THE SOLUTIONS OF CERTAIN HYPERGEOMETRIC EQUATIONS

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1. Certain hypergeometric functions of three variables have recently been defined by me in one of my papers (Shanti Saran, 1955). The elementary properties of these functions, including the partial differential equations, have been given in that paper. Later, the Pochhammer type of integral representations for the ten hypergeometric functions defined in Shanti Saran (1955) have been obtained (Shanti Saran, 1955). In this paper I have discussed the nature of the different solutions of the partial differential equations satisfied by two of these functions, viz. F_E and F_K, at the various regular singularities. The other systems of differential equations can be treated in a similar manner. In the first instance, I have obtained the series solution of the differential systems at the regular singularities at the origin. Later the general solution of these differential equations by the help of Pochhammer type of integrals has been obtained and the behaviour of these differential equations at various other singularities have been discussed.

The functions F_E and F_K are defined as

(1)
$$F_E(a_1, a_1, a_1, b_1, b_2, b_2\,;\; c_1, c_2, c_3\,;\; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1, m+n+p)(b_1, m)(b_2, n+p)}{(1, m)(1, n)(1, p)(c_1, m)(c_2, n)(c_3, p)}\, x^m y^n z^p,$$

and

(2)
$$F_K(a_1, a_2, a_2, b_1, b_2, b_1\,;\; c_1, c_2, c_3\,;\; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)(a_2, n+p)(b_1, m+p)(b_2, n)}{(1, m)(1, n)(1, p)(c_1, m)(c_2, n)(c_3, p)}\, x^m y^n z^p.$$

2. The Differential Equations satisfied by F_E and F_K

The differential equations satisfied by F_E and F_K are given by

(1)
$$F_E \begin{cases} [\theta(\theta+c_1-1)-x(\theta+\phi+\psi+a_1)(\theta+b_1)]W = 0 \\ [\phi(\phi+c_2-1)-y(\theta+\phi+\psi+a_1)(\phi+\psi+b_2)]W = 0 \\ [\psi(\psi+c_3-1)-z(\theta+\phi+\psi+a_1)(\phi+\psi+b_2)]W = 0 \end{cases}$$

and

(2)
$$F_K \begin{cases} [\theta(\theta+c_1-1)-x(\theta+a_1)(\theta+\psi+b_1)]W = 0 \\ [\phi(\phi+c_2-1)-y(\phi+\psi+a_2)(\phi+b_2)]W = 0 \\ [\psi(\psi+c_3-1)-z(\phi+\psi+a_2)(\theta+\psi+b_1)]W = 0 \end{cases}$$

respectively.

Using the classical method of solution by series, we assume a solution to be of the form

$$W = x^g y^h z^k \sum_{m,\, n,\, p\, =\, 0}^{\infty} A_{m,\, n,\, p} x^m y^n z^p,$$

where g, h, k are suitable constants.

The indicial equations of the systems 2(1—2) are given by

$$g(g+c_1-1) = 0$$
$$h(h+c_2-1) = 0$$
$$k(k+c_3-1) = 0.$$

These give

$$g = 0 \text{ or } 1-c_1$$
$$h = 0 \text{ or } 1-c_2$$
$$k = 0 \text{ or } 1-c_3.$$

The above roots of the indicial equation lead to the following eight possible sets of values of the parameters g, h, k :—

$g = 0$	0	0	$1-c_1$	$1-c_1$	0	$1-c_1$	$1-c_1$
$h = 0$	$1-c_2$	0	0	$1-c_2$	$1-c_2$	0	$1-c_2$
$k = 0$	0	$1-c_3$	0	0	$1-c_3$	$1-c_3$	$1-c_3$.

These lead to the following general solutions of 2(1) and 2(2) valid in the neighbourhood of the origin :—

(3) $$\begin{aligned} W = {} & AF_E(a_1, a_1, a_1, b_1, b_2, b_2\,;\; c_1, c_2, c_3\,;\; x, y, z) \\ & +B_1 x^{1-c_1} F_E(1-c_1+a_1,\, 1-c_1+a_1,\, 1-c_1+a_1,\, 1-c_1+b_1,\, b_2,\, b_2\,; \\ & \qquad 2-c_1,\, c_2,\, c_3\,;\; x, y, z) \\ & +B_2 y^{1-c_2} F_E(1-c_2+a_1,\, 1-c_2+a_1,\, 1-c_2+a_1,\, b_1,\, 1-c_2+b_2,\, 1-c_2+b_2\,; \\ & \qquad c_1,\, 2-c_2+c_3\,;\; x, y, z) \\ & +B_3 z^{1-c_3} F_E(1-c_3+a_1,\, 1-c_3+a_1,\, 1-c_3+a_1,\, b_1,\, 1-c_3+b_2,\, 1-c_3+b_2\,;\; c_1, \\ & \qquad c_2,\, 2-c_3\,;\; x, y, z) \\ & +D_1 x^{1-c_1} y^{1-c_2} F_E(2-c_1-c_2+a_1,\, 2-c_1-c_2+a_1,\, 2-c_1-c_2+a_1,\, 1-c_1+b_1, \\ & \qquad 1-c_2+b_2,\, 1-c_2+b_2\,;\; 2-c_1,\, 2-c_2,\, c_3\,;\; x, y, z) \\ & +D_2 y^{1-c_2} z^{1-c_3} F_E(2-c_2-c_3+a_1,\, 2-c_2-c_3+a_1,\, 2-c_2+c_3+a_1,\, b_1, \\ & \qquad 2-c_2-c_3+b_2,\, 2-c_2-c_3+b_1\,;\; c_1,\, 2-c_2,\, 2-c_3\,;\; x, y, z) \\ & +D_3 x^{1-c_1} z^{1-c_3} F_E(2-c_1-c_3+a_1,\, 2-c_1-c_3+a_1,\, 2-c_1-c_3+a_1,\, 1-c_1+b_1, \\ & \qquad 1-c_3+b_2,\, 1-c_3+b_2\,;\; 2-c_1,\, c_2,\, 2-c_3\,;\; x, y, z) \\ & +E x^{1-c_1} y^{1-c_2} z^{1-c_3} F_E(3-c_1-c_2-c_3+a_1,\, 3-c_1-c_2-c_3+a_1,\, 3-c_1-c_2-c_3 \\ & +a_1,\, 1-c_1+b_1,\, 2-c_2-c_3+b_2,\, 2-c_2-c_3+b_2\,;\; 2-c_1,\, 2-c_2,\, 2-c_3\,;\; x, y, z) \end{aligned}$$

and

$$\begin{aligned} W = {} & AF_K(a_1, a_2, a_2, b_1, b_2, b_1 ; c_1, c_2, c_3 ; x, y, z) \\ & +B_1 x^{1-c_1} F_K(1-c_1+a_1, a_2, a_2, 1-c_1+b_1, b_2, 1-c_1+b_1 ; \\ & \qquad 2-c_1, c_2, c_3 ; x, y, z) \\ & +B_2 y^{1-c_2} F_K(a_1, 1-c_2+a_2, 1-c_2+a_2, b_1, 1-c_2+b_2, b_1 ; c_1, 2-c_2, c_3 ; \\ & \qquad x, y, z) \\ & +B_3 z^{1-c_3} F_K(a_1, 1-c_3+a_2, 1-c_3+a_2, 1-c_3+b_1, b_2, 1-c_3+b_1 ; \\ & \qquad c_1, c_2, 2-c_3 ; x, y, z) \\ & +D_1 x^{1-c_1} y^{1-c_2} F_K(1-c_1+a_1, 1-c_2+a_2, 1-c_2+a_2, 1-c_1+b_1, 1-c_2+b_2 ; \\ & \qquad 1-c_1+b_1 ; 2-c_1, 2-c_2, c_3 ; x, y, z) \\ & +D_2 y^{1-c_2} z^{1-c_3} F_K(a_1, 2-c_2-c_3+a_1, 2-c_2-c_3+a_1, 1-c_2+b_1, 1-c_2+b_2, \\ & \qquad 1-c_3+b_1 ; c_1, 2-c_2, 2-c_3 ; x, y, z) \\ & +D_3 x^{1-c_1} z^{1-c_3} F_K(1-c_1+a_1, 1-c_3+a_2, 1-c_3+a_2, 2-c_1-c_3+b_1, \\ & \qquad b_2, 2-c_1-c_3+b_1 ; 2-c_1, c_2, 2-c_3 ; x, y, z) \\ & +E x^{1-c_1} y^{1-c_2} z^{1-c_3} F_K(1-c_1+a_1, 2-c_2-c_3+a_2, 2-c_2-c_3+a_2, \\ & \qquad 2-c_1-c_3+b_1, b_2, 2-c_1-c_3+b_1, 2-c_1, 2-c_2, 2-c_3 ; x, y, z). \end{aligned}$$

3. In this section Pochhammer-integral representations (Shanti Saran, 1955) are used to investigate the solutions of the differential systems 2(1) and 2(2).

Consider first the following integral representation (Shanti Saran, 1955) of F_E.

(1) $$\begin{aligned} & F_E(a_1, a_1, a_1, b_1, b_2, b_2 ; c_1, c_2, c_3 ; x, y, z) \\ & = \frac{\Gamma(k)\,\Gamma(k')\,\Gamma(2-k-k')}{(2\pi i)^2} \times \\ & \times \int_C (-t)^{-k}(t-1)^{-k'} {}_2F_1(k, b_1 ; c_1 ; x/t)\, F_4(k', b_2 ; c_2, c_3 ; y/1-t, z/1-t)\, dt, \end{aligned}$$

where $a_1 = k+k'-1$. The contour C denotes a Pochhammer double-loop slung round the points 0 and 1 such that $|t| > |x|$, and $|\sqrt{y/1-t}|+|\sqrt{z/1-t}| < 1$ along the contour.

The above integral suggests that

(2) $$W = \int_C t^{-k}(1-t)^{-k'} f_1(x/t)\, f_2(y/1-t, z/1-t)\, dt,$$

should be a solution of the system 2(1) satisfied by F_E where C is some closed contour and $f_1(u)$ and $f_2(v, w)$ are the solutions of the differential equations

(3) $$[\theta_1(\theta_1+c_1-1)-u(\theta_1+k)(\theta_1+b_1)]\, f_1(u) = 0$$

(4) $$\begin{cases} [\phi_1(\phi_1+c_2-1)-v(\phi_1+\psi_1+k')(\phi_1+\psi_1+b_2)]\, f_2(v, w) = 0 \\ [\psi_1(\psi_1+c_3-1)-w(\phi_1+\psi_1+k')(\phi_1+\psi_1+b_2)]\, f_2(v, w) = 0 \end{cases}$$

respectively, where $\theta_1, \phi_1, \psi_1 \equiv u\partial/\partial u, v\partial/\partial v, w\partial/\partial w$ with $u \equiv x/t$, $v \equiv y/1-t$, $w \equiv z/1-t$.

Now, denoting the differential system 2(1) by $L_1(W)=0$, $L_2(W)=0$ and $L_3(W)=0$ respectively, we get from 3(2), with $a_1=k+k'-1$, that

$$L_1(W)=\int_C t^{-k}(1-t)^{-k'}[\{f_2(\theta(\theta+c_1-1)-x(\theta+k)(\theta+b_1)\}f_1 \\ -x(\phi+\psi+k'-1)f_2(\theta+b_1)f_1]\,dt.$$

Since $u=x/t$, $v=y/1-t$ and $w=z/1-t$, we have on using 3(3) that

$$L_1(W)=\int_C t^{-k}(1-t)^{-k'}\Big[f_2\Big\{\frac{x}{t}(\theta_1+k)(\theta_1+b_1)-x(\theta_1+k)(\theta_1+b_1)\Big\}f_1 \\ -x(\phi_1+\psi_1+k'-1)f_2(\theta_1+b_1)f_1\Big]\,dt$$

$$=x\int_C t^{-k-1}(1-t)^{-k'}[f_2(1-t)(\theta_1+k)(\theta_1+b_1)f_1 \\ -(\phi_1+\psi_1+k'-1)f_2(\theta_1+b_1)f_1]\,dt$$

$$=x\int_C t^{-k-1}(1-t)^{-k'}[f_2(1-t)\{t\partial/\partial t(t\partial/\partial t-b_1)f_1 \\ -kf_2(1-t)(t\partial/dt-b_1)f_1 \\ -(1-k')\,tf_2(t\partial/\partial t-b_1)f_1 \\ -t(1-t)\,\partial f_2/\partial t(t\partial/\partial t-b_1)f_1]\,dt.$$

This gives on simplification

(5) $$L_1(W)=x\int_C d\,[t^{-k}(1-t)^{1-k'}f_2\,\{t\partial f_1/dt-b_1f_1\}].$$

Similarly

(6) $$L_2(W)=y\int_C d\,[t^{1-k}(1-t)^{-k'}\{f_1(v\partial/\partial v+b_2)f_2\}],$$

and

(7) $$L_3(W)=z\int_C d\,[t^{1-k}(1-t)^{-k'}f_1(w\partial/\partial w+b_2)f_2].$$

Now 3(5–7) show that 3(2) will certainly be a solution of 2(1) whenever C is either a closed contour or else an open contour at the two ends of which

$$[t^{-k}(1-t)^{1-k'}f_2(t\partial/\partial t-b_1)f_1]$$
$$[t^{1-k}(1-t)^{-k'}f_1(v\partial/\partial v+b_2)f_2]$$

and

$$[t^{1-k}(1-t)^{-k'}f_1(w\partial/\partial w+b_2)f_2]$$

vanish where f_1 and f_2 are any of the solutions of the systems 3(4) and 3(5) respectively.

Next, from the Pochhammer type of integral representation of F_K (Shanti Saran, 1955), namely

(8) $$F_K(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;x,y,z) \\ =\frac{\Gamma(k)\Gamma(k')\Gamma(2-k-k')}{(2\pi i)^2}\times$$

$$\times\int_C(-t)^{-k}(t-1)^{-k'}{}_2F_1(k,a_1;c_1;x/t)F_2(a_2,b_2,k';c_2,c_3;y,z/1-t)\,dt$$

where $|t| > |x|$, $|y|+|z/1-t| < 1$ along the contour and $b_1 = k+k'-1$, we see that the general solution of 2(2) will be of the type

$$W = \int_C t^{-k}(1-t)^{-k'} f_1(x/t) f_2(y, z/1-t)\, dt \tag{9}$$

provided the contour C is a closed contour or an open contour at the two ends of which

$$\left[t^{-k}(1-t)^{1-k'} f_2(t\partial/\partial t - a_1) f_1\right]$$

and

$$\left[t^{1-k}(1-t)^{-k'} f_1(v\partial/\partial v + a_2) f_2\right]$$

vanish.

Now, let us use in 3(2) the following branches of f_1 and f_2 valid in the vicinity of point $y = 1/x = 1/z = 0$, namely

$$u^{-b_1}{}_2F_1(b_1, 1+b_1-c_1, 1-b_1-k\,;\, 1/u)$$

and

$$w^{-b_2} F_4(1+b_2-c_3, b_2\,;\, c_2, 1+b_2-k'\,;\, v/w, 1/w)$$

respectively, we obtain a solution of 2(1) given by

$$W = Ax^{-b_1}z^{-b_2} F_R(1+b_2-c_3, b_1, 1+b_2-c_3, b_2, 1+b_1-c_1, b_2\,;$$
$$c_2, 1+b_1+b_2-a_1, 1+b_1+b_2-a_1\,;\, y/z, 1/x, 1/z)$$

which is regular at $y = 1/x = 1/z = 0$.

Next, in 3(9) if we use a branch of f_1 in the vicinity of $1/x = 0$, we obtain a solution of 2(2) regular in the neighbourhood of $1/x = y = z = 0$. In fact, we get a power series in $1/x$, y, z, namely

$$W = A \sum_{m,n,p=0}^{\infty} \frac{(a_1, m)(a_2, n+p)(1+a_1-c_1, m)(b_2, n)}{(1, m)(1, n)(1, p)(1+a_1-b_1, m-p)(c_2, n)(c_3, p)} x^{-m} y^n (-z)^p,$$

where A is a constant. This function can be said to be a generalised Horn's function of three variables. For $y = 0$, this reduces to Horn's function H_2.

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Bibliography

Appell, P. et Kampé de Fériet, J. (1926). Fonctions hypergéométriques et hyperspheriques. Polynomes d'Hermite. Gauthier-Villars, Paris.

Erdelyi, A. (1953). Higher transcendental functions. Vol. I.

——— (1941). Integration of the differential equations of Appell's function F_4. *Quart. Jour. of Math.* (*Oxford*), **12**, 68–77.

Shanti Saran (1955). Hypergeometric functions of three variables. *Ganita*, **5**, No. 2.

——— (1955*a*). Integrals associated with hypergeometric functions of three variables. *Proc. Nat. Inst. Sc. of India*, **21A**, No. 2, 1955.

——— (1955*b*). A study of certain hypergeometric functions of three variables. *Thesis* approved for the degree of Ph.D., Lucknow University, pp. 76–100.

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