ON A QUESTION OF J. M. WHITTAKER *

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(*Received June 18, 1955; read January 1, 1956*)

§ 1. *The question.* J. M. Whittaker (1954) has asked the question:

(i) *Can* $F(z) = \sum_{n=0}^{\infty} a_n f(nz)$, *where* $f(z)$ *is an integral function either rational or transcendental, vanish identically except in the case where* $f(z)$ *is rational?*

He has shown that, in the excepted case, the answer to this question is in the affirmative, and also stated that, in the case of transcendental $f(z)$, (i) is equivalent to the question:

(ii) *Can a function of the form* $g(e^z)$, *where* g *is entire, have an infinite number of zero Taylor coefficients?*

We propose to answer (ii) in the affirmative by constructing a sequence a_n, $n = 1, 2, 3, \ldots$, such that

$$g(z) = \sum_{n=1}^{\infty} a_n z^n$$

is an entire function and the function

$$g(e^z) = \sum_{n=0}^{\infty} b_n z^n, \quad \text{where} \quad b_n = \frac{1}{\lfloor n} \cdot \sum_{r=1}^{\infty} a_r \cdot r^n,$$

has its coefficients $b_n = 0$ for an increasing sequence m_i $(i = 1, 2, \ldots)$ of values of n, i.e.

$$\sum_{r=1}^{\infty} a_r \cdot r^{m_i} = 0,\ i = 1, 2, 3, \ldots,\ m_1 < m_2 < m_3 < \ldots \tag{1}$$

In other words, our object is to define a_r $(r = 1, 2, \ldots)$ and an increasing sequence of positive integers n_i $(i = 1, 2, \ldots)$ so that we have

$$\sum_{r=1}^{r=2^{2^{n_j}}} a_r \cdot r^{m_k} = 0 \text{ where } 1 \leqslant k \leqslant j \text{ for every } j = 1, 2, \ldots \tag{2}$$

which, together with the following condition

$$|a_r \cdot \lfloor r| < 1, \text{ for all large } r, \tag{3}$$

ensures (1) and is sufficient to make $\Sigma a_r z^r$ an entire function.

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* The late Dr. T. Vijayaraghavan drew my attention to Whittaker's question and also indicated the lines on which one could proceed to answer it. Unfortunately Dr. Vijayaraghavan passed away before I could have the benefit of his collaboration in details and so I am alone to be blamed for the shortcomings of the present note (K. Padmavally).

§ 2. *A definition of the a_r's satisfying* (2). We may suppose that $a_r = 0$ for $r \neq 2^{2^k}$, $k = 0, 1, 2, \ldots\ldots$ so that we have merely to define $a_r \equiv a(r)$ for $r = 2^{2^k}$

$$= 2_2[k] \text{ (say)}.$$

Let

$$n_i = \sum_{r=1}^{i} r = \frac{i(i+1)}{2}, \quad m_i = 4 \cdot 2_2[n_i], \quad i = 1, 2, 3, \ldots. \tag{4}$$

Also let $a(2_2[0]) = a(2)$, $a(2_2[1]) = a(4)$ be chosen so that (2) holds for $j = 1$ and $k = 1$, i.e. so that

$$a(2) \,.\, 2^{m_1} + a(4) \,.\, 4^{m_1} = 0, \tag{5}$$

or e.g.

$$a(2) = 1, \quad a(2^2) = -2^{-m_1} = -2^{-16}.$$

Then let $a(2_2[k])$ be defined, for the range $n_1 < k \leqslant n_2$ of values of k, by means of the equations:

$$a(2_2[2]) \,.\, (2_2[2])^{m_1} + a(2_2[3]) \,.\, (2_2[3])^{m_1} = 0\,; \tag{6a}$$

$$a(2_2[2]) \,.\, (2_2[2])^{m_2} + a(2_2[3]) \,.\, (2_2[3])^{m_2} + \{a(2) \,.\, 2^{m_2} + a(4) \,.\, 4^{m_2}\} = 0, \tag{6b}$$

so that (2) holds for $j = 2$ and $k = 1, 2$, as a result of (5) added to (6*a*) taken in conjunction with (6*b*). We have so far defined $a(2_2[k])$ for $0 \leqslant k \leqslant n_2$, ensuring that (2) holds for $j = 1, 2$. Proceeding in this way we can define $a(2_2[k])$ inductively for $0 \leqslant k \leqslant n_{i+1}$, $(i \geqslant 2)$, ensuring that (2) holds for $j = 1, 2, \ldots. i+1$. The procedure consists in assuming that $a(2_2[k])$ has been defined for $0 \leqslant k \leqslant n_i$ so that (2) holds for $j = 1, 2, \ldots. i$, and then determining $a(2_2[k])$ for $n_i < k \leqslant n_{i+1}$ by the $i+1$ equations:

$$\sum_{\lambda=1}^{i+1} a(2_2[n_i+\lambda]) \,.\, (2_2[n_i+\lambda])^{m_k} = 0 \quad \text{for } k = 1, 2, \ldots. i, \tag{7a}$$

$$\sum_{\lambda=1}^{i+1} a(2_2[n_i+\lambda]) \,.\, (2_2[n_i+\lambda])^{m_{i+1}} + \sum_{\nu=0}^{n_i} a(2_2[\nu])(2_2[\nu])^{m_{i+1}} = 0. \tag{7b}$$

(7*a*) and (7*b*) and the assumption we have made together ensure that (2) holds for $j = 1, 2, \ldots. i+1$. For, the assumption in question is that

$$\sum_{\nu=0}^{n_j} a(2_2[\nu]) \,.\, (2_2[\nu])^{m_k} = 0 \quad \text{for } j = 1, 2, \ldots. i \tag{8}$$

where $1 \leqslant k < j$ for each j. (8) with $j = i$, $1 \leqslant k \leqslant i$, gives us i equations. Adding to each of these i equations the corresponding equation of (7*a*) with the same value of k, and taking the resulting i equations along with (7*b*), we see that (2) holds for $j = i+1$, and every k such that $1 \leqslant k \leqslant i+1$. This means, in view of (8) again, that (2) holds for each k such that $1 \leqslant k \leqslant j$, when $j = 1, 2, \ldots. i+1$ successively. Therefore finally we have defined $a(2_2[k])$ successively over the ranges $0 \leqslant k \leqslant n_1$, $n_1 < k \leqslant n_2$, $n_2 < k \leqslant n_3$, ... so that (2) holds for every positive integer j.

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§ 3. *Proof that the a_r's defined satisfy* (3). Solving equations (7*a*) and (7*b*) determinantally, we get $a(r)$ in the following form for $2_2[n_i] < r \leqslant 2_2[n_{i+1}]$:

$$(9) \qquad a(2_2[n_i+\lambda]) = \frac{\det((A_{\mu,k}))}{\det((B_{\mu,k}))}, \; \lambda = 1, 2, \ldots . \, i+1,$$

where

$$(10) \qquad A_{\mu,k} = \begin{cases} (2_2[n_i+\mu])^{m_k} \text{ for } \begin{cases} \mu = 1, 2 \ldots . \lambda-1, \lambda+1, \ldots . i+1, \\ k = 1, 2, \ldots . i+1; \end{cases} \\ 0 \text{ for } \begin{cases} \mu = i+2, \\ k = 1, 2, \ldots . i; \end{cases} \\ -\sum_{\nu=0}^{n_i} a(2_2[\nu])(2_2[\nu])^{m_{i+1}} \text{ for } \begin{cases} \mu = i+2, \\ k = i+1; \end{cases} \end{cases}$$

$$(11) \qquad B_{\mu,k} = (2_2[n_i+\mu])^{m_k} \text{ for } \begin{cases} \mu = 1, 2, \ldots . i+1, \\ k = 1, 2, \ldots . i+1. \end{cases}$$

The inequality $|a(r)| \leqslant 1$ evidently holds for $r \leqslant 2_2[n_1]$. We proceed to show that, if it holds for $r \leqslant 2_2[n_i]$, then it holds for $2_2[n_i] < r \leqslant 2_2[n_{i+1}]$ in the stronger form $|a(r) \underline{|r}| < 1$. We then conclude by induction that (3) holds for $r > 2_2[n_1]$.

From (9) and (10) we get, for $\lambda = 1, 2, \ldots . i+1$,

$$(12) \qquad |a(2_2[n_i+\lambda])| = \frac{\left| \sum_{\nu=0}^{n_i} a(2_2[\nu]) \,.\, (2_2[\nu])^{m_{i+1}} \right| . \left| \det((C_{\mu,k})) \right|}{|\det((B_{\mu,k}))|}$$

where

$$(13) \qquad C_{\mu,k} = (2_2[n_i+\mu])^{m_k} \text{ for } \begin{cases} \mu = 1, 2, \ldots . \lambda-1, \lambda+1, \ldots . i+1, \\ k = 1, 2, \ldots . i, \end{cases}$$

the set $\mu = 1, 2, \ldots . \lambda-1$ being empty in the case $\lambda = 1$.

Now

$$|\det((B_{\mu,k}))| \geqslant \begin{cases} |\text{largest absolute value of a term in its expansion} \\ -(\text{sum of absolute values of other terms})| \end{cases}$$

$$= \left| \prod_{k=1}^{i+1} B_{k,k} - \sum \left(\prod_{k=1}^{i+1} B_{\mu_k, k} \right) \right|$$

by Lemma 1 of § 4, the Σ including all terms ($\underline{|i+1}-1$ in number) made up of elements $B_{\mu_k, k}$ such that $\mu_k \neq k$ for at least one k. Hence by Lemma 1 again,

$$(14) \qquad |\det((B_{\mu,k}))| > \prod_{k=1}^{i+1} B_{k,k} \left\{ 1 - \sum \frac{1}{2\underline{|i+1}} \right\} > \frac{1}{2} \prod_{k=1}^{i+1} B_{k,k}.$$

Next

$$|\det((C_{\mu,k}))| \leqslant \begin{cases} (\text{largest absolute value of a term in its expansion}) \times \\ \times (\text{total number of terms}) \end{cases}$$

$$(15) \qquad < \left(\prod_{k=1}^{\lambda-1} C_{k,k} \right) \left(\prod_{k=\lambda}^{i} C_{k+1,k} \right) . \underline{|i}$$

by Lemma 2 of § 4. Using (14) and (15) in (12), we obtain

$$\left| a(2_2[n_i+\lambda]) \right| < \left| \sum_{\nu=0}^{n_i} a(2_2[\nu]) \cdot (2_2[\nu])^{m_{i+1}} \right| \left(\prod_{k=1}^{\lambda-1} C_{k,k} \right) \cdot \left(\prod_{k=\lambda}^{i} C_{k+1,k} \right) \cdot 2 \underline{|i} \Big/ \prod_{k=1}^{i+1} B_{k,k}.$$

Now assuming that $|a(r)| \leqslant 1$ for $r \leqslant 2_2[n_i]$ and using Lemma 3 of § 4, we get

$$\begin{aligned} | a(2_2[n_i+\lambda]) | &\leqslant \sum_{\nu=0}^{n_i} (2_2[\nu])^{m_{i+1}} \cdot 2^{(m_i - m_{i+1}) \cdot 2^{n_i+i+1}} \cdot 2 \underline{|i} \\ &< 2 \cdot 2^{m_{i+1} \cdot 2^{n_i}} \cdot 2^{(m_i - m_{i+1}) \cdot 2^{n_i+i+1}} \cdot 2 \underline{|i} \\ &< 4 \cdot \underline{|i} \cdot 2^{(m_i - m_{i+1}/2)\, 2^{n_i+i+1}} \end{aligned}$$

Hence for $\lambda = 1, 2, \ldots . i+1$, we have

$$\begin{aligned} \left| a(2_2[n_i+\lambda]) \cdot \underline{|2_2[n_i+\lambda]} \right| &\leqslant \left| a(2_2[n_i+\lambda]) \cdot \underline{|2^{2^{n_i+i+1}}} \right| \\ &= \left| a(2_2[n_i+\lambda]) \cdot \underline{|m_{i+1}/4} \right| \\ &< 4 \underline{|i} \cdot 2^{(m_i - m_{i+1}/2) \cdot 2^{n_i+i+1}} \cdot (m_{i+1}/4)^{m_{i+1}/4} \\ &= 4 \underline{|i} \cdot 2^{(m_i - m_{i+1}/4) \cdot 2^{n_i+i+1}} * \\ &\leqslant 4 \underline{|i} \cdot 2^{-2^{n_i+i+1}} < 1. \end{aligned}$$

This is (3) for $2_2[n_i] < r \leqslant 2_2[n_{i+1}]$ and (as explained already) it leads to (3) for $r > 2_2[n_1]$.

§ 4. *The results on determinants assumed in § 3 and their proofs.*

LEMMA 1. *If $B_{\mu,k}$ is defined by (11), then*

$$\prod_{k=1}^{i+1} B_{k,k} \Big/ \prod_{k=1}^{i+1} B_{\mu_k, k} > 2_2[n_i+1] > 2 \underline{|i+1}$$

where the denominator is the absolute value of any term in the expansion of det $((B_{\mu,k}))$, made up of elements $B_{\mu_k, k}$ belonging to the k^{th} column and any corresponding chosen $\mu_k{}^{th}$ row such that $\mu_k \neq k$ for at least one k.

Proof. Consider the nonnull set of integers k such that $\mu_k \neq k$. If j is the largest of these integers, then either $j = i+1$ or $2 \leqslant j \leqslant i$. In the case of both alternatives

(16) $$\mu_j \leqslant j-1, \ \mu_k \leqslant j \ (1 \leqslant k \leqslant j-1).$$

(16) is obvious in the case of the first alternative, and (16) follows, in the case of the second alternative, from the consideration that we have to choose, the μ_j^{th} row

* $(m_i - m_{i+1}/4)$ is negative and an integer and hence it cannot exceed -1.

ruling out the j^{th} and all subsequent rows, and the $\mu_k{}^{th}$ row $(1 \leqslant k \leqslant j-1)$ ruling out the $(j+1)^{th}$ and all subsequent rows. Under both alternatives it is evident that

$$\prod_{k=1}^{i+1} B_{k,k} \Big/ \prod_{k=1}^{i+1} B_{\mu_k,k} = \prod_{k=1}^{j} B_{k,k} \Big/ \prod_{k=1}^{j} B_{\mu_k,k}$$

$$= \prod_{k=1}^{j} (l_k/l_{\mu_k})^{m_k} \quad \{l_k = 2_2[n_i+k]\}$$

$$= (l_j/l_{\mu_j})^{m_j} \cdot \prod_{k=1}^{j-1} (l_k/l_{\mu_k})^{m_k}$$

$$\geqslant (l_j/l_{\mu_j})^{m_j} \cdot \prod_{k=1}^{j-1} (l_1/l_{\mu_k})^{m_k}$$

$$\geqslant (l_j/l_{j-1})^{m_j} (l_1/l_j)^{m_{k'}} \prod_{k=1}^{j-1}{}' (l_1/l_{j-1})^{m_k} \tag{17*}$$

where k' is the value of k for which $\mu_k = j$ and $\prod_{k=1}^{j-1}{}'$ is a product of factors for the values of k from 1 to $j-1$, omitting k'. This is so since by (16), $l_{\mu_j} \leqslant l_{j-1}$ and $l_{\mu_k} \leqslant l_j$ $(1 \leqslant k \leqslant j-1)$, equality prevailing in the last relation for only one μ_k (on account of our having to choose different μ_k's to be different). Now $(l_1/l_{j-1}) \leqslant 1$ for $j \geqslant 2$ and so $(l_1/l_{j-1})^{m_k} \geqslant (l_1/l_{j-1})^{m_{j-1}}$ for $1 \leqslant k \leqslant j-1$. Thus, from (17),

$$\prod_{k=1}^{i+1} B_{k,k} \Big/ \prod_{k=1}^{i+1} B_{\mu_k,k} \geqslant (l_j/l_{j-1})^{m_j} \cdot (l_1/l_j)^{m_{j-1}} \cdot (l_1/l_{j-1})^{(j-2)m_{j-1}}.$$

Since $l_j = l_{j-1}^2$ by definition, the above step gives

$$\prod_{k=1}^{i+1} B_{k,k} \Big/ \prod_{k=1}^{i+1} B_{\mu_k,k} \geqslant (l_{j-1})^{m_j - 2m_{j-1} - (j-2)m_{j-1}} \cdot (l_1)^{m_{j-1} + (j-2)m_{j-1}}$$

$$\geqslant l_1^{(j-1)m_{j-1}} > l_1 = 2^{2^{n_i+1}} > 2\underline{|i+1}. \tag{18}$$

Lemma 2. *If $C_{\mu,k}$ is defined by* (13), *then*

$$\left(\prod_{k=1}^{\lambda-1} C_{k,k}\right)\left(\prod_{k=\lambda}^{i} C_{k+1,k}\right) \Big/ \prod_{k=1}^{i} C_{\mu_k,k} > 2^{2^{n_i+1}} > 1$$

where the denominator is the absolute value of any term in the expansion of det $((C_{\mu,k}))$ *not made up of diagonal elements alone, and where, in the special case $\lambda = i+1$, the product $\prod_{k=\lambda}^{i} C_{k+1,k}$ is empty, i.e. the product is replaced by* 1 *in the above relation.*

The proof is exactly like that of Lemma 1.

Lemma 3. *In the notation of Lemmas* 1, 2,

$$\left(\prod_{k=1}^{\lambda-1} C_{k,k}\right)\left(\prod_{k=\lambda}^{i} C_{k+1,k}\right) \Big/ \prod_{k=1}^{i+1} B_{k,k} < 2^{(m_i - m_{i+1}) \cdot 2^{n_i+i+1}}$$

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* If $j = 2$, then $\prod_{k=1}^{j-1}{}'$ is empty, but (18) obviously holds.

Proof. The left-hand member of the inequality to be proved is

$$\left(\prod_{k=1}^{\lambda-1} l_k^{m_k}\right)\left(\prod_{k=\lambda}^{i} l_{k+1}^{m_k}\right)\Big/\prod_{k=1}^{i+1} l_k^{m_k} \quad \{l_k = 2_2[n_i+k]\}$$

$$= \prod_{k=\lambda}^{i} l_{k+1}^{m_k-m_{k+1}} < l_{i+1}^{m_i-m_{i+1}} = 2^{(m_i-m_{i+1})2^{n_i+i+1}}.$$

In the special case $\lambda = i+1$, the result of the Lemma is still true, for

$$\prod_{k=1}^{i} C_{k,k}\Big/\prod_{k=1}^{i+1} B_{k,k} = l_{i+1}^{-m_{i+1}} < l_{i+1}^{m_i-m_{i+1}} = 2^{(m_i-m_{i+1})2^{n_i+i+1}}.$$

Reference

Whittaker, J. M. (1954). A note on the series $\Sigma a_n f(nz)$. *Duke Journal*, **21**, 571–573.

Issued June 22, 1956.