ON THE CHANGE IN SHAPE OF A GRAVITATING FLUID SPHERE IN A UNIFORM EXTERNAL ELECTRIC FIELD

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1. Introduction

The problem of the gravitational instability of an infinitely conducting, incompressible fluid sphere under the influence of a *magnetic field* has been recently investigated by Chandrasekhar and Fermi (1953), G. Gjellestad (1954), and Auluck and Kothari (1955). It is found that the sphere is not a form of equilibrium but becomes a spheroid in the presence of the magnetic field.

In this note we shall study the problem of the stability of a conducting, gravitating, incompressible fluid sphere in a uniform external *electric field* by two different methods, namely, the Energy method due to Chandrasekhar and Fermi (1953) and Equilibrium method due to Ferraro (1954). The two methods, of course, give identical results, i.e. the sphere is unstable under uniform external electric field and tends to become a prolate spheroid, in the direction of the field, of ellipticity, $(\epsilon/R \rightarrow 0)$.

$$\epsilon/R = \frac{5}{2}\frac{E^2R^4}{GM^2} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1)$$

where E is the strength of the electric field, M and R denote the mass and the radius of the fluid sphere, and G the gravitational constant.

In the energy method we subject the sphere to a small deformation and calculate the change in the total energy due to the deformation. The change in the total energy consists of two terms, one representing the change in the electric energy and the other, the change in the gravitational energy of the sphere. It shall be shown that the change in the electric energy is proportional to a small quantity ϵ/R, while the change in the gravitational energy is proportional to the square of this quantity. The signs of the energy terms are such that the sphere will tend to become elongated in the direction of the given electric field. The condition that for stability the total change in energy should be minimum, then gives the expression for the ellipticity of the spheroid.

In the equilibrium method, we obtain the electric pressure and the gravitational pressure at the surface of the sphere and for stability the net pressure at the surface of the sphere must vanish.

2. Formulation of the Problem

We assume that the uniform and constant external electric field is due to a charge of strength Ed^2 situated on the axis of symmetry, and at a large distance d $(d >> R)$ from the centre C of the sphere (Fig. 1). This device makes the field uniform near the sphere and at the same time the external field vanishes at infinity. The introduction of the sphere with no initial electric field necessitates a superimposition of a field due to a dipole of moment ER^3 placed at the centre of the sphere, over the uniform external field E along the Z-direction, the direction of the superimposed dipole field being antiparallel to the initial field.

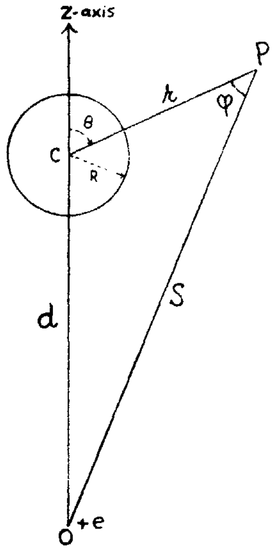

FIGURE 1

The components of the electric field in the radial and the transverse directions can be written down as ($\mu = \cos\theta$, θ being the polar angle),

$$^0E_r = \frac{Ed^2}{s^2}\cos\phi + \frac{2ER^3}{r^3}\mu$$

and .. .. .. (2)

$$^0E_\theta = -\frac{Ed^2}{s^2}\sin\phi + \frac{ER^3}{r^3}(1-\mu^2)^{\frac{1}{2}}$$

where $$\cos\phi = \frac{r+d\mu}{s} \text{ and } \sin\phi = \frac{d}{s}(1-\mu^2)^{\frac{1}{2}}. \quad \text{.. .. (3)}$$

Here s denotes the distance of an external point P (r, θ, ϕ) from the charge at O, ϕ being the angle that OP makes with CP.

Let us give a P_l-deformation to the sphere, so that the boundary given by $r = R$ changes to one given by

$$r = R + \epsilon P_l(\mu) \quad (l > 1) \quad \text{.. .. .. (4)}$$

where ϵ is a constant $(\epsilon/R < < 1)$, R is the radius of the undeformed sphere and $P_l(\mu)$ is the Legendre function of first kind and of order l.

3. The Expression for the Ellipticity by the Energy Method

We shall first calculate the change, $\Delta\mathcal{E}$, in the electric energy. Since the electric potential ϕ defined by

$$E = -\operatorname{grad}\phi$$

has to satisfy the Laplace's equation

$$\nabla^2 \phi = 0. \qquad \qquad (5)$$

The change in the components of the electric field due to deformation (4) can be put down as

$$\delta E_r = E \frac{\epsilon}{R} \sum_{n=1} a_n n (n+1) \left(\frac{R}{r}\right)^{n+2} P_n(\mu),$$

and

$$\delta E_\theta = E \cdot \frac{\epsilon}{R} \sum_{n=1} a_n n (R/r)^{n+2} P_n^1(\mu) \qquad \qquad (6)$$

where $P_n^1(\mu)$ is defined by

$$P_n^1(\mu) = (1-\mu^2)^{\frac{1}{2}} \frac{dP_n(\mu)}{d\mu}. \qquad \qquad (7)$$

The constants a_n are determined by the boundary condition that the tangential component of the electric field to the deformed sphere is zero. The tangential component of the electric field to the deformed boundary is, to the first order in ϵ,

given by $\left\{E_\theta\right\}_{R+\epsilon P_l} - \left\{E_r\right\}_R \frac{\epsilon}{R} P_l^1(\mu)$.

The boundary condition of the electric field requires that the above expression should be equal to zero.

Hence

$$\left\{E_\theta\right\}_{R+\epsilon P_l} - \left\{E_r\right\}_R \frac{\epsilon}{R} P_l^1(\mu) = 0. \qquad \qquad (8)$$

At the surface $d \sim s$, and using equations (2) and (6) in (8) we get

$$\sum_n a_n n P_n^1(\mu) = \frac{3}{(2l+1)} \left[(l+1) P_{l+1}^1 + l P_{l-1}^1\right]$$

which shows that

$$a_n = 0 \text{ if } n \neq l \pm 1$$

and

$$a_{l+1} = \frac{3}{(2l+1)} \qquad \qquad (9)$$

$$a_{l-1} = \frac{3l}{(2l+1)(l-1)}.$$

Thus the components of the electric field outside the deformed sphere can be written as (using equations (3) and (9))

$$E_r = E\left[\frac{d^2}{s^3}(r+d\mu) + \frac{2R^3}{r^3}\mu\right] + \frac{3}{(2l+1)} \frac{\epsilon}{R} E \left[\left(\frac{R}{r}\right)^{l+1} l^2 P_{l-1}(\mu) + \left(\frac{R}{r}\right)^{l+3} (l+2)(l+1) P_{l+1}(\mu)\right] \qquad (10)$$

and

$$E_\theta = -E\left[\frac{d^3}{s^3}-\frac{R^3}{r^3}\right](1-\mu^2)^{\frac{1}{2}}+\frac{3}{(2l+1)}\frac{\epsilon}{R}E\left[\left(\frac{R}{r}\right)^{l+1} l\; P^{1}_{l-1}(\mu) + (l+1)\left(\frac{R}{r}\right)^{l+3} P^{1}_{l+1}(\mu)\right]$$

To the first order in ϵ, the change, $\Delta\mathcal{E}$, in the electric energy is given by

$$\Delta\mathcal{E} = -\int_R^{R+\epsilon P_l}\int_{-1}^{+1}\int_0^{2\pi}\frac{1}{8\pi}\left[{}^0E_r^2+{}^0E_\theta^2\right]r^2\,dr\,d\mu\,d\phi$$
$$+2\int_R^{\infty}\int_{-1}^{+1}\int_0^{2\pi}\frac{1}{8\pi}\left[{}^0E_r\,\delta E_r+{}^0E_\theta\,\delta E_\theta\right]r^2\,dr\,d\mu\,d\phi. \quad .. \quad (11)$$

Substituting equation (10) in (11), we get

$$\Delta\mathcal{E} = -\frac{E^2R^6}{2}\int_R^{R+\epsilon P_l}\int_{-1}^{+1}\frac{1+P_2}{r^4}\,dr\,d\mu - E^2R^3\int_R^{R+\epsilon P_l}\int_{-1}^{+1}\frac{rP_1+dP_2}{rs}\,dr\,d\mu$$
$$+\frac{3}{2}E^2\frac{\epsilon}{R}\int_R^{\infty}\int_{-1}^{+1}\left[\frac{d^2}{s^3}\frac{l^2}{(2l+1)}\left(\frac{R}{r}\right)^{l+1}(rP_{l-1}+dP_l)\right.$$
$$+\frac{d^2}{s^3}\frac{(l+1)(l+2)}{(2l+1)}\left(\frac{R}{r}\right)^{l+3}(rP_{l+1}+dP_{l+2})$$
$$+\frac{l^2}{(2l+1)(2l-1)}\left(\frac{R}{r}\right)^{l+4}\{(l+1)P_l+3(l-1)P_{l-2}\}$$
$$\left.+\frac{(l+1)(l+2)}{(2l+1)(2l+3)}\left(\frac{R}{r}\right)^{l+6}\{3(l+1)P_l+(l+3)P_{l+2}\}\right]r^2\,dr\,d\mu. \quad .. \quad (12)$$

After evaluating the integrals we obtain for the change, $\Delta\mathcal{E}$, in the electric energy

$$\Delta\mathcal{E} = -\frac{3}{5}E^2R^3(\epsilon/R) \quad \text{for} \quad l=2$$

and

$$= 0 \quad \text{for} \quad l\neq 2. \quad .. \quad .. \quad .. \quad .. \quad (13)$$

The change in the gravitational potential energy, $\Delta\Omega$, due to a P_l-deformation has been calculated by Chandrasekhar and Fermi (1953) and is given by

$$\Delta\Omega_l = \frac{3(l-1)}{(2l+1)^2}\frac{GM^2}{R}(\epsilon/R)^2. \quad .. \quad .. \quad .. \quad (14)$$

The change in the gravitational energy is seen to be of second order in ϵ for a P_l-deformation.

For a P_2-deformation, the total change in energy, ΔW, in a gravitating sphere is given by

$$\Delta W_2 = \Delta \mathcal{E}_2 + \Delta \Omega_2$$

$$= -\frac{3}{5} E^2 R^3 (\epsilon/R) + \frac{3}{25}\frac{GM^2}{R}(\epsilon/R)^2 \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

where we have inserted the subscript 2 in order to emphasize that we are dealing with a P_2-deformation.

As a function of ϵ/R, ΔW_2 has a minimum for a value of ϵ/R determined by the equation

$$\frac{6}{25}\frac{GM^2}{R}(\epsilon/R) - \frac{3}{5}E^2R^3 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

or

$$\epsilon/R = \frac{5}{2}\frac{E^2R^4}{GM^2}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

4. The Expression for Ellipticity by the Equilibrium Method

The electric pressure at the surface of the sphere is given by

$$(p_E)_s = (E^2/8\pi)_s = \frac{1}{8\pi}\left[\{E_r\}_s^2 + \{E_\theta\}_s^2\right]. \quad \ldots \quad \ldots \quad (18)$$

From equations (2), we have for a P_l-deformation

$$(E_r)_s^2 = 9E^2\mu^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

and

$$(E_\theta)_s^2 = 0$$

which shows that the contribution to the electric pressure $(P_E)_s$ comes only from the radial component of the electric field. Thus the electric pressure at the surface is

$$[p_E]_s = \frac{9E^2\mu^2}{8\pi}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

Now the gravitational pressure of a uniform spheroidal mass is of the form

$$\rho\Omega_s = G\pi\rho^2 r^2[\alpha_0 \sin^2\theta + \gamma_0 \cos^2\theta] \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

where $r = R + \epsilon P_l$

and

$$\alpha_0 = \frac{2}{3} + \frac{6}{15}\epsilon/R$$

$$\gamma_0 = \frac{2}{3} - \frac{4}{5}\epsilon/R\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

Substituting (22) in (21), we get for the gravitational pressure at the surface

$$\rho\Omega_s = G\pi\rho^2R^2\left[1 + 2\frac{\epsilon}{R}P_l(\mu)\right]\left[\frac{2}{3} - \frac{4}{5}\frac{\epsilon}{R}P_2(\mu)\right] \quad \ldots \quad \ldots \quad (23)$$

or in terms of the mass M of the sphere

$$= \frac{9GM^2}{16\pi R^4}\left[1+2\frac{\epsilon}{R}P_l(\mu)\right]\left[\frac{2}{3}-\frac{4}{5}\frac{\epsilon}{R}P_2(\mu)\right]$$

$$= \frac{9GM^2}{16\pi R^4}\left[\frac{2}{3}+4\frac{\epsilon}{R}\left(\frac{1}{3}P_l(\mu)-\frac{1}{5}P_2(\mu)\right)\right]. \qquad \ldots \quad \ldots \quad (24)$$

Therefore the change in the gravitational pressure at the surface under a P_2-deformation of the sphere is given by (up to first order in ϵ/R)

$$[p_s]_G = \frac{3}{10}\frac{GM^2}{\pi R^4}(\epsilon/R)\,P_2(\mu). \qquad \ldots \quad \ldots \quad \ldots \quad (25)$$

For the surface to be a form of equilibrium the angular part $\frac{9}{20}\frac{GM^2}{\pi R^4}\frac{\epsilon}{R}\mu^2$ in gravitational pressure at the surface must be balanced by the electric pressure at the sphere (equation (20)). Thus

$$\frac{9E^2}{8\pi}\mu^2 = \frac{9}{20}\frac{GM^2\mu^2}{\pi R^4}(\epsilon/R)$$

which yields the following expression for the ellipticity of the prolate spheroid,

$$\epsilon/R = \frac{5}{2}\frac{E^2R^4}{GM^2}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

This expression is exactly identical with the expression (17) for the ellipticity of the prolate spheroid deduced using the energy method.

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Summary

The stability of a conducting, gravitating, incompressible fluid sphere in a uniform external electric field is discussed by two different methods—the Energy method and the Equilibrium method. The results obtained by both methods show that the stable configuration is a prolate of ellipticity,

$$\epsilon/R = \frac{5}{2}\frac{E^2R^4}{GM^2}.$$

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