THE PROBLEM OF MOTION IN GENERAL RELATIVITY

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1. Introduction

In the early stages of the development of general relativity, it was assumed that a geodesic principle governs the motion of the particles. But later it was recognized that the field equations of general relativity are strong enough to determine the co-ordinates of the particles which are sources of the field. The problem of motion consists in solving the field equations so that the field is regular everywhere except at the particles. To do this, one has to take recourse to an approximation method. The problem of deriving the equations of motion from the field equations of empty space alone was tackled by Einstein, Infeld and Hoffmann in 1938, and Einstein and Infeld in 1940 and 1949. But from the earlier methods it does not follow that the field equations can be solved to any arbitrary high approximation. The method given in 1949 was considered to be satisfactory from the logical standpoint. We shall see in § 2 that this method also is not satisfactory from the mathematical point of view. A new version of the approximation method will be given in § 3 and § 4.

2. A Mathematical Discrepancy in the Method of Approximation given in 1949

The method of approximation starts with the expansion of the field variables $\gamma_{\mu\nu}$ which are defined by

$$\left.\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu}+h_{\mu\nu}, \\ \gamma_{\mu\nu} &= h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}h_{\rho\sigma}, \\ \eta_{11} &= \eta_{22}=\eta_{33}=-\eta_{00}=-1, \\ \eta_{\mu\nu} &= o, \quad \mu\neq\nu, \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.1)$$

where $g_{\mu\nu}$ is the symmetric metric tensor of space-time. Here and in what follows the dummy suffix summation convention is generally adopted, exceptions being explicitly stated. Latin indices refer to space-co-ordinates running over the values 1, 2, 3 and the Greek indices refer to both space and time, running over the values 0, 1, 2, 3. The suffix 0 refer to the time co-ordinate. The power-series development for $\gamma_{\mu\nu}$ to represent the field of a non-radiating system of particles is known to be given by

$$\left.\begin{aligned} \gamma_{00} &= \lambda^2 \underset{2}{\gamma_{00}}+\lambda^4 \underset{4}{\gamma_{00}}+\cdots+\lambda^{2l}\underset{2l}{\gamma_{00}}+\cdots, \\ \gamma_{0m} &= \lambda^3 \underset{3}{\gamma_{0m}}+\lambda^5 \underset{5}{\gamma_{0m}}+\cdots+\lambda^{2l+1}\underset{2l+1}{\gamma_{0m}}+\cdots, \\ \gamma_{mn} &= \lambda^4 \underset{4}{\gamma_{mn}}+\lambda^6 \underset{6}{\gamma_{mn}}+\cdots+\lambda^{2l+2}\underset{2l+2}{\gamma_{mn}}+\cdots, \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.2)$$

where the parameter λ is of the order of smallness of v/c, v being a typical velocity of the particles and c the velocity of light. The indices at the bottom indicate the order of smallness. The derivative of a field quantity ϕ with respect to x^s is of the same order as ϕ while the derivative of ϕ with respect to x^0 is of the order of $\lambda\phi$. To treat the derivatives with respect to the time-co-ordinate on the same footing as the derivatives with respect to space-co-ordinates x^s, a new time τ ($= \lambda x^0$) is introduced so that $\partial\phi/\partial\tau$ is of the same order of ϕ. The derivatives of any entity with respect to (τ, x^s) are denoted by a comma. Thus

$$\left.\begin{aligned} \partial\phi/\partial\tau &= \frac{1}{\lambda}\,\partial\phi/\partial x^0 = \phi_{,0}\,; \\ \partial\phi/\partial x^s &= \phi_{,s}\,. \end{aligned}\right\} \qquad (2.3)$$

Instead of the usual field equations

$$R_{\mu\nu} \equiv -\frac{\partial\Gamma^{\sigma}_{\mu\nu}}{\partial x^{\sigma}} + \frac{\partial\Gamma^{\sigma}_{\mu\sigma}}{\partial x^{\nu}} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho} = 0, \qquad (2.4)$$

a set of linear combinations of the field equations defined by

$$\Phi_{\mu\nu} + 2\Lambda_{\mu\nu} \equiv 2\left(R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R_{\rho\sigma}\right) = 0, \qquad (2.5)$$

where

$$\left.\begin{aligned} \Phi_{00} &= -\gamma_{00,\,ss}, \\ \Phi_{0m} &= -(\gamma_{0m,\,s} - \gamma_{0s,\,m})_{,\,s}, \\ \Phi_{mn} &= -(\gamma_{mn,\,s} - \gamma_{ms,\,n})_{,\,s} + (\delta_{mr}\gamma_{ns,\,s} - \delta_{mn}\gamma_{rs,\,s})_{,\,r}, \end{aligned}\right\} \qquad (2.6)$$

and

$$\left.\begin{aligned} 2\Lambda_{00} &= \gamma_{sr,\,sr} + 2\Lambda'_{00}, \\ 2\Lambda_{0m} &= \lambda(\gamma_{ms,\,s0} - \gamma_{00,\,m0}) + 2\Lambda'_{0m}, \\ 2\Lambda_{mn} &= -\lambda(\gamma_{0m,\,0n} + \gamma_{0n,\,0m} - 2\delta_{mn}\gamma_{0s,\,0s}) + \lambda^2\gamma_{mn,\,00} \\ &\quad - \lambda^2\delta_{mn}\gamma_{00,\,00} + 2\Lambda'_{mn}. \end{aligned}\right\} \qquad (2.7)$$

is to be solved by the method of approximation. Then the field equations (2.5) are split according to the powers of λ and are given as

$$\underset{2l}{\Phi_{00}} + 2\underset{2l}{\Lambda_{00}} = 0, \qquad (2.8a)$$

$$\underset{2l+1}{\Phi_{0m}} + 2\underset{2l+1}{\Lambda_{0m}} = 0, \qquad (2.8b)$$

$$\underset{2l+2}{\Phi_{mn}} + 2\underset{2l+2}{\Lambda_{mn}} = 0, \qquad (2.8c)$$

$$l = 1, 2, 3 \cdots$$

These equations are to be solved step by step without introducing singularities in empty space. A solution of the field equations (2.8a) for $l = 1$, viz.,

$$-\underset{2}{\gamma_{00}}{}_{,\,ss} = 0 \qquad (2.9)$$

is given by

$$\overset{k}{\psi} = 1/\overset{k}{r} = \left[\left(x^s - \overset{k}{\xi^s}\right)\left(x^s - \overset{k}{\xi^s}\right)\right]^{-\frac{1}{2}}, \quad \underset{2}{\gamma_{00}} = -4\sum_{k=1}^{N}\underset{2}{\overset{k}{m}}\,\overset{k}{\psi}, \qquad (2.10)$$

so that it may represent the field of N particles. $\overset{k}{\xi}{}^s$ in the above formula are the space-co-ordinates of the kth particle at time τ. Starting with the solution (2.10) the field equations (2.8) for all l are to be solved consistently. In the process of solving the field equations (2.8b) and (2.8c) it becomes inevitable to have to add single poles $-4\underset{2l}{\overset{k}{m}}\overset{k}{\psi}$ and dipoles $-\underset{2l}{\overset{k}{S}}_r\overset{k}{\psi}_{,r}$ to γ_{00}. Here $\underset{2l}{\overset{k}{m}}$ and $\underset{2l}{\overset{k}{S}}_r$ are functions of time which will be determined by the integrability conditions of the field equations (2.8b) and (2.8c). The integrability conditions of the equations (2.8c) that determine $\underset{2l}{\overset{k}{S}}_r$ are

$$\oint_{\overset{k}{S}} 2\underset{2l+2}{\Lambda}_{mr}\, n_r\, ds = 0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.11)$$

where $\overset{k}{S}$ is a two-dimensional surface enclosing the kth particle and n_r is the unit normal vector to the surface $\overset{k}{S}$. Equations (2.11) after the integration give

$$\underset{2l+2}{\overset{k}{C}}_m - \underset{2l}{\overset{\ddot{k}}{S}}_m + \underset{2l+2}{\overset{k}{A}}_m = 0, \qquad \ldots \quad \ldots \quad \ldots \quad (2.12)$$

where

$$\underset{2l+2}{\overset{k}{A}}_m = \sum_{p=1}^{N}{}' \left(\underset{2}{\overset{p}{m}}\underset{2l}{\overset{k}{S}}_r - \underset{2}{\overset{k}{m}}\underset{2l}{\overset{p}{S}}_r\right)\overset{p,k}{\psi}_{,rm} + \left(\underset{2}{\overset{k}{S}}_t\underset{2l}{\overset{p}{S}}_r - \underset{2}{\overset{p}{S}}_r\underset{2l}{\overset{k}{S}}_t\right)\overset{p,k}{\psi}_{,trm},$$

$$\overset{p,k}{\psi} = \left(\overset{p}{\psi}\right)(x^s = \overset{k}{\xi}{}^s),\ \overset{p,k}{\psi}_{,r}\ldots = \left(\overset{p}{\psi}_{,r}\ldots\right)(x^s = \overset{k}{\xi}{}^s),$$

$\sum_{p=1}^{N}{}'$ stands for the summation of p from 1 to N excepting k and a dot above an entity stands for differentiation with respect to time thus:

$$\underset{2l}{\overset{\ddot{k}}{S}}_m = d^2\left(\underset{2l}{\overset{k}{S}}_m\right)\Big/ d\tau^2.$$

$\underset{2l}{\overset{k}{C}}_m$ in the equation (2.12) are certain known functions of the co-ordinates of the particles and their derivatives with respect to time τ.

By the device of adding single poles and dipoles to γ_{00} at each stage of the approximation the field equations can be solved to any desired degree of accuracy. Finally dipoles are removed by the use of the condition

$$\sum_{l=1} \lambda^{2l}\underset{2l}{\overset{k}{S}}_r = 0, \qquad \ldots \quad \ldots \quad \ldots \quad (2.13)$$

which are the equations of motion of the particles.

The equation (2.13) implies that either

$$\underset{2l}{\overset{k}{S}}_r = 0,\ l = 1, 2, 3, \ldots, \qquad \ldots \quad \ldots \quad \ldots \quad (2.14)$$

or that at least one of $\underset{2l}{\overset{k}{S}}_r\,(l = 1, 2, \ldots)$ is a function of λ. The equations (2.14) imply

$$\underset{2l+2}{\overset{k}{C}}_m = 0,\ l = 1, 2, 3, \ldots, \qquad \ldots \quad \ldots \quad \ldots \quad (2.15)$$

which are to be satisfied by $\overset{k}{\xi}{}^s$. $\overset{k}{\xi}{}^s$ are $3N$ functions of time. The infinite number of equations (2.15) will not in general be satisfied by any system of functions $\overset{k}{\xi}{}^s$.

Thus we are left with no alternative but to regard $\underset{2b}{\overset{k}{S}}{}_r$ as a function of λ. This implies that $\underset{2b}{\gamma_{00}}$ is a function of λ since we have added dipoles $-\underset{2b}{\overset{k}{S}}{}_r \overset{k}{\psi}{}_{,r}$ to $\underset{2b}{\gamma_{00}}$. But the expansion (2.2) for $\gamma_{\mu\nu}$ implies that $\underset{l}{\gamma}{}_{\mu\nu}$ are independent of λ which will be vitiated if $\underset{2b}{\overset{k}{S}}{}_r$ is a function of λ. Actually the device of adding dipoles was to avoid the infinite number of equations (2.15) and get the equations

$$\sum_{l=1} \lambda^{2l+2} \underset{2l+2}{\overset{k}{C}}{}_m = 0. \qquad \ldots \qquad (2.16)$$

This will be possible only by the violation of the assumption implied in the expansion (2.2), viz. that $\underset{l}{\gamma}{}_{\mu\nu}$ are independent of λ. Thus the method of adding and annihilating the dipole field also do not satisfactorily solve the problem of motion.

The fact that $\underset{2l}{\gamma_{00}}$ is required to be independent of λ was overlooked from the beginning. In the equations of motion, one expects Newtonian terms in the first approximation; and in the second approximation, those terms that account for the perihelion motion of Mercury. The expected equations of motion are of the type

$$\lambda^4 \left(\underset{2}{\overset{k}{m}} \ddot{\overset{k}{\xi}}{}^s - \sum_{p=1}^{N}{}' \underset{2}{\overset{k}{m}} \underset{2}{\overset{p}{m}} \overset{p,k}{\psi}{}_{,s} \right) + \lambda^6 \underset{6}{\overset{k}{C}}{}s + \cdots = 0. \qquad \ldots \qquad (2.17)$$

The equation (2.17) implies that $\overset{k}{\xi}{}^s$ are functions of the parameter λ. But a solution of the equation (2.9) was given by (2.10) which is a function of λ since $\overset{k}{\xi}{}^s$ is a function of λ. To avoid this inconsistency in the procedure one may assume at the outset that $\overset{k}{\xi}{}^s$, the co-ordinates of the kth particle and $\overset{k}{M}$, the mass of the kth particle, are functions of λ; and seek solutions of the field equations (2.8) for which $\underset{l}{\gamma}{}_{\mu\nu}$ are independent of λ. In § 3 and § 4 we shall see how the field equations can be solved consistently to any arbitrary order of approximation.

3. The Development of $\overset{k}{\xi}{}^s$ and $\overset{k}{M}$

With the power-series development (2.2) for $\gamma_{\mu\nu}$ the field equations (2.5) are expanded in powers of λ and split according to powers of λ. In the l *th* stage of the approximation the equations to be solved are

$$\underset{2l}{\Phi_{00}} + 2\underset{2l}{\Lambda_{00}} = 0, \qquad \ldots \qquad (3.1a)$$

$$\underset{2l+1}{\Phi_{0m}} + 2\underset{2l+1}{\Lambda_{0m}} = 0, \qquad \ldots \qquad (3.1b)$$

$$\underset{2l+2}{\Phi_{mn}} + 2\underset{2l+2}{\Lambda_{mn}} = 0, \qquad \ldots \qquad (3.1c)$$

where

$$\underset{2l}{\Phi_{00}} \equiv -\underset{2l}{\gamma_{00,\,ss}}, \qquad \text{(3.2}a\text{)}$$

$$\underset{2l+1}{\Phi_{0m}} \equiv -\Big(\underset{2l+1}{\gamma_{0m,\,s}}-\underset{2l+1}{\gamma_{0s,\,m}}\Big)_{,\,s}, \qquad \text{(3.2}b\text{)}$$

$$\underset{2l+2}{\Phi_{mn}} \equiv -\Big(\underset{2l+2}{\gamma_{mn,\,s}}-\underset{2l+2}{\gamma_{ms,\,n}}\Big)_{,\,s}+\Big(\delta_{ms}\underset{2l+2}{\gamma_{nr,\,r}}-\delta_{mn}\underset{2l+2}{\gamma_{sr,\,r}}\Big)_{,\,s}, \qquad \text{(3.2}c\text{)}$$

and

$$\left.\begin{aligned}
2\underset{2l}{\Lambda_{00}} &\equiv \underset{2l}{\gamma_{rs,\,rs}}+2\underset{2l}{\Lambda'_{00}},\\
2\underset{2l+1}{\Lambda_{0m}} &\equiv \Big(-\underset{2l}{\gamma_{00,\,0m}}+\underset{2l}{\gamma_{ms,\,s0}}\Big)+2\underset{2l+1}{\Lambda'_{0m}},\\
2\underset{2l+2}{\Lambda_{mn}} &\equiv -\underset{2l+1}{\gamma_{0m,\,0n}}-\underset{2l+1}{\gamma_{0n,\,0m}}+2\delta_{mn}\underset{2l+1}{\gamma_{0s,\,0s}}\\
&\quad+\underset{2l}{\gamma_{mn,\,00}}-\delta_{mn}\underset{2l}{\gamma_{00,\,00}}+2\underset{2l+2}{\Lambda'_{mn}}.
\end{aligned}\right\} \qquad \text{(3.3)}$$

In the above formulae $\underset{b}{\Lambda'_{\mu\nu}}$ contains all the non-linear terms of order b in the field equations (2.5). The field equations are to be solved so that $\gamma_{\mu\nu}$ are regular everywhere except at the positions of the particles, tending to zero as $r \rightarrow \infty$.

If a solution of the field equations in a specially chosen co-ordinate system is given, one can find out the solution of the field equations in any co-ordinate system connected with the chosen one by a given co-ordinate transformation. Hence, without any loss of generality, the field equations will now be solved in a co-ordinate system distinguished from others by the co-ordinate conditions:

$$\left.\begin{aligned}
\gamma_{0m,\,m}-\gamma_{00,0} &= 0,\\
\gamma_{mn,n} &= 0.
\end{aligned}\right\} \qquad \text{(3.4)}$$

We consider that the presence of the function

$$-4\overset{k}{M}\overset{k}{\psi} \qquad \text{(3.5)}$$

in γ_{00} is crucial for the representation of the field of N particles which are at the positions $\overset{k}{\xi}{}^{s}$ at time τ. The character of the entire solution will depend on the harmonic function (3.5).

Schwarzschild's solution in isotropic co-ordinates to represent the field of a particle of mass m is given by

$$\left.\begin{aligned}
\gamma_{00} &= -\frac{4m}{r}-\frac{2m^2}{r^2}+\cdots,\\
\gamma_{0m} &= 0,\\
\gamma_{mn} &= \delta_{mn}\frac{m^2}{r^2}+\cdots.
\end{aligned}\right\}$$

or as

$$\left.\begin{aligned}
\gamma_{00} &= -4m/r-2m^2/r^2+\cdots,\\
\gamma_{0m} &= 0,\\
\gamma_{mn} &= \delta_{mn}\,m^2/r^2+\cdots.
\end{aligned}\right\} \qquad \text{(3.6)}$$

In considering (3.4) and (3.5) we are seeking a solution for the field of N particles analogous to the solution (3.6).

Since $\overset{k}{M}$ is at the most of the order of λ^2, the expansions for $\overset{k}{M}$ and $\overset{k}{\xi}{}^s$ may be taken as

$$\left.\begin{aligned}\overset{k}{M} &= \lambda^2 \underset{2}{\overset{k}{M}}+\lambda^3 \underset{3}{\overset{k}{M}}+\lambda^4 \underset{4}{\overset{k}{M}}+\cdots,\\ \overset{k}{\xi}{}^s &= \underset{0}{\overset{k}{\xi}}{}^s+\lambda \underset{1}{\overset{k}{\xi}}{}^s+\lambda^2 \underset{2}{\overset{k}{\xi}}{}^s+\cdots,\end{aligned}\right\} \qquad (3.7)$$

When (3.7) is substituted in $-4\overset{k}{M}\overset{k}{\psi}$ one gets, on expansion,

$$-4\left(\lambda^2\underset{2}{\overset{k}{M}}+\lambda^3\underset{3}{\overset{k}{M}}+\cdots\right)\overset{k}{\bar{\psi}}+4\left(\sum_{l=2}^{\infty}\lambda^l\underset{l}{\overset{k}{M}}\right)\left(\sum_{b=1}^{\infty}\lambda^b\underset{b}{\overset{k}{\xi}}{}^s\right)\overset{k}{\bar{\psi}}{}_{,s}+\cdots, \qquad (3.8)$$

where

$$\overset{k}{\bar{\psi}} = (\overset{k}{R})^{-1} = \left[(x^s-\underset{0}{\overset{k}{\xi}}{}^s)(x^s-\underset{0}{\overset{k}{\xi}}{}^s)\right]^{-\frac{1}{2}}$$

Since $-4\overset{k}{M}\overset{k}{\psi}$ is to be present in γ_{00}, according to the expansions (2.2) and (3.7), one finds that

$$\underset{2l+1}{\overset{k}{M}} = \underset{2l+1}{\overset{k}{\xi}}{}^s = 0, \qquad (3.9)$$

for all positive integral values of l. Thus in accordance with the power-series development (2.2) for $\gamma_{\mu\nu}$, the power-series development for $\overset{k}{\xi}{}^s$ and $\overset{k}{M}$ is

$$\overset{k}{\xi}{}^s = \underset{0}{\overset{k}{\xi}}{}^s+\lambda^2\underset{2}{\overset{k}{\xi}}{}^s+\lambda^4\underset{4}{\overset{k}{\xi}}{}^s+\cdots+\lambda^{2l}\underset{2l}{\overset{k}{\xi}}{}^s+\cdots, \qquad (3.10a)$$

$$\overset{k}{M} = \lambda^2\underset{2}{\overset{k}{M}}+\lambda^4\underset{4}{\overset{k}{M}}+\cdots+\lambda^{2l}\underset{2l}{\overset{k}{M}}+\cdots. \qquad (3.10b)$$

4. The General Outline of the Method

By the use of (3.10) the function $-4\overset{k}{M}\overset{k}{\psi}$ may be expanded in powers of λ as

$$\begin{aligned}-4\overset{k}{M}\overset{k}{\psi} = {}&\lambda^2\left(-4\underset{2}{\overset{k}{M}}\overset{k}{\bar{\psi}}\right)+\lambda^4\left(-4\underset{4}{\overset{k}{M}}\overset{k}{\bar{\psi}}+4\underset{2}{\overset{k}{M}}\underset{2}{\overset{k}{\xi}}{}^s\overset{k}{\bar{\psi}}{}_{,s}\right)+\cdots\\ &+\lambda^{2l}\Bigg[-4\underset{2l}{\overset{k}{M}}\overset{k}{\bar{\psi}}+4\sum_{a=1}^{l}\underset{2(l-a)}{\overset{k}{M}}\underset{2a}{\overset{k}{\xi}}{}^s\overset{k}{\bar{\psi}}{}_{,s}\\ &\qquad -2\sum_{a=1}^{l}\sum_{b=1}^{l}\underset{2(l-a-b)}{\overset{k}{M}}\underset{2a}{\overset{k}{\xi}}{}^s\underset{2b}{\overset{k}{\xi}}{}^r\overset{k}{\bar{\psi}}{}_{,sr}+\cdots\Bigg]+\cdots, \end{aligned} \qquad (4.1a)$$

where $\underset{-2a}{\overset{k}{M}}$ for $a = 0, 1, 2, \ldots$, is considered to be zero. The expansion (4.1a) is valid in the region given by

$$\left(x^s-\underset{0}{\overset{k}{\xi}}{}^s\right)\left(x^s-\underset{0}{\overset{k}{\xi}}{}^s\right) > \left(\overset{k}{\xi}{}^s-\underset{0}{\overset{k}{\xi}}{}^s\right)\left(\overset{k}{\xi}{}^s-\underset{0}{\overset{k}{\xi}}{}^s\right). \qquad (4.1b)$$

The potential γ_{00} may be divided into two parts as

$$\gamma_{00} = \bar{\gamma}_{00} - 4\overset{k}{M}\,\overset{k}{\psi}, \qquad \ldots \qquad (4.2)$$

where $\bar{\gamma}_{00}$ stands for the function which does not contain any harmonic function. The expansion (2.2) for $\gamma_{\mu\nu}$ now takes the form

$$\left.\begin{aligned} \gamma_{00} &= \lambda^2\Big(\underset{2}{\bar{\gamma}_{00}} - 4\underset{2}{\overset{k}{M}}\,\overset{k}{\bar{\psi}}\Big) + \lambda^4\Big(\underset{4}{\bar{\gamma}_{00}} - 4\underset{4}{\overset{k}{M}}\,\overset{k}{\bar{\psi}} + 4\underset{2}{\overset{k}{M}}\,\underset{2}{\overset{k}{\xi}}{}^s\,\overset{k}{\bar{\psi}}_{,s}\Big) + \cdots, \\ \gamma_{0m} &= \lambda^3\underset{3}{\gamma_{0m}} + \lambda^5\underset{5}{\gamma_{0m}} + \cdots + \lambda^{2l+1}\underset{2l+1}{\gamma_{0m}} + \cdots, \\ \gamma_{mn} &= \lambda^4\underset{4}{\gamma_{mn}} + \lambda^6\underset{6}{\gamma_{mn}} + \cdots + \lambda^{2l+2}\underset{2l+2}{\gamma_{mn}} + \cdots, \end{aligned}\right\} \qquad \ldots \qquad (4.3)$$

where

$$\underset{2l}{\gamma_{00}} = \underset{2l}{\bar{\gamma}_{00}} - 4\underset{2l}{\overset{k}{M}}\,\overset{k}{\bar{\psi}} + 4\sum_{a=1}^{l}\underset{2(l-a)}{\overset{k}{M}}\,\underset{2a}{\overset{k}{\xi}}{}^s\,\overset{k}{\bar{\psi}}_{,s}$$

$$-2\sum_{a=1}^{l}\sum_{b=1}^{l}\underset{2(l-a-b)}{\overset{k}{M}}\,\underset{2a}{\overset{k}{\xi}}{}^s\,\underset{2b}{\overset{k}{\xi}}{}^r\,\overset{k}{\bar{\psi}}_{,sr} + \cdots. \qquad \ldots \qquad (4.4)$$

The variables $\underset{2l}{\bar{\gamma}_{00}}, \underset{2l+1}{\gamma_{0m}}, \underset{2l+2}{\gamma_{mn}}, \underset{2l}{\overset{k}{M}}$ and $\underset{2l-2}{\overset{k}{\xi}}{}^s$ appear for the first time in the field equations at the l *th* stage of the approximation. Hence these variables are to be determined by the equations (3.1). For the integrability of (3.1*b*) and (3.1*c*) certain integrals over closed surfaces, each enclosing one and only one of the particles, should vanish. The equations provided by the vanishing of these integrals are expected to determine $\underset{2l}{\overset{k}{M}}$ and $\underset{2l-2}{\overset{k}{\xi}}{}^s$, while the equations (3.1) determine the field variables $\underset{2l}{\bar{\gamma}_{00}}, \underset{2l+1}{\gamma_{0m}}, \underset{2l+2}{\gamma_{mn}}$.

For the uniqueness of the solution, it was assumed that no new harmonic function is to be added to a field variable at any stage of the approximation. The equation (3.1*a*) for $l = 1$ is

$$\underset{2}{\bar{\gamma}_{00}}{}_{,ss} = 0, \qquad \ldots \qquad (4.5a)$$

which gives the unique solution

$$\underset{2}{\bar{\gamma}_{00}} = 0 \qquad \ldots \qquad (4.5b)$$

according to the assumption about the addition of harmonic functions. It is to be shown that the field equations can be solved to any desired degree of accuracy. We shall prove this by the method of induction. Let us assume that the field equations are solved up to the $(l-1)$th stage of the approximation. It is to be shown that the equations (3.1) can be solved. Since the field equations in the lower approximations are assumed to be solved, the variables

$$\underset{2a}{\bar{\gamma}_{00}},\ \underset{2a+1}{\gamma_{0m}},\ \underset{2a+2}{\gamma_{mn}},\ \underset{2a}{\overset{k}{M}},\ \underset{2a-2}{\overset{k}{\xi}}{}^s, \qquad \ldots \qquad (4.6)$$

$$a = 1, 2, \ldots, (l-1),$$

may be considered as known functions of the co-ordinates (x^s, τ).

By the use of (4.4), the equations (3.1) are written out thus:

$$-\underset{2l}{\bar{\gamma}}_{00,\,ss}+2\underset{2l}{\Lambda}_{00}=0, \qquad (4.7a)$$

$$\underset{2l+1}{\Phi}_{0m}+2\underset{2l+1}{\bar{\Lambda}}_{0m}+\underset{2l+1}{C}_{0m}=0, \qquad (4.7b)$$

$$\underset{2l+2}{\Phi}_{mn}+2\underset{2l+2}{\bar{\Lambda}}_{mn}+\underset{2l+2}{C}_{mn}=0, \qquad (4.7c)$$

where

$$\underset{2l+1}{C}_{0m}=4\left(\underset{2l}{\overset{k}{M}}\,\overset{k}{\bar{\psi}}\right)_{,\,0m}-4\left(\underset{2}{\overset{k}{M}}\,\underset{2l-2}{\overset{k}{\xi^s}}\,\overset{k}{\bar{\psi}}\right)_{,\,0ms} \qquad (4.7d)$$

and

$$\begin{aligned}\underset{2l+2}{C}_{mn}=&-4\left[\underset{2}{\overset{k}{M}}\left(\underset{2l-2}{\overset{k}{\xi^m}}\,\overset{k}{\bar{\psi}}_{,\,n}+\underset{2l-2}{\overset{k}{\xi^n}}\,\overset{k}{\bar{\psi}}_{,\,m}-\delta_{mn}\underset{2l-2}{\overset{k}{\xi^s}}\,\overset{k}{\bar{\psi}}_{,\,s}\right)\right]_{,\,00}\\&-\underset{2}{\gamma}_{00,\,n}\left(\underset{2}{\overset{k}{M}}\,\underset{2l-2}{\overset{k}{\xi^s}}\,\overset{k}{\bar{\psi}}_{,\,sm}\right)-\underset{2}{\gamma}_{00,\,m}\left(\underset{2}{\overset{k}{M}}\,\underset{2l-2}{\overset{k}{\xi^s}}\,\overset{k}{\bar{\psi}}_{,\,sn}\right)\\&+3\delta_{mn}\underset{2}{\gamma}_{00,\,s}\left(\underset{2}{\overset{k}{M}}\,\underset{2l-2}{\overset{k}{\xi^r}}\,\overset{k}{\bar{\psi}}_{,\,rs}\right)\\&-2\underset{2}{\gamma}_{00,\,mn}\left(\underset{2}{\overset{k}{M}}\,\underset{2l-2}{\overset{k}{\xi^s}}\,\overset{k}{\bar{\psi}}_{,\,s}\right)-2\underset{2}{\gamma}_{00}\left(\underset{2}{\overset{k}{M}}\,\underset{2l-2}{\overset{k}{\xi^s}}\,\overset{k}{\bar{\psi}}_{,\,smn}\right). \qquad (4.7e)\end{aligned}$$

The function $\underset{2l+1}{\Lambda}_{0m}$ is split into two parts, viz. $\underset{2l+1}{\bar{\Lambda}}_{0m}$, containing all those terms that are not dependent on $\underset{2l}{\overset{k}{M}}$ and $\underset{2l-2}{\overset{k}{\xi^s}}$ and $\underset{2l+1}{C}_{0m}$, containing all those terms that are dependent on $\underset{2l}{\overset{k}{M}}$ and $\underset{2l-2}{\overset{k}{\xi^s}}$. But the function $\underset{2l+2}{\Lambda}_{mn}$ is split into two parts, viz. $\underset{2l+2}{\bar{\Lambda}}_{mn}$, containing those and only those terms that are not dependent on $\underset{2l-2}{\overset{k}{\xi^s}}$, and $\underset{2l+2}{C}_{mn}$, containing all those terms that are dependent on $\underset{2l-2}{\overset{k}{\xi^s}}$.

The $\underset{2l}{\Lambda}_{00}$ contains only known functions (4.6) of the co-ordinates. The equation (4.7a), which is Poisson's equation in $\underset{2l}{\bar{\gamma}}_{00}$, determines $\underset{2l}{\bar{\gamma}}_{00}$ uniquely. Now the equation (4.7b) is to be solved for $\underset{2l+1}{\gamma}_{0m}$. Since

$$\underset{2l+1}{\Phi}_{0m,\,m}=0,$$

$$\oint_{\overset{k}{S}}\underset{2l+1}{\Phi}_{0m}\,n_m\,dS=0,$$

the conditions

$$2\underset{2l+1}{\bar{\Lambda}}_{0m,\,m}+\underset{2l+1}{C}_{0m,\,m}=0, \qquad (4.8a)$$

$$\oint_{\overset{k}{S}}\left(2\underset{2l+1}{\bar{\Lambda}}_{0m}+\underset{2l+1}{C}_{0m}\right)n_m\,dS=0, \qquad (4.8b)$$

are to be satisfied for the integrability of (4.7b). Since we have to take $\overset{k}{S}$ in the regular region for the γ's and since our procedure implies the expansion (4·1a), the surface $\overset{k}{S}$ is to be taken in the region given by (4.1b). This will be understood throughout the work. The four identities between the field equations, together with the field equations in the lower approximations, ensure the condition (4.8a), which in turn ensures that the left-hand side of (4.8b) is independent of the space-co-ordinates x^s. Since $\underset{2l+1}{\bar{\Lambda}_{0m}}$ contain only known functions of the co-ordinates, the left-hand side of (4.8b) can be evaluated and the contribution from $\underset{2l+1}{\bar{\Lambda}_{0m}}$ and $\underset{2l+1}{C_{0m}}$ is given by

$$\frac{1}{4\pi}\oint_{\overset{k}{S}} \underset{2l+1}{C_{0m}}\, n_m\, dS = -4\overset{k}{\underset{2l}{M}}, \qquad (4.9)$$

$$\frac{1}{4\pi}\oint_{\overset{k}{S}} 2\underset{2l+1}{\bar{\Lambda}_{0m}}\, n_m\, dS = \overset{k}{\underset{2l+1}{C_0}}, \qquad (4.10)$$

where $\overset{k}{\underset{2l+1}{C_0}}$ is a known function of $\overset{k}{\underset{2a}{\xi^s}}$, $a = 0, 1, 2, \ldots (l-2)$. Thus the integrability condition (4·8b) gives

$$4\overset{k}{\underset{2l}{M}} - \overset{k}{\underset{2l+1}{C_0}} = 0. \qquad (4.11)$$

The equation (4.11) can always be solved for $\overset{k}{\underset{2l}{M}}$. Hence the integrability condition (4.8b) can be satisfied by choosing $\overset{k}{\underset{2l}{M}}$ according to the equation (4.11). Now the equation (4.7b), together with the condition (3.4), can be solved uniquely for $\underset{2l+1}{\gamma_{0m}}$. So far we have seen that $\underset{2l}{\bar{\gamma}_{00}}$, $\overset{k}{\underset{2l}{M}}$ and $\underset{2l+1}{\gamma_{0m}}$ become known from the equations (4.7a), (4.11) and (4.7b). The equation (4.7c) remains to be solved, the unknowns being $\underset{2l+2}{\gamma_{mn}}$ and $\overset{k}{\underset{2l-2}{\xi^s}}$.

Since $\underset{2l+2}{\Phi_{mn}}$ satisfies the equations

$$\underset{2l+2}{\Phi_{mn,\,n}} = 0,$$

$$\oint_{\overset{k}{S}} \underset{2l+2}{\Phi_{mn}}\, n_n\, dS = 0,$$

the conditions,

$$2\underset{2l+2}{\bar{\Lambda}_{mn,\,n}} + \underset{2l+2}{C_{mn,\,n}} = 0, \qquad (4.12a)$$

$$\frac{1}{4\pi}\oint_{\overset{k}{S}} \left(2\underset{2l+2}{\bar{\Lambda}_{mn}} + \underset{2l+2}{C_{mn}}\right) n_n\, dS = 0, \qquad (4.12b)$$

are to be fulfilled for the integrability of (4.7c). The four identities, together with the field equations in the lower approximations, ensure the equation (4.12a). The equation (4.12a) ensures that the left-hand side of (4.12b) is at most a function of time. Since $\underset{2l+2}{\bar{\Lambda}}_{mn}$ is a known function of the co-ordinates the contribution to the integral coming from $\underset{2l+2}{\bar{\Lambda}}_{mn}$ can be evaluated. Therefore, we write

$$\frac{1}{4\pi}\oint_{\overset{k}{S}} 2\underset{2l+2}{\bar{\Lambda}}_{mn}\, n_n\, dS = \overset{k}{\underset{2l+2}{C}}_m, \qquad \ldots \quad (4.13a)$$

and assume that we have calculated C's. From the equation (4.7e) it may be seen that

$$\frac{1}{4\pi}\oint_{\overset{k}{S}} \underset{2l+2}{C}_{mn}\, n_n\, dS = 4\left(\overset{k}{\underset{2}{M}}\,\overset{k}{\underset{2l-2}{\xi}}{}^m\right)_{,00} + 4\overset{k}{\underset{2}{M}}\sum_{p=1}^{N}{}'\,\overset{p}{\underset{2}{M}}\left(\overset{p}{\underset{2l-2}{\xi}}{}^r - \overset{k}{\underset{2l-2}{\xi}}{}^r\right)\overset{\overline{p,k}}{\psi}_{,rm}, \qquad (4.13b)$$

where

$$\overset{\overline{p,k}}{\psi} = \left(\overset{p}{\psi}\right)_{(x^s = \overset{k}{\underset{0}{\xi}}{}^s)}, \quad \overset{\overline{p,k}}{\psi}_{,r\ldots} = \left(\overset{p}{\psi}_{,r\ldots}\right)_{(x^s = \overset{k}{\underset{0}{\xi}}{}^s)}.$$

Hence the integrability condition (4.12b) gives

$$4\left(\overset{k}{\underset{2}{M}}\,\overset{k}{\underset{2l-2}{\xi}}{}^m\right)_{,00} + 4\overset{k}{\underset{2}{M}}\sum_{p=1}^{N}{}'\,\overset{p}{\underset{2}{M}}\left(\overset{p}{\underset{2l-2}{\xi}}{}^r - \overset{k}{\underset{2l-2}{\xi}}{}^r\right)\overset{\overline{p,k}}{\psi}_{,rm} + \overset{k}{\underset{2l+2}{C}}_m = 0. \qquad \ldots \quad (4.14)$$

$\overset{k}{\underset{2l-2}{\xi}}{}^s$ can be chosen such that the condition (4.14) is satisfied and hence the equations (4.7c), together with the co-ordinate conditions (3.4), can be solved for $\underset{2l+2}{\gamma}_{mn}$. Thus from the equations (4.7), (4.11) and (4.14) the variables $\underset{2l}{\bar{\gamma}}_{00}$, $\underset{2l+1}{\gamma}_{0m}$, $\underset{2l+2}{\gamma}_{mn}$, $\overset{k}{\underset{2l}{M}}$ and $\overset{k}{\underset{2l-2}{\xi}}{}^s$ become known. Since the field equations are satisfied by the Minkowskian metric $\eta_{\mu\nu}$ the field equations can be solved to any order of approximation. In solving the field equations we get the equations of the type (4.14) which determine $\overset{k}{\underset{2l-2}{\xi}}{}^s$. Thus the equations (4.14) for $l = 1, 2, 3 \ldots$, give the equations of motion of the particles. The co-ordinates of the particles given by

$$\overset{k}{\xi}{}^s = \overset{k}{\underset{0}{\xi}}{}^s + \lambda^2\overset{k}{\underset{2}{\xi}}{}^s + \ldots + \lambda^{2l-2}\overset{k}{\underset{2l-2}{\xi}}{}^s + \ldots, \qquad \ldots \quad (4.15)$$

become known functions of time up to the term $\overset{k}{\underset{2l-2}{\xi}}{}^s$ when the field equations are solved up to the order $(2l+1)$. The equations (4.15) are the integrated equations of motion.

$-4\overset{k}{\underset{2}{M}}\,\overset{k}{\underset{2l-2}{\xi}}{}^s$ serves the same purpose as $\overset{k}{\underset{2l}{S}}_r$ given in the method of approximation followed by Einstein and Infeld in 1949. $\overset{k}{\underset{2l-2}{\xi}}{}^s$ is a part of $\overset{k}{\xi}{}^s$, the co-ordinates of the k *th* particle. Thus this method is based on a purely mathematical deduction and no physical concept was used to get the equations of motion.

5. Equations of Motion up to the Sixth Order

By direct calculations the integrability conditions (4.14) for $l = 1$ and $l = 2$ can be found out and are given by

$$4\left(\overset{k}{\underset{2}{M}}\,\overset{\ddot{k}}{\underset{0}{\xi}}{}^{m}-\overset{k}{\underset{2}{M}}\sum_{p=1}^{N}{}'\,\overset{p}{\underset{2}{M}}\,\overset{p,k}{\overline{\psi}}{}_{,m}\right)=0, \qquad (5.1)$$

and

$$\overset{k}{\underset{2}{M}}\left[\overset{\ddot{k}}{\underset{2}{\xi}}{}^{m}-\sum_{p=1}^{N}{}'\,\overset{p}{\underset{2}{M}}\left(\overset{k}{\underset{2}{\xi}}{}^{s}-\overset{p}{\underset{2}{\xi}}{}^{s}\right)\overset{p,k}{\overline{\psi}}{}_{,sm}\right]$$
$$+\sum_{p=1}^{N}{}'\,\overset{k}{\underset{2}{M}}\,\overset{p}{\underset{2}{M}}\left[-\tfrac{1}{2}\overset{p,k}{R}{}_{,srm}\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{r}+\tfrac{1}{2}\overset{\ddot{p}}{\underset{0}{\xi}}{}^{s}\,\overset{p,k}{R}{}_{,sm}-4\,\overset{p,k}{\overline{\psi}}\sum_{q=1}^{N}{}''\,\overset{q}{\underset{2}{M}}\,\overset{q,p}{\overline{\psi}}{}_{,m}\right.$$
$$+\overset{p,k}{\overline{\psi}}{}_{,s}\left\{4\,\overset{\dot{k}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{k}}{\underset{0}{\xi}}{}^{m}+4\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{m}-4\,\overset{\dot{k}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{m}-3\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{k}}{\underset{0}{\xi}}{}^{m}\right\}$$
$$+\overset{p,k}{\overline{\psi}}{}_{,m}\left\{-\overset{\dot{k}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{k}}{\underset{0}{\xi}}{}^{s}-\tfrac{3}{2}\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{s}+4\,\overset{\dot{k}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{s}+4\sum_{q=1}^{N}{}'\,\overset{q}{\underset{2}{M}}\,\overset{q,k}{\overline{\psi}}\right.$$
$$\left.\left.+\sum_{q=1}^{N}{}''\,\overset{q}{\underset{2}{M}}\,\overset{q,p}{\overline{\psi}}\right\}\right]=0. \qquad (5.2)$$

The equations (5.1) and (5.2) together with (4.15) give the equations of motion

$$\lambda^{4}\,\overset{k}{\underset{2}{M}}\left(\overset{\ddot{k}}{\xi}{}^{m}-\sum_{p=1}^{N}{}'\,\overset{p}{\underset{2}{M}}\,\overset{p,k}{\overline{\psi}}{}_{,m}\right)=-\lambda^{6}\sum_{p=1}^{N}{}'\,\overset{k}{\underset{2}{M}}\,\overset{p}{\underset{2}{M}}\left[-\tfrac{1}{2}\overset{p,k}{R}{}_{,srm}\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{r}+\tfrac{1}{2}\overset{\ddot{p}}{\underset{0}{\xi}}{}^{s}\,\overset{p,k}{R}{}_{,sm}\right.$$
$$-4\,\overset{p,k}{\overline{\psi}}\sum_{q=1}^{N}{}''\,\overset{q}{\underset{2}{M}}\,\overset{q,p}{\overline{\psi}}{}_{,m}+\overset{p,k}{\overline{\psi}}{}_{,s}\left\{4\,\overset{\dot{k}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{k}}{\underset{0}{\xi}}{}^{m}+4\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{m}\right.$$
$$\left.-4\,\overset{\dot{k}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{m}-3\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{k}}{\underset{0}{\xi}}{}^{m}\right\}+\overset{p,k}{\overline{\psi}}{}_{,m}\left\{-\overset{\dot{k}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{k}}{\underset{0}{\xi}}{}^{s}-\tfrac{3}{2}\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{s}\right.$$
$$\left.\left.+4\,\overset{\dot{k}}{\underset{0}{\xi}}{}^{s}\,\overset{\dot{p}}{\underset{0}{\xi}}{}^{s}+4\sum_{q=1}^{N}{}'\,\overset{q}{\underset{2}{M}}\,\overset{q,k}{\overline{\psi}}+\sum_{q=1}^{N}{}''\,\overset{q}{\underset{2}{M}}\,\overset{q,p}{\overline{\psi}}\right\}\right]. \qquad (5.3)$$

The equations (5.3) are correct up to the sixth order of approximation. This is the equation which was solved by H. P. Robertson (1938) to find the perihelion motion of the planet Mercury. The $\overset{k}{\underset{0}{\xi}}{}^{s}$ which are appearing on the right-hand side of the equation (5.3) are the solutions of the Newtonian equations (5.1).

Our calculations show that $\overset{k}{\underset{2}{M}}$ is a constant, whereas $\overset{k}{\underset{4}{M}}$, $\overset{k}{\underset{6}{M}}$, etc. are functions of time. This result taken along with (3.9) shows how the associated mass parameter $\overset{k}{M}$ varies with time when the gravitational interaction is taken account of. This effect for a slowly varying field is in agreement with Mach's conjecture regarding the inertia of a particle.

Abstract

A mistake has been noticed in the current theory of approximations giving the equations of motion of N particles in a relativistic gravitational field. The genesis of the mistake has been shown and the method of approximation is modified so as to give correct results. The new equations have the same form as the old ones considered by Robertson but in substance there is a difference. In the procedure given here the splitting up of terms of the field equations is the same as of the 1949 theory and the co-ordinate conditions are used in a form free from singularities. The calculations also show in what manner Mach's conjecture regarding the inertia of a particle is verified.

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