INTERNAL BALLISTICS FOR POWER LAW OF BURNING WITH MOST GENERAL FORM FUNCTION

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1. Introduction

All the present theories of internal ballistics of guns assume a linear law of burning of the propellants as it renders the mathematical treatment of the ballistic problem comparatively easy. Since the experiments showed that many propellants burn according to the power law of burning, Clemmow in 1928 gave numerical solution of the equations of internal ballistics for power law of burning, assuming simple quadratic form function.

Now in the present-day advancement of the ballistic theories the need of a better form function has arisen. The simple quadratic form function is not applicable when we are considering a more digressive burning surface or when the propellant is a chopped one, e.g. chopped cord. Even for the most simple case of a cord we have a cubic form function although we generally take a quadratic one. The most general form function can be of the form $Z = \phi(f) \equiv (1-f)[1+\chi(f)]$, where $\chi(0) = 0$ which satisfies the two conditions that when $Z = 0, f = 1$ and when $Z = 1, f = 0$.

In the present communication the author has extended the results of Clemmow to the case of general form function $Z = \phi(f)$ and has deduced the results for other form functions, such as general cubic form function, for spherical propellants and for quadratic form function.

2. Fundamental equations

The four fundamental equations of internal ballistics for power law of burning, and having general form function, are:

Energy equation is

$$FCZ = p\left\{A(x+l)-CZ\left(b-\frac{1}{\delta}\right)\right\}+\tfrac{1}{2}(\gamma-1)w_1v^2 \qquad (1)$$

where $w_1 = 1{\cdot}05w+\frac{1}{3}C$.

Dynamical equation is

$$w_1\frac{dv}{dt} = Ap. \qquad (2)$$

Form function is

$$Z = \phi(f), \text{ where } \phi(f) \text{ is such that when } Z = 0, f = 1 \text{ and when } Z = 1, f = 0$$
$$\equiv (1-f)[1+\chi(f)], \text{ where } \chi(f) \text{ is zero for } f = 0. \qquad (3)$$

Also there is a condition on equation (3) that it should give one positive root less than unity.

Rate of burning equation is

$$D\frac{df}{dt} = -\beta p^{\alpha}. \qquad (4)$$

Making the following dimensionless transformation

$$\xi = 1+\frac{x}{l}$$

$$\eta = \frac{vAD}{FC\beta}\left(\frac{FC}{Al}\right)^{1-\alpha}$$

$$\zeta = \frac{pAl}{FC}$$

and

$$M = \frac{A^2D^2}{FC\beta^2 w_1}\left(\frac{FC}{Al}\right)^{2-2\alpha}$$

the above equations reduce to

$$Z = \zeta(\xi - BZ)+\tfrac{1}{2}(\gamma-1)\eta^2/M \qquad (5)$$

$$\eta\frac{d\eta}{d\xi} = M\zeta \qquad (6)$$

$$Z = \phi(f) \qquad (7)$$

and

$$\eta\frac{df}{d\xi} = -\zeta^{\alpha}. \qquad (8)$$

3. Solution of the equations for $Z = \phi(f)$ with $B = 0$

Now the variable η can be easily eliminated, by differentiating (5) and using (6) we have

$$dZ = \xi\, d\zeta + \gamma\zeta\, d\xi$$
$$= \xi^{1-\gamma}\, d(\zeta\xi^{\gamma}). \qquad (9)$$

From (6) and (8) we get

$$\frac{d\eta}{df} = -M\zeta^{1-\alpha} \qquad (10)$$

and equation (8) can be written as

$$\eta = -\zeta^{\alpha}\frac{d\xi}{df}. \qquad (11)$$

Substituting the value of η from (11) in (10) we get

$$\frac{d}{df}\left(\zeta^{\alpha}\frac{d\xi}{df}\right) = M\zeta^{1-\alpha}. \qquad (12)$$

Now make the following substitutions:

$$Y = \zeta\xi^{\gamma} \qquad (13)$$

so that equation (9) reduces to

$$dZ = \xi^{1-\gamma}\, dY \qquad (14)$$

Mathematically, the function $\phi(f)$ being known, we can express f as a function of Z, $f = \phi_1(Z)$ (say).

Therefore differentiating this function we get

$$df = \phi_1'(Z)\, dZ. \qquad (15)$$

From (9), (12) and (15) we get

$$\frac{d}{dZ}\cdot\frac{dZ}{df}\left[\zeta^{\alpha}\frac{d\xi}{dZ}\cdot\frac{dZ}{df}\right] = M\zeta^{1-\alpha}$$

or

$$\frac{1}{\phi_1'(Z)}\frac{d}{dZ}\left[\zeta^{\alpha}\frac{1}{\phi_1'(Z)}\frac{d\xi}{dZ}\right] = M\zeta^{1-\alpha}. \qquad (16)$$

Performing this differentiation we get

$$\frac{1}{\phi_1'(Z)}\left[\xi''\zeta^{\alpha}\frac{1}{\phi_1'(Z)} + \alpha\zeta^{\alpha-1}\,\zeta'\xi'\frac{1}{\phi_1'(Z)} + \zeta^{\alpha}\,\xi'\frac{d}{dZ}\left\{\frac{1}{\phi_1'(Z)}\right\}\right] = M\zeta^{1-\alpha}. \qquad (17)$$

This can be written as

$$\left\{\frac{1}{\phi_1'(Z)}\right\}^2\left[\xi'' + \alpha\xi'\frac{\zeta'}{\zeta}\right] + \xi'\frac{1}{\phi_1'(Z)}\frac{d}{dZ}\left\{\frac{1}{\phi_1'(Z)}\right\} = M\zeta^{1-2\alpha}. \qquad (18)$$

Differentiating (14) we get

$$(\gamma-1)\frac{\xi'}{\xi} = \frac{Y''}{Y'}. \qquad (19)$$

Differentiating this equation (19) again we get

$$\frac{\xi''}{\xi'} = \frac{Y'''}{Y''} + (2-\gamma)\frac{Y''}{(\gamma-1)Y'} \qquad (20)$$

and from (13)

$$\frac{\zeta'}{\zeta} = \frac{Y'}{Y} - \gamma\frac{\xi'}{\xi} = \frac{Y'}{Y} - \frac{\gamma Y''}{(\gamma-1)Y'}. \qquad (21)$$

Substituting these values in (18) we get

$$\left\{\frac{1}{\phi_1'(Z)}\right\}^2\left[\frac{Y'''}{Y''} + (n-2)\frac{Y''}{Y'} + \alpha\frac{Y'}{Y}\right] + \frac{1}{\phi_1'(Z)}\frac{d}{dZ}\left\{\frac{1}{\phi_1'(Z)}\right\}$$

$$= \frac{Q(Y')^{2-2n}}{Y''\,Y^{2\alpha-1}} \qquad (22)$$

where $n = \dfrac{\gamma(1-\alpha)}{\gamma-1}$ and $Q = (\gamma-1)M$.

For the particular function $\phi(f)$ we can find out $\phi_1(Z)$ and hence the equation (22) can always be integrated numerically to give a series of values of Y, Y' and Y'' in terms of Z, for a given propellant, i.e. given n and α and a given value of Q representing the loading conditions. For finite shot-start pressure the initial conditions are

$$Y_0 = \zeta_0,\ Z = Z_0 = \zeta_0$$

and also from (14) we have $Y' = 1$ and $Y'' = 0$.

For zero shot-start pressure the initial conditions are

$$Y_0 = 0,\ Z_0 = 0,\ Y' = 1 \text{ and } Y'' = 0.$$

We can develop a series solution of the equation (22) if the particular form of $\phi(f)$ is assumed and the initial calculations can be checked up.

Here the shot-travel is given by equation (14) as

$$\xi^{\gamma-1} = Y'. \qquad \text{(23)}$$

The pressure is given by equation (13) as

$$\zeta = \frac{Y}{\xi^\gamma} = \frac{Y}{(Y')^{\frac{\gamma}{\gamma-1}}}. \qquad \text{(24)}$$

For maximum pressure $\frac{d\zeta}{dZ} = 0$ and therefore differentiating equation (23) and putting $\frac{d\zeta}{dZ} = 0$ we have

$$\frac{Y'}{Y} = \frac{\gamma}{\gamma-1}\frac{Y''}{Y'} \qquad \text{(25)}$$

and this equation can be solved numerically from the tabulated values of Y, Y' and Y''.

At all-burnt position, $Z = 1$.

Therefore the velocity can be obtained from the energy equation in the following form:

$$\eta^2 = 2M(Z-\zeta\xi)/(\gamma-1)$$
$$= 2M\left(Z-\frac{Y}{Y'}\right)\Big/(\gamma-1).$$

Hence

$$v^2 = 2FC\left(Z-\frac{Y}{Y'}\right)\Big/(\gamma-1)w_1. \qquad \text{(26)}$$

After all-burnt there is simple adiabatic expansion and

$$\zeta\xi^\gamma = Y_2 \qquad \text{(27)}$$

and therefore muzzle velocity is given by

$$v_3{}^2 = \frac{2FC}{(\gamma-1)w_1}\left(1-\frac{Y_2}{\xi_3{}^{\gamma-1}}\right). \qquad \text{(28)}$$

Now in further analysis we will assume different forms of the function $\phi(f)$ and show how these equations are modified.

4. Solution for cubic form function

$$Z = (1-f)(1+\theta f+\theta' f^2).$$

This form function can be written as

$$Z = 1+(\theta-1)f+(\theta'-\theta)f^2-\theta' f^3$$

or

$$f^3+\frac{(\theta-\theta')}{\theta'}f^2+\frac{(1-\theta)}{\theta'}f+\frac{(Z-1)}{\theta'} = 0. \qquad \text{(29)}$$

This further reduces to the form

$$f'^3+p_1f'+q_1=0 \qquad (30)$$

where $p_1=\frac{1-\theta}{\theta'}-\frac{1}{3}\left(\frac{\theta-\theta'}{\theta'}\right)^2$

and $q_1=\frac{(Z-1)}{\theta'}-\frac{(1-\theta)(\theta-\theta')}{3\theta'^2}+\frac{2}{27}\frac{(\theta-\theta')^3}{\theta'^3}$.

The solutions of the equation (30) are

$$f_1'=(A)^{\frac{1}{3}}+(B)^{\frac{1}{3}}$$

$$f_2'=\omega(A)^{\frac{1}{3}}+\omega^2(B)^{\frac{1}{3}}$$

$$f_3'=\omega^2(A)^{\frac{1}{3}}+\omega(B)^{\frac{1}{3}}$$

where $A=-\frac{q_1}{2}+\sqrt{\frac{q_1^2}{4}+\frac{p_1^3}{27}}$.

$$B=-\frac{q_1}{2}-\sqrt{\frac{q_1^2}{4}+\frac{p_1^3}{27}}.$$

Hence

$$f_1=\left[(A)^{\frac{1}{3}}+(B)^{\frac{1}{3}}\right]-\frac{(\theta-\theta')}{3\theta'}$$

$$f_2=\left[\omega(A)^{\frac{1}{3}}+\omega^2(B)^{\frac{1}{3}}\right]-\frac{(\theta-\theta')}{3\theta'}$$

$$f_3=\left[\omega^2(A)^{\frac{1}{3}}+\omega(B)^{\frac{1}{3}}\right]-\frac{(\theta-\theta')}{3\theta'}.$$

Now the values of the roots depend on the values of θ and θ', so that let us consider the most general root given by

$$f=\frac{K_1}{\theta_1}\left[-\{Z+k\}+\sqrt{(Z+k)^2+k_1}\right]^{\frac{1}{3}}+\frac{K_2}{\theta_1}\left[-\{Z+k\}-\sqrt{(Z+k)^2+k_1}\right]^{\frac{1}{3}}-K_3$$

where $k=\frac{2}{27}\frac{(\theta-\theta')^3}{\theta'^2}-\frac{(1-\theta)(\theta-\theta')}{3\theta'}-1,\ \theta_1=(2\theta')^{\frac{1}{3}}$

$$k_1=\frac{4\theta'^2p_1}{27} \quad \text{and} \quad k_3=\frac{\theta-\theta'}{3\theta'}.$$

Also K_1 and K_2 can have values 1, 1; ω, ω^2 and ω^2, ω respectively. Let us make the substitution $(Z+k)=Z_1$ (say)

$$\therefore f=\frac{K_1}{\theta_1}\left[-Z_1+\sqrt{Z_1^2+k_1}\right]^{\frac{1}{3}}+\frac{K_2}{\theta_1}\left[-Z_1-\sqrt{Z_1^2+k_1}\right]^{\frac{1}{3}}-K_3.$$

Here $\phi_1(Z_1)=\frac{K_1}{\theta_1}\left[-Z_1+\sqrt{Z_1^2+k_1}\right]^{\frac{1}{3}}+\frac{K_2}{\theta_1}\left[-Z_1-\sqrt{Z_1^2+k_1}\right]^{\frac{1}{3}}-K_3 \qquad (32)$

$$\therefore\ \frac{1}{\phi_1'(Z_1)} = \frac{3\theta_1\sqrt{Z_1^2+k_1}}{\left[-K_1\left\{-Z_1+\sqrt{Z_1^2+k_1}\right\}^{\frac{1}{3}}+K_2\left\{-Z_1-\sqrt{Z_1^2+k_1}\right\}^{\frac{1}{3}}\right]}$$

or
$$\left\{\frac{1}{\phi_1'(Z_1)}\right\}^2 = \frac{9\theta_1^2(Z_1^2+k_1)}{\left[-K_1\left\{-Z_1+\sqrt{Z_1^2+k_1}\right\}^{\frac{1}{3}}+K_2\left\{-Z_1-\sqrt{Z_1^2+k_1}\right\}^{\frac{1}{3}}\right]^2} \quad \ldots \quad (33)$$

Also

$$\frac{1}{\phi_1'(Z_1)}\frac{d}{dZ_1}\left\{\frac{1}{\phi_1'(Z_1)}\right\} = \frac{9\theta_1^2 Z_1\left\{-K_1\left(-Z_1+\sqrt{Z_1^2+k_1}\right)^{\frac{1}{3}}+K_2\left(-Z_1-\sqrt{Z_1^2+k_1}\right)^{\frac{1}{3}}\right\} - 3\theta_1^2\sqrt{Z_1^2+k_1}\left\{K_1\left(-Z_1+\sqrt{Z_1^2+k_1}\right)^{\frac{1}{3}}+K_2\left(-Z_1-\sqrt{Z_1^2+k_1}\right)^{\frac{1}{3}}\right\}}{\left[-K_1\left\{-Z_1+\sqrt{Z_1^2+k_1}\right\}^{\frac{1}{3}}+K_2\left\{-Z_1-\sqrt{Z_1^2+k_1}\right\}^{\frac{1}{3}}\right]^3} \quad \ldots \quad (34)$$

Hence our equation (22) reduces to:

$$\frac{(Z_1^2+k_1)}{\left[-K_1\left\{-Z_1+\sqrt{Z_1^2+k_1}\right\}^{\frac{1}{3}}+K_2\left\{-Z_1-\sqrt{Z_1^2+k_1}\right\}^{\frac{1}{3}}\right]^2}\left[\frac{Y'''}{Y''}+(n-2)\frac{Y''}{Y'}+\alpha\frac{Y'}{Y}\right]+$$
$$+\frac{\left[9\theta_1^2 Z_1\left\{-K_1\left(-Z_1+\sqrt{Z_1^2+k_1}\right)^{\frac{1}{3}}+K_2\left(-Z_1-\sqrt{Z_1^2+k_1}\right)^{\frac{1}{3}}\right\} - 3\theta_1^2\sqrt{Z_1^2+k_1}\left\{K_1\left(-Z_1+\sqrt{Z_1^2+k_1}\right)^{\frac{1}{3}}+K_2\left(-Z_1-\sqrt{Z_1^2+k_1}\right)^{\frac{1}{3}}\right\}\right]}{9\theta_1^2\left[-K_1\left\{-Z_1+\sqrt{Z_1^2+k_1}\right\}^{\frac{1}{3}}+K_2\left\{-Z_1-\sqrt{Z_1^2+k_1}\right\}^{\frac{1}{3}}\right]^3} = \frac{Q_1(Y')^{2-2n}}{Y''\,Y^{2\alpha-1}} \quad \ldots \quad (35)$$

where $n = \dfrac{\gamma(1-\alpha)}{(\gamma-1)}$ and $Q_1 = \dfrac{(\gamma-1)M}{9\theta_1^2}$.

In this case this is the equation which gives Y, Y' and Y'' in terms of Z_1.

Further analysis is the same as has already been discussed in section 3.

As an example let us consider the case of a cord. In this case Z is a cubic in f given by

$$Z = (1-f)\left(1+f+\frac{f^2}{\lambda}\right)$$

λ is generally of the order of 200.

$$\therefore Z = (1-f)(1+f+{\cdot}005f^2).$$

This can be written as

$$\cdot005f^3+\cdot995f^2+(Z-1) = 0$$
$$\text{or}\quad f^3+199f^2+200(Z-1) = 0.$$

Here there is one change of sign, because $(Z-1)$ is negative; hence there is only one positive root. Also the discriminant of the transformed cubic is positive, therefore $K_1 = K_2 = 1$. Now we can set up our equation (35) in this case and can integrate this numerically.

5. Solution for the cubic form function

$$Z = (1-f)(1+f+f^2).$$

This form function can be written as

$$Z = (1-f^3). \qquad \ldots \quad (37)$$

From this equation we get

$$f = (1-Z)^{\frac{1}{3}}. \qquad \ldots \quad (38)$$

Therefore our

$$\phi_1(Z) = (1-Z)^{\frac{1}{3}}.$$

Hence our equation (22) reduces to

$$9(1-Z)^{\frac{4}{3}}\left[\frac{Y'''}{Y''}+(n-2)\frac{Y''}{Y'}+\alpha\frac{Y'}{Y}\right]-6(1-Z)^{\frac{1}{3}} = \frac{Q(Y')^{2-2n}}{Y''Y^{2\alpha-1}}. \qquad \ldots \quad (39)$$

This can be further written as

$$(1-Z)^{\frac{4}{3}}\left[\frac{Y'''}{Y''}+(n-2)\frac{Y''}{Y'}+\alpha\frac{Y'}{Y}\right]-\frac{2}{3}(1-Z)^{\frac{1}{3}} = \frac{Q_1(Y')^{2-2n}}{Y''\,Y^{2\alpha-1}} \qquad \ldots \quad (40)$$

where $n = \dfrac{\gamma(1-\alpha)}{(\gamma-1)}$ and $Q_1 = \dfrac{(\gamma-1)M}{9}$.

In this case, this is the equation which gives Y, Y' and Y'' in terms of Z. Further analysis we have to do exactly in the same way as done in section 3.

6. Solution for the general quadratic form function

$$Z = (1-f)(1+\theta f).$$

This form function can be written as

$$Z = 1+(\theta-1)f-\theta f^2. \qquad \ldots \quad (41)$$

From this equation we get

$$f = \left[\frac{-(\theta-1)}{2\theta}+\frac{\sqrt{(\theta+1)^2-4\theta Z}}{2\theta}\right]$$

(Since we have to consider positive root, we take the positive sign.)

Therefore

$$\phi_1(Z) = \left[\frac{-(\theta-1)+\sqrt{(\theta+1)^2-4\theta Z}}{2\theta}\right] \qquad \ldots \quad (42)$$

Hence our equation (22) reduces to

$$\left[(\theta+1)^2-4\theta Z\right]\left[\frac{Y'''}{Y''}+(n-2)\frac{Y''}{Y'}+\alpha\frac{Y'}{Y}\right]-2\theta = \frac{Q(Y')^{2-2n}}{Y''\,Y^{2\alpha-1}}. \qquad \ldots \quad (43)$$

This can be further modified as has been done in Clemmow's paper.

Let us have

$$qY = q\zeta\xi^{\gamma} = Y_1 \text{ (say)}$$

and

$$Z_1 = qZ \quad \text{where} \quad q = \frac{4\theta}{(1+\theta)^2}.$$

Therefore the equation (43) modifies to

$$\left(1-Z_1\right)\left[\frac{Y_1'''}{Y_1''}+(n-2)\frac{Y_1''}{Y_1'}+\alpha\frac{Y_1'}{Y_1}\right]-\frac{1}{2}=\frac{Q(Y_1')^{2-2n}}{Y_1''Y_1^{2\alpha-1}}. \quad .. \quad (44)$$

This is the equation which gives on numerical integration the values of Y, Y' and Y'' in terms of Z_1.

The rest of the treatment is the same as done by Clemmow.

7. Solution for the form function $Z = (1-f)$.

Here our $\phi_1(Z) = (1-Z)$.

Hence our equation (22) reduces to

$$\frac{Y'''}{Y''}+(n-2)\frac{Y''}{Y'}+\alpha\frac{Y'}{Y}=\frac{M(\gamma-1)\,(Y')^{2-2n}}{Y''\,Y^{2\alpha-1}} \quad .. \quad .. \quad .. \quad (45)$$

and we have to proceed exactly in the same way as in section 3. The series solution for this equation has been given by Clemmow.

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Summary

In this communication the author deals with the problem of internal ballistics of guns for power law of burning and using propellants which have the most general form function, $Z = \phi(f)$; where function $\phi(f)$ is such that when $Z = 0, f = 1$ and when $Z = 1, f = 0$. Further from this solution the author has deduced the results for a general cubic form function given by $Z = (1-f)(1+\theta f+\theta' f^2)$, and from this for a cubic $Z = (1-f^3)$ which represents the form function for spherical propellants. Also the results for a general quadratic form function given by $Z = (1-f)(1+\theta f)$ have been deduced.

References

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