac{t}{p}\right) dt$$

$$\times \int_0^\infty e^{-\frac{1}{2}(p+t)x} \{(p+t)x\}^{\mu-\frac{1}{2}} W_{\sigma,\mu}\{(p+t)x\}\, x^{2m-2\mu} h(x)\, dx$$

$$= \frac{p^{k-\sigma+m-\mu+1}}{\Gamma(\sigma-k+m-\mu)} \int_0^\infty t^{\sigma-k+m-\mu-1} (p+t)^{-1} f(p+t)$$

$$\times {}_2F_1\left(\tfrac{1}{2}-k+m,\ \sigma+\mu-k-m;\ \sigma-\mu-k+m;\ -\frac{t}{p}\right) dt.$$

The change in the order of integration is permissible by the application of de la Vallée Poussin's theorem (Bromwich, 1908, p. 457) when the generalised Laplace transforms of the functions involved exist and the resulting integral is absolutely convergent.

When $\sigma = -\mu+\frac{1}{2}$, the theorem may be put in the following form (Rathie, 1955).

COROLLARY. If

$$\phi(p) \underset{k,\ m}{\overset{v}{=}} h(x)$$

and

$$f(p) \doteqdot x^{\lambda} h(x)$$

then

$$\phi(p) = \frac{p^{m+k+\frac{1}{2}}}{\Gamma(\lambda-m-k+\frac{1}{2})} \int_0^\infty t^{\lambda-m-k-\frac{1}{2}}(p+t)^{-1} f(p+t)$$

$$\times {}_2F_1\left(\tfrac{1}{2}-k+m,\ \tfrac{1}{2}-k-m;\ \lambda-k-m+\tfrac{1}{2};\ -\frac{t}{p}\right) dt \quad .. \quad (27)$$

provided that $R(\lambda-m-k+\frac{1}{2}) > 0$ and the integral is convergent.

4. We shall now use the theorem to evaluate two infinite integrals.

Example 1. Taking (6)

$$h(x) = e^{-ax} x^{\nu-1}$$

$$\underset{k,\ m}{\overset{v}{=}} \frac{\Gamma(\nu)\Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})} p^{1-\nu} {}_2F_1\left(\nu,\ \nu+2m;\ \nu+m-k+\tfrac{1}{2};\ -\frac{a}{p}\right)$$

$$= \phi(p),\ R(\nu) > 0,\ R(\nu+2m) > 0,$$

we have

$$x^{2m-2\mu} h(x) = e^{-ax} x^{2m-2\mu+\nu-1}$$

$$\underset{\sigma,\ \mu}{\overset{v}{=}} \frac{\Gamma(2m+\nu)\Gamma(2m+\nu-2\mu)}{\Gamma(2m+\nu-\mu-\sigma+\frac{1}{2})} p^{2\mu-2m-\nu+1}$$

$$\times {}_2F_1\left(2m+\nu,\ 2m+\nu-2\mu;\ 2m+\nu-\mu-\sigma+\tfrac{1}{2};\ -\frac{a}{p}\right)$$

$$= f(p),\ R(\nu+2m) > 0,\ R(\nu+2m-2\mu) > 0.$$

Applying the theorem and replacing $\frac{1}{2}-k+m$ by α, $\sigma+\mu-k-m$ by β, $\sigma-\mu-k+m$ by γ, and $\nu+2m$ by δ, we get

$${}_2F_1\left(\nu,\ \delta : \nu+\alpha;\ -\frac{a}{p}\right) = \frac{\Gamma(\nu+\alpha)\Gamma(\nu+\gamma-\beta)}{\Gamma(\nu)\Gamma(\gamma)\Gamma(\alpha+\nu-\beta)} p^{\nu-\beta}$$

$$\times \int_0^\infty t^{\gamma-1}(p+t)^{\beta-\gamma-\nu} {}_2F_1\left(\nu+\gamma-\beta,\ \delta;\ \nu+\alpha-\beta;\ -\frac{a}{p+t}\right)$$

$$\times {}_2F_1\left(\alpha,\ \beta;\ \gamma;\ -\frac{t}{p}\right) dt, \quad .. \quad (28)$$

valid, by analytic continuation (A.C.), for $R(\gamma) > 0$, $R(\alpha+\nu-\beta) > 0$, $R(\nu) > 0$, $R(p) > R(a) > 0$.

Example 2. Take (25)

$$h(x) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} x^{k-m-\frac{3}{2}} {}_2F_1\left(a, b; c; -\frac{1}{x}\right)$$

$$\stackrel{v}{=} \sum_{k,\, m} \sum_{m,\, -m} \left\{ \frac{\Gamma(-2m)\Gamma(a+k+m-\frac{1}{2})\Gamma(b+k+m-\frac{1}{2})}{\Gamma(c+k+m-\frac{1}{2})} p^{m} \right.$$

$$\left. \times {}_2F_2(a+k+m-\tfrac{1}{2}, b+k+m-\tfrac{1}{2}; c+k+m-\tfrac{1}{2}, 1+2m; p) \right\}$$

$$= \phi(p),\ R(a+k\pm m-\tfrac{1}{2}) > 0, R(b+k\pm m-\tfrac{1}{2}) > 0.$$

Then (Rathie, 1952, p. 240)

$$x^{\lambda}h(x) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} x^{\lambda+k-m-\frac{3}{2}} {}_2F_1\left(a, b; c; -\frac{1}{x}\right)$$

$$\doteqdot \frac{\Gamma(a)\Gamma(b)\Gamma(\lambda+k-m-\frac{1}{2})}{\Gamma(c)} p^{\frac{3}{2}+m-k-\lambda}$$

$$\times {}_2F_2(a, b; c, m-k-\lambda+\tfrac{3}{2}; p)$$

$$+ \frac{\Gamma(a+\lambda+k-m-\frac{1}{2})\Gamma(b+\lambda+k-m-\frac{1}{2})\Gamma(\frac{1}{2}+m-k-\lambda)}{\Gamma(c+\lambda+k-m-\frac{1}{2})} p$$

$$\times {}_2F_2(a+\lambda+k-m-\tfrac{1}{2}, b+\lambda+k-m-\tfrac{1}{2}; c+\lambda+k-m-\tfrac{1}{2}, \lambda+k-m+\tfrac{1}{2}; p)$$

$$= f(p),\ R(a+\lambda+k-m-\tfrac{1}{2}) > 0,\ R(b+\lambda+k-m-\tfrac{1}{2}) > 0.$$

Applying the corollary and replacing $\frac{1}{2}-k+m$ by α, $\frac{1}{2}-k-m$ by β, $\lambda-m-k+\frac{1}{2}$ by γ, we get after a little simplification

$$\sum_{\alpha,\beta} \frac{\Gamma(a-\beta)\Gamma(b-\beta)\Gamma(\beta-\alpha)}{\Gamma(c-\beta)} p^{\alpha} {}_2F_2(a-\beta, b-\beta; c-\beta, 1+\alpha-\beta; p)$$

$$= \frac{1}{\Gamma(\gamma)} \int_0^{\infty} t^{\gamma-1} {}_2F_1\left(\alpha, \beta; \gamma; -\frac{t}{p}\right) \times$$

$$\times \left[\frac{\Gamma(a)\Gamma(b)\Gamma(\gamma-\alpha-\beta)}{\Gamma(c)} (p+t)^{\alpha+\beta-\gamma} {}_2F_2(a, b; c, 1+\alpha+\beta-\gamma; p+t) \right.$$

$$+ \frac{\Gamma(a+\gamma-\alpha-\beta)\Gamma(b+\gamma-\alpha-\beta)\Gamma(\alpha+\beta-\gamma)}{\Gamma(c+\gamma-\alpha-\beta)} \times$$

$$\left. \times {}_2F_2(a+\gamma-\alpha-\beta, b+\gamma-\alpha-\beta; c+\gamma-\alpha-\beta, 1+\gamma-\alpha-\beta; p+t) \right] dt, \quad \ldots \quad (29)$$

valid, by A.C., for $R(a-\alpha) > 0$, $R(b-\alpha) > 0$, $R(a-\beta) > 0$, $R(b-\beta) > 0$, $R(\gamma) > 0$, $R(p) > 0$.

As a particular case of this result, if we take $b=c$ and use the relation

$$E(\alpha, \beta :: x) = \sum_{\alpha,\, \beta} \Gamma(\beta-\alpha)\Gamma(\alpha)x^{\alpha} {}_1F_1(\alpha; \alpha-\beta+1; x) \quad \ldots \quad \ldots \quad (30)$$

we get

$$E(a-\alpha, a-\beta :: p) = \frac{p^{a-\alpha-\beta}}{\Gamma(\gamma)} \int_0^{\infty} t^{\gamma-1}(p+t)^{\alpha+\beta-a-\gamma} E(a, a+\gamma-\alpha-\beta :: p+t) \times$$

$$\times {}_2F_1\left(\alpha, \beta; \gamma; -\frac{t}{p}\right) dt, \quad \ldots \quad (31)$$

$R(\gamma) > 0$, $R(a-\alpha) > 0$, $R(a-\beta) > 0$, $R(p) > 0$.

Since

$$E(\tfrac{1}{2}-k-m,\ \tfrac{1}{2}-k+m :: x) = \Gamma(\tfrac{1}{2}-k-m)\Gamma(\tfrac{1}{2}-k+m)x^{-k}e^{\frac{1}{2}x}W_{k,m}(x) \qquad (32)$$

the above result is equivalent to the following integral representation for Whittaker function (cf. Meijer, 1941, p. 601)

$$W_{k,m}(p) = \frac{\Gamma(\frac{1}{2}-\lambda-\mu)\Gamma(\frac{1}{2}-\lambda+\mu)}{\Gamma(2k-2\lambda)\Gamma(\frac{1}{2}-k-m)\Gamma(\frac{1}{2}-k+m)}\, p^{\lambda+\mu-k+\frac{1}{2}}$$

$$\times \int_0^\infty t^{2k-2\lambda-1}(p+t)^{-\mu-\frac{1}{2}}e^{\frac{1}{2}t}W_{\lambda,\mu}(p+t)$$

$$\times\, {}_2F_1\left(k+m-\lambda-\mu,\ k-m-\lambda-\mu;\ 2k-2\lambda;\ -\frac{t}{p}\right) dt, \quad .. \quad (33)$$

$R(2k-2\lambda) > 0$, $R(\frac{1}{2}-k\pm m) > 0$, $R(p) > 0$.

THEOREM II. If

$$\phi(p) \doteqdot h(x)$$

$$p^{2-\lambda}h(p) \underset{k,\ m}{\overset{v}{=}} g(x)$$

and

$$p^{2-\mu}g(p) \doteqdot f(x)$$

then

$$\phi(p) = p^{1-\lambda}\int_0^\infty x^{-\mu}F_{\lambda,\ \mu}(px)f(x)\, dx,$$

provided that $R(\lambda) > 0$, $R(\lambda+2m) > 0$, $R(\mu) > 0$, $R(\mu+2m) > 0$, and the integral is convergent, and

$$F_{\lambda,\ \mu}(x) = \sum_{\lambda,\ \mu} \frac{\Gamma(\lambda)\Gamma(\lambda+2m)\Gamma(\mu-\lambda)}{\Gamma(\lambda+m-k+\frac{1}{2})}\, x^{\lambda}$$

$$\times\, {}_2F_2(\lambda,\ \lambda+2m;\ \lambda+m-k+\tfrac{1}{2},\ 1-\mu+\lambda;\ x).$$

Proof. We know that if (Rathie, 1952, p. 243)

$$\phi(p) \doteqdot h(x)$$

and

$$p^{2-\lambda}h(p) \underset{k,\ m}{\overset{v}{=}} g(x)$$

then

$$\phi(p) = \frac{\Gamma(\lambda)\Gamma(\lambda+2m)}{\Gamma(\lambda+m-k+\frac{1}{2})}\, p\int_0^\infty t^{-\lambda}{}_2F_1\left(\lambda,\ \lambda+2m;\ \lambda+m-k+\tfrac{1}{2};\ -\frac{p}{t}\right) g(t)\, dt.$$

But

$$t^{2-\mu}g(t) = t\int_0^\infty e^{-tx}f(x)\, dx.$$

Hence

$$\phi(p) = \frac{\Gamma(\lambda)\Gamma(\lambda+2m)}{\Gamma(\lambda+m-k+\frac{1}{2})} p \int_0^\infty t^{\mu-\lambda-1} {}_2F_1\left(\lambda,\ \lambda+2m;\ \lambda+m-k+\tfrac{1}{2};\ -\frac{p}{t}\right) \times \left\{ \int_0^\infty e^{-tx} f(x)\, dx \right\} dt$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda+2m)}{\Gamma(\lambda+m-k+\frac{1}{2})} p \int_0^\infty f(x)\, dx \int_0^\infty e^{-tx} t^{\mu-\lambda-1} {}_2F_1\left(\lambda,\ \lambda+2m;\ \lambda+m-k+\tfrac{1}{2};\ -\frac{p}{t}\right) dt$$

$$= p^{1-\lambda} \int_0^\infty x^{-\mu} F_{\lambda,\ \mu}(px) f(x)\, dx,\ R(\mu) > 0,\ R(\mu+2m) > 0,$$

by virtue of the integral (Rathie, 1952, p. 239)

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \int_0^\infty e^{-px} x^{\alpha-1} {}_2F_1\left(a,\ b;\ c;\ -\frac{1}{x}\right) dx = p^{-\alpha} \frac{\Gamma(a)\Gamma(b)\Gamma(\alpha)}{\Gamma(c)} {}_2F_2(a,\ b;\ c,\ 1-\alpha;\ p)$$

$$+ \frac{\Gamma(a+\alpha)\Gamma(b+\alpha)\Gamma(-\alpha)}{\Gamma(c+\alpha)} {}_2F_2(a+\alpha,\ b+\alpha;\ c+\alpha,\ 1+\alpha;\ p), \quad \ldots \quad (34)$$

$R(a+\alpha) > 0$, $R(b+\alpha) > 0$, $R(p) > 0$.

The reversion of the order of integration is justified when the Laplace and generalised transforms of the functions involved exist and the resulting integral is absolutely convergent.

It is easily seen that

$$F_{\lambda,\ \mu}(x) \sim A + Bx^{-2m} \text{ for large } x,$$

and

$$F_{\lambda,\ \mu}(x) = ax^{\lambda} + bx^{\mu} \text{ for small } x.$$

Example 1. Take (McLachlan and Humbert, 1950, p. 47)

$$h(x) = 2x^{\lambda+m-1} K_{2m}(2\sqrt{x})$$

$$\doteqdot \Gamma(\lambda)\Gamma(\lambda+2m)\, e^{\frac{1}{2p}} p^{\frac{3}{2}-\lambda-m} W_{\frac{1}{2}-\lambda-m,\ m}\left(\frac{1}{p}\right)$$

$$= pE\left(\lambda,\ \lambda+2m :: \frac{1}{p}\right)$$

$$= \phi(p),\ R(\lambda) > 0,\ R(\lambda+2m) > 0;$$

then (13)

$$p^{2-\lambda} h(p) = 2p^{m+1} K_{2m}(2\sqrt{p})$$

$$\underset{k,\ m}{\overset{v}{=}} e^{-\frac{1}{x}} x^{k-m-\frac{3}{2}} = g(x),$$

and (McLachlan and Humbert, 1950, p. 42)

$$p^{2-\mu} g(p) = e^{-\frac{1}{p}} p^{k-m-\mu+\frac{1}{2}}$$

$$\doteqdot x^{\frac{1}{2}(m+\mu-k-\frac{1}{2})} J_{m+\mu-k-\frac{1}{2}}(2\sqrt{x})$$

$$= f(x).$$

Hence the theorem gives

$$E\left(\lambda, \lambda+2m :: \frac{1}{p}\right) = p^{-\lambda} \int_0^\infty x^{\frac{1}{2}(m-\mu-k-\frac{1}{2})} J_{m+\mu-k-\frac{1}{2}}(2\sqrt{x}) F_{\lambda, \mu}(px)\, dx, \quad .. \quad (35)$$

valid, by A.C., for $R(\lambda+m-k+\frac{1}{2})>0$, $R(\mu+m-k+\frac{1}{2})>0$, $R(\mu+k-m)>0$, $R(\mu+k+3m)>0$, $R(p)>0$.

When $k = -m+\frac{1}{2}$,

$$F_{\lambda, \mu}(x) = \sum_{\lambda, \mu} \Gamma(\lambda)\Gamma(\mu-\lambda) x^{\lambda}\, {}_1F_1(\lambda;\, 1-\mu+\lambda;\, x)$$

$$= E(\lambda, \mu :: x) \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (36)$$

and we then have

$$E\left(\lambda, \lambda+2m :: \frac{1}{p}\right) = p^{-\lambda} \int_0^\infty x^{\frac{1}{2}(2m-\mu-1)} J_{2m+\mu-1}(2\sqrt{x}) E(\lambda, \mu :: px)\, dx, \quad (37)$$

$R(\lambda+2m)>0$, $R(\mu+2m)>0$, $R(\mu-2m+\frac{1}{2})>0$, $R(p)>0$,

which is equivalent to a result given by Erdelyi.

Example 2. Next take (34)

$$h(x) = \frac{\Gamma(\nu)\Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})} x^{\lambda-\nu-1}\, {}_2F_1\left(\nu, \nu+2m;\, \nu+m-k+\tfrac{1}{2};\, -\frac{1}{x}\right)$$

$$\doteqdot \frac{\Gamma(\nu)\Gamma(\nu+2m)\Gamma(\lambda-\nu)}{\Gamma(\nu+m-k+\frac{1}{2})} p^{1+\nu-\lambda}\, {}_2F_2(\nu, \nu+2m;\, \nu+m-k+\tfrac{1}{2}, 1+\nu-\lambda;\, p)$$

$$+ \frac{\Gamma(\lambda)\Gamma(\lambda+2m)\Gamma(\nu-\lambda)}{\Gamma(\lambda+m-k+\frac{1}{2})} p\, {}_2F_2(\lambda, \lambda+2m;\, \lambda+m-k+\tfrac{1}{2}, 1+\lambda-\nu;\, p)$$

$$= p^{1-\lambda} F_{\lambda, \nu}(p) = \phi(p),\ R(\lambda)>0,\ R(\lambda+2m)>0.$$

Then

$$p^{2-\lambda} h(p) = \frac{\Gamma(\nu)\Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})} p^{1-\nu}\, {}_2F_1\left(\nu, \nu+2m;\, \nu+m-k+\tfrac{1}{2};\, \frac{1}{p}\right)$$

$$\underset{k,\, m}{\overset{\nu}{=}} e^{-x} x^{\nu-1} = g(x),\ R(\nu)>0,\ R(\nu+2m)>0,$$

and

$$p^{2-\mu} g(p) = e^{-p} p^{\nu-\mu+1}$$

$$\doteqdot \frac{(x-1)^{\mu-\nu-1}}{\Gamma(\mu-\nu)} = f(x),\ R(\mu-\nu)>0,\ x>1.$$

Applying the theorem we get

$$F_{\lambda, \nu}(p) = \frac{1}{\Gamma(\mu-\nu)} \int_1^\infty x^{-\mu}(x-1)^{\mu-\nu-1} F_{\lambda, \mu}(px)\, dx, \quad .. \quad .. \quad (38)$$

$R(\mu-\nu)>0$, $R(\nu)>0$, $R(\nu+2m)>0$, $R(p)>0$,

which by the substitution $px = p+t$ gives

$$F_{\lambda, \nu}(p) = \frac{p^{\nu}}{\Gamma(\mu-\nu)} \int_0^\infty t^{\mu-\nu-1}(p+t)^{-\mu} F_{\lambda, \mu}(p+t)\, dt, \quad .. \quad .. \quad (39)$$

$R(\mu-\nu)>0$, $R(\nu)>0$, $R(\nu+2m)>0$, $R(p)>0$.

When $k = -m+\frac{1}{2}$, this yields by virtue of (36)

$$E(\lambda, \nu :: p) = \frac{p^{\nu}}{\Gamma(\mu-\nu)} \int_0^{\infty} t^{\mu-\nu-1}(p+t)^{-\mu} E(\lambda, \mu :: p+t)\, dt, \quad \ldots \quad (40)$$

$R(\mu-\nu) > 0$, $R(\nu) > 0$, $R(p) > 0$.

7. In this section we shall derive a few recurrence relations for the generalised transform and use them to obtain some recurrence formulae for the E-function.

THEOREM III. If

$$\phi_{k,m}(p) \underset{k,\,m}{\overset{v}{=}} h(x)$$

where $h(x)$ is independent of k and m, then

$$\phi_{k,m}(p) = \phi_{k-\frac{1}{2},\,m+\frac{1}{2}}(p) + (\tfrac{1}{2}-k-m)\phi_{k-1,\,m}(p) \quad \ldots \quad \ldots \quad (A)$$

and if

$$\phi_{k,m,\lambda}(p) \underset{k,\,m}{\overset{v}{=}} x^{\lambda} h(x)$$

$h(x)$ being independent of λ, k, and m, then

$$\phi_{k,m,\lambda}(p) = p\phi_{k-\frac{1}{2},\,m-\frac{1}{2},\,\lambda+1}(p) + (\tfrac{1}{2}-k+m)\phi_{k-1,\,m,\,\lambda}(p) \quad \ldots \quad (B)$$

and

$$\frac{d^n}{dp^n}\left\{\frac{1}{p}\phi_{k,m,\lambda}(p)\right\} = \frac{(-1)^n}{p}\phi_{k+\frac{1}{2}n,\,m-\frac{1}{2}n,\,\lambda+n}(p), \quad \ldots \quad \ldots \quad (C)$$

provided that the integrals involved are uniformly convergent and n is a positive integer.

Proof. (*a*) Since (Whittaker and Watson, 1935, p. 352)

$$W_{k,m}(z) = z^{\frac{1}{2}} W_{k-\frac{1}{2},\,m+\frac{1}{2}}(z) + (\tfrac{1}{2}-k-m) W_{k-1,\,m}(z) \quad \ldots \quad \ldots \quad (41)$$

we have

$$\phi_{k,m}(p) = p\int_0^{\infty} e^{-\frac{1}{2}px}(px)^{m-\frac{1}{2}} W_{k,m}(px)\, h(x)\, dx$$

$$= p\int_0^{\infty} e^{-\frac{1}{2}px}(px)^{m-\frac{1}{2}} \Big\{(px)^{\frac{1}{2}} W_{k-\frac{1}{2},\,m+\frac{1}{2}}(px) + $$

$$+ (\tfrac{1}{2}-k-m) W_{k-1,\,m}(px)\Big\} h(x)\, dx$$

$$= \phi_{k-\frac{1}{2},\,m+\frac{1}{2}}(p) + (\tfrac{1}{2}-k-m)\phi_{k-1,\,m}(p).$$

(*b*) The formula (*B*) may similarly be obtained by using

$$W_{k,m}(z) = z^{\frac{1}{2}} W_{k-\frac{1}{2},\,m-\frac{1}{2}}(z) + (\tfrac{1}{2}-k+m) W_{k-1,\,m}(z) \quad \ldots \quad \ldots \quad (42)$$

in

$$\phi_{k,m,\lambda}(p) = p\int_0^{\infty} e^{-\frac{1}{2}px}(px)^{m-\frac{1}{2}} W_{k,m}(px)\, x^{\lambda} h(x)\, dx.$$

(*c*) The formula (*C*) follows from the above equation on differentiation n times with respect to p and using the result

$$\frac{d^n}{dx^n}\left\{x^{m-\frac{1}{2}} e^{-\frac{1}{2}x} W_{k,m}(x)\right\} = (-1)^n x^{m-\frac{1}{2}(n+1)} e^{-\frac{1}{2}x} W_{k+\frac{1}{2}n,\,m-\frac{1}{2}n}(x) \quad \ldots \quad (43)$$

provided that differentiation under the sign of integration is permissible.

To illustrate the application of the above formulae, we take

$$h(x) = x^{\gamma-1}E\left(\alpha_1, \ldots, \alpha_{r-2} : \beta_1, \ldots, \beta_{s-1} : \frac{1}{x}\right)$$

$$\underset{k,\,m}{\overset{v}{=}} p^{1-\gamma}E(\alpha_1, \ldots, \alpha_{r-2}, \gamma, \gamma+2m : \beta_1, \ldots, \beta_{s-1}, \gamma+m-k+\tfrac{1}{2} : p)$$

$$= \phi_{k,m}(p).$$

Applying the formula (A) and then replacing γ by α_{r-1}, $\gamma+2m$ by α_r, and $\gamma+m-k+\frac{1}{2}$ by β_s, we get

$$E(\alpha_1, \ldots, \alpha_r : \beta_1, \ldots, \beta_s : p) = E(\alpha_1, \ldots, \alpha_r+1 : \beta_1, \ldots, \beta_s+1 : p) + \\ +(\beta_s-\alpha_r)E(\alpha_1, \ldots, \alpha_r : \beta_1, \ldots, \beta_s+1 : p) \quad (44)$$

This result may also be derived from known recurrence relations for E-functions (MacRobert, 1941, p. 260).

A special case of this is

$$E(\alpha, \beta, \gamma : \delta : p) = E(\alpha, \beta, \gamma+1 : \delta+1 : p) + (\delta-\gamma)E(\alpha, \beta, \gamma : \delta+1 : p) \quad (45)$$

Now if we take $\gamma = \frac{1}{2}-k-m$, $\alpha = \delta = m-k-\frac{1}{2}$, and $\beta = m-k+\frac{1}{2}$ we get

$$E(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m :: p) = E(m-k-\tfrac{1}{2}, \tfrac{3}{2}-k-m :: p) + \\ +(2m-1)E(m-k-\tfrac{1}{2}, \tfrac{1}{2}-k-m :: p) \quad .. \quad (46)$$

which by virtue of (32) yields

$$(m-k-\tfrac{1}{2})\,W_{k,m}(p) = (\tfrac{1}{2}-k-m)\,W_{k,m-1}(p) + (2m-1)p^{-\frac{1}{2}}\,W_{k+\frac{1}{2},m-\frac{1}{2}}(p) \quad (47)$$

This new recurrence formula for Whittaker function leads us to another recurrence formula for the generalised transform, viz.

If
$$\phi_{k,m,\lambda}(p) \underset{k,\,m}{\overset{v}{=}} x^{\lambda}h(x)$$

and $h(x)$ is independent of λ, k and m, then

$$(m-k-\tfrac{1}{2})\phi_{k,m,\lambda}(p) = (\tfrac{1}{2}-k-m)p\phi_{k,m-1,\lambda+1}(p) + (2m-1)\phi_{k+\frac{1}{2},m-\frac{1}{2},\lambda}(p) \quad (D)$$

To use this formula we take

$$x^{\lambda}h(x) = x^{\lambda-1}E\left(\alpha_1, \ldots, \alpha_{r-2}, : \beta_1, \ldots, \beta_{s-1} : \frac{1}{x}\right)$$

$$\underset{k,\,m}{\overset{v}{=}} p^{1-\lambda}E(\alpha_1, \ldots, \alpha_{r-2}, \lambda, \lambda+2m : \beta_1, \ldots, \beta_{s-1}, \lambda+m-k+\tfrac{1}{2} : p)$$

$$= \phi_{k,m,\lambda}(p).$$

The formulae (D) and (C) then give us, on replacing λ by α_{r-1}, $\lambda+2m$ by α_r and $\lambda+m-k+\frac{1}{2}$ by β_s

$$(\beta_s-\alpha_{r-1}-1)E(\alpha_1, \ldots, \alpha_r : \beta_1, \ldots, \beta_s : p) \\ = (\beta_s-\alpha_r)E(\alpha_1, \ldots, \alpha_{r-1}+1, \alpha_r-1 : \beta_1, \ldots, \beta_s : p) + \\ +(\alpha_r-\alpha_{r-1}-1)E(\alpha_1, \ldots, \alpha_r-1 : \beta_1, \ldots, \beta_s-1 : p) \quad .. \quad (48)$$

and

$$\frac{d^n}{dp^n}\{p^{-\alpha_r}E(\alpha_1, \ldots, \alpha_r : \beta_1, \ldots, \beta_s : p)\}$$
$$= (-1)^n p^{-\alpha_r - n} E(\alpha_1, \ldots, \alpha_r + n : \beta_1, \ldots, \beta_s : p) \quad (49)$$

and when $n=1$,

$$pE'(\alpha_1, \ldots, \alpha_r : \beta_1, \ldots, \beta_s : p) = \alpha_r E(\alpha_1, \ldots, \alpha_r : \beta_1, \ldots, \beta_s : p) -$$
$$- E(\alpha_1, \ldots, \alpha_r + 1 : \beta_1, \ldots, \beta_s : p) \quad .. \quad (50)$$

An interesting special case of (48) is

$$(\alpha - \beta - 1)E(\beta, \gamma :: p) + (\gamma - \alpha)E(\beta + 1, \gamma - 1 :: p)$$
$$= (\gamma - \beta - 1)E(\alpha, \beta, \gamma - 1 : \alpha - 1 : p) \quad .. \quad (51)$$

From (49) we deduce that

$$\frac{d^n}{dp^n}\{p^{-\beta}E(\alpha, \beta :: p)\} = (-1)^n p^{-\beta - n} E(\alpha, \beta + n :: p) \quad .. \quad .. \quad (52)$$

which is equivalent to

$$\frac{d^n}{dp^n}\left\{p^{-m-\frac{1}{2}}e^{\frac{1}{2}p}W_{k,m}(p)\right\} = \frac{\Gamma(\frac{1}{2}-k+m+n)}{\Gamma(\frac{1}{2}-k+m)}(-1)^n p^{-\frac{1}{2}-m-\frac{1}{2}n}e^{\frac{1}{2}p}W_{k-\frac{n}{2}, m+\frac{n}{2}}(p) \quad (53)$$

and when $n = 1$,

$$pW'_{k,m}(p) = (\tfrac{1}{2}+m-\tfrac{1}{2}p)W_{k,m}(p) - (\tfrac{1}{2}-k+m)p^{\frac{1}{2}}W_{k-\frac{1}{2}, m+\frac{1}{2}}(p). \quad .. \quad (54)$$

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