ON THE ORTHOGONAL SUB-SET OF APPELL POLYNOMIALS

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INTRODUCTION

An Appell set of polynomials $\{P_n(x)\}$ where $P_n(x)$ is of degree n in x is defined by the relation

$$A(t)\, e^{tx} = \sum_{n=0}^{\infty} t^n P_n(x) \tag{1}$$

where $A(t)$ can be expanded as an infinite series in ascending powers of t.

The particular cases of the polynomials belonging to this class are Hermite, Euler and Bernoulli polynomials and the corresponding function $A(t)$ in each case is respectively

$$e^{-\frac{1}{4}t^2},\ \frac{2}{1+e^t} \text{ and } \frac{t}{e^t-1}.$$

Various authors [Hahn (1935), Krall (1936), Meixner (1934), Sohat (1936), Webster (1937)] have attacked the problem of finding the orthogonal sub-set of Appell set of polynomials. In the present paper we use the integral representation for Appell polynomials and their generating function recently given by the author (Singh, 1954) in the form

$$P_n(x) = \frac{x^n}{n!}\int_0^{\infty} {}_{r+1}F_r \left\{ \begin{matrix} -n_1 a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix};\ \frac{-t}{x} \right\} d\beta(t) \tag{2}$$

and

$$A(t) = \int_0^{\infty} {}_rF_r \left\{ \begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_r \end{matrix};\ ut \right\} d\beta(t) \tag{3}$$

respectively, where $\beta(t)$ is such that

$$\mu_n = \int_0^{\infty} t^n\, d\beta(t), \quad n = 0, 1, 2, 3 \ldots \tag{4}$$

exist and $\mu_0 \neq 0$, to prove the above property and to sum a few infinite series involving Appell polynomials when the generating function $A(t)$ is an even function of t.

§ 1. THE ORTHOGONAL SUB-SET

We use, here, the well-known property that the necessary and sufficient condition for polynomials $\{P_n(x)\}$ to be orthogonal is that they satisfy a difference equation of the type

$$P_n(x) = (a_n x + b_n)\, P_{n-1}(x) + c_n\, P_{n-2}(x), \tag{5}$$

where a_n, b_n and c_n are constants independent of x and $a_n \neq 0$, $c_n \neq 0$. We note here that the limits of integration in (2), (3) and (4) can without any difficulty be made from a to b such that $\infty \geqslant b > a \geqslant -\infty$.

Expanding ${}_{r+1}F_r \left\{ \begin{matrix} -n_1 a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix}; \frac{-t}{x} \right\}$ in (2) and integrating term by term we can rewrite (2) as

$$P_n(x) = \frac{x^n}{n!}\mu_0 + \frac{x^{n-1}}{(n-1)!}\frac{a_1 a_2 \ldots a_r}{b_1 b_2 \ldots b_r}\mu_1$$

$$(6) \qquad + \sum_{l=0}^{n-2} \frac{(a_1)_l \ldots (a_r)_l}{(b_1)_l \ldots (b_r)_l} \frac{\mu_{l+2}}{(l+2)!} \frac{x^{n-l-2}}{(n-l-2)!},$$

and similarly we can write

$$(a_n x + b_n) P_{n-1}(x) = \frac{a_n x^n}{(n-1)!}\mu_0 + \frac{x^{n-1}}{(n-2)!}\left\{\mu_0 \frac{b_n}{n-1} + \frac{a_1 \ldots a_r}{b_1 \ldots b_r} a_n\right\}$$

$$(7) \qquad + \sum_{l=0}^{n-2} \frac{(a_1)_{l+1} \ldots (a_r)_{l+1}}{(b_1)_{l+1} \ldots (b_r)_{l+1}} \frac{x^{n-l-2}}{(n-l-3)!}\left\{\frac{b_n \mu_{l+1}}{(l+1)!\,(n-l-2)!} + a_n \frac{(a_1+l+1) \ldots (a_r+l+1)}{(b_1+l+1) \ldots (b_r+l+1)} \frac{\mu_{l+2}}{l+2}\right\}.$$

If $\{P_n(x)\}$ is an orthogonal set we will get from (5), on comparing the coefficients of various powers of x, on both the sides, a set of $(n+1)$ equations from which $(n+1)$ unknown quantities μ_r can be found out.

Comparing the coefficients of x^n and x^{n-1} in (6) and (7) we get

$$(8) \qquad a_n = \frac{1}{n},$$

and

$$(9) \qquad b_n = \frac{1}{n} \frac{\mu_1}{\mu_0} \frac{a_1 a_2 \ldots a_r}{b_1 b_2 \ldots b_r}.$$

A comparison of the coefficients of remaining powers of x in (5) gives

$$(10) \qquad \prod_{m=1}^{r} \frac{(a_m+l)(a_m+l+1)}{(b_m+l)(b_m+l+1)} \frac{\mu_{l+2}}{n} - \prod_{m=1}^{r} \frac{a_m(a_m+l)}{b_m(b_m+l)} \frac{1}{n} \frac{\mu_1}{\mu_0} \mu_{l+1} = c_n(l+1)\mu_l,$$
$$l = 1, 2, \ldots, n-2,$$

which, as expected, is a difference equation of second order in μ_l. On substituting for μ_n from (4) in (10) we get for $l=1, 2, \ldots, n-2$,

$$(11) \qquad \prod_{m=1}^{r} \frac{(a_m+l)(a_m+l+1)}{(b_m+l)(b_m+l+1)} \int_a^b t^{l+2} d\beta(t) - \prod_{m=1}^{r} \frac{a_m(a_m+l)}{b_m(b_m+l)} \frac{\mu_1}{\mu_0} \int_a^b t^{l+1} d\beta(t)$$
$$= nc_n(l+1) \int_a^b t^l d\beta(t) = -nc_n \int_a^b t^{l+1} \frac{d^2\beta(t)}{dt^2}\, dt$$

where we have taken $\beta(t)$ to be absolutely continuous and such that at the limits

$$(12) \qquad t^{r+1} \frac{d\beta(t)}{dt} = 0, \qquad r = 0, 1, 2, \ldots, n-2.$$

Transposing all the terms in (11) to one side it becomes

$$\int_a^b t^l \left[t \left\{ \prod_{m=1}^{r} \frac{(a_m+l)(a_m+l+1)}{(b_m+l)(b_m+l+1)}\, t - \prod_{m=1}^{r} \frac{a_m(a_m+l)}{b_m(b_m+l)} \frac{\mu_1}{\mu_0} \right\} \frac{d\beta(t)}{dt} + nc_n t \frac{d^2\beta(t)}{dt^2} \right] dt = 0. \tag{13}$$

The solution of (13) can be written as

$$\left\{ \prod_{m=1}^{r} \frac{(a_m+l)(a_m+l+1)}{(b_m+l)(b_m+l+1)}\, t - \prod_{m=1}^{r} \frac{a_m(a_m+l)}{b_m(b_m+l)} \frac{\mu_1}{\mu_0} \right\} \frac{d\beta(t)}{dt} + nc_n \frac{d^2\beta(t)}{dt^2} = T(t) \tag{14}$$

where $T(t)$ is such that

$$\int_a^b t^n T(t)\, dt = 0 \qquad n = 0, 1, 2, 3 \dots \tag{15}$$

It is well known that $T(t) \equiv 0$ if a and b both are finite, otherwise it is not unique. Since $\beta(t)$ is to be independent of n and l, we must have

$$nc_n = \theta \quad \text{(a constant)}$$

$$a_m = b_m \qquad m = 1, 2, 3, \dots, r.$$

(14) thus reduces to

$$\theta \frac{d^2\beta(t)}{dt^2} + \left(t - \frac{\mu_1}{\mu_0}\right) \frac{d\beta(t)}{dt} = T(t),$$

which on integration gives

$$\frac{d\beta(t)}{dt} = \exp\left\{ -\frac{1}{2\theta}\left(t - \frac{\mu_1}{\mu_0}\right)^2 \right\} \left[c + \int^t \frac{T(u)\, du}{\exp\left\{ -\frac{1}{2\theta}\left(u - \frac{\mu_1}{\mu_0}\right)^2 \right\}} \right]. \tag{16}$$

The limits of integration a and b are now to be found such that they satisfy the equation (12) and the equations

$$\mu_0 = \int_a^b d\beta(t)$$

$$= \int_a^b \exp\left\{ -\frac{1}{2\theta}\left(t - \frac{\mu_1}{\mu_0}\right)^2 \right\} \left[c + \int^t \frac{T(u)\, du}{\exp\left\{ -\frac{1}{2\theta}\left(u - \frac{\mu_1}{\mu_0}\right)^2 \right\}} \right] dt, \tag{17}$$

and

$$\mu_1 = \int_a^b t \exp\left\{ -\frac{1}{2\theta}\left(t - \frac{\mu_1}{\mu_0}\right)^2 \right\} \left[c + \int^t \frac{T(u)\, du}{\exp\left\{ -\frac{1}{2\theta}\left(u - \frac{\mu_1}{\mu_0}\right)^2 \right\}} \right] dt. \tag{18}$$

Equation (17) serves to determine the value of c. Using (17) we can write (18) as

$$\mu_1 = \int_a^b \left(t-\frac{\mu_1}{\mu_0}\right) \exp\left\{-\frac{1}{2\theta}\left(t-\frac{\mu_1}{\mu_0}\right)^2\right\} \left[c+\int^t \frac{T(u)\,du}{\exp\left\{-\frac{1}{2\theta}\left(u-\frac{\mu_1}{\mu_0}\right)^2\right\}}\right] dt+\mu_1.$$

Thus we have

$$\int_a^b \left(t-\frac{\mu_1}{\mu_0}\right) \exp\left\{-\frac{1}{2\theta}\left(t-\frac{\mu_1}{\mu_0}\right)^2\right\} \left[c+\int^t \frac{T(u)\,du}{\exp\left\{-\frac{1}{2\theta}\left(u-\frac{\mu_1}{\mu_0}\right)^2\right\}}\right] dt = 0,$$

or

$$c\int_a^b \left(t-\frac{\mu_1}{\mu_0}\right) \exp\left\{-\frac{1}{2\theta}\left(t-\frac{\mu_1}{\mu_0}\right)^2\right\} dt$$

$$+\left[-\theta \exp\left\{-\frac{1}{2\theta}\left(t-\frac{\mu_1}{\mu_0}\right)^2\right\}\int^t \frac{T(u)}{\exp\left\{-\frac{1}{2\theta}\left(u-\frac{\mu_1}{\mu_0}\right)^2\right\}}\right]_a^b +\theta\int_a^b T(u)\,du = 0.$$

The last term vanishes by virtue of (15) and from (12) and (15) we can see that the limits of integration will be $-\infty$ and ∞. This ensures that the remaining terms in the above expression vanish. Thus, without loss of generality, we can take μ_1 to be zero. This entails that all moments μ_{2n+1} of odd order will vanish and accordingly the polynomials will be symmetric. Since $\frac{d\beta(t)}{dt}$ will now be symmetric in t, if we put

(19) $$T_1(t) = \frac{d\beta(t)}{dt}+c_1 e^{-t^2/2\theta},$$

then $T_1(t)$ would be symmetric in t and

(20) $$\int_{-\infty}^{\infty} T_1(t)\, t^i\, dt = 0, \qquad i \text{ an odd integer.}$$

We now choose c_1 such that

(21) $$\int_{-\infty}^{\infty} T_1(t)\, dt = 0.$$

Integrating by parts, when i is an integer

(22) $$\int_{-\infty}^{\infty} \frac{T_1(x)}{e^{-x^2/2\theta}} \frac{d}{dx}\{x^i e^{-x^2/2\theta}\}\, dx$$

$$= \left[x^i\left\{\frac{d\beta(x)}{dx}+c_1 e^{-x^2/2\theta}\right\}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} T(x) x^i\, dx = 0,$$

by virtue of (12) and (15). But the left hand side of (22) can be written as

$$\int_{-\infty}^{\infty} T_1(x)\left\{i\, x^{i+1} - \frac{1}{\theta} x^{i+1}\right\} dx = 0 \qquad i = 1, 2, 3, \ldots .$$

Putting $i = 1, 3, 5, \ldots$ and using (20) and (21) we see that all moments of even order of $T_1(x)$ vanish. Thus we have shown that except for a function whose moments vanish the determining function reduces to $e^{-\frac{t^2}{2\theta}}$. Substituting this value in (3) we get

$$A(t) = 2\sqrt{2\pi\theta}\, c \exp\left(\frac{\theta t^2}{2}\right)$$

which is the same as the generating function for Hermite polynomials.

§ 2. Infinite Sums.

If the generating function is an even function then it is easy to see that

$$\sum_{n=0}^{\infty} t^n P_{2n}(x) = A(\sqrt{t}) \cosh(x\sqrt{t})$$

and

$$\sum_{n=0}^{\infty} t^n P_{2n+1}(x) = \frac{1}{\sqrt{t}} A(\sqrt{t}) \sinh(x\sqrt{t}).$$

The particular cases of the above sums when $P_n(x)$ is Hermite polynomial are well known. We will sum two infinite series when the generating function involved is an even function of its argument.

Result 1. Whenever the integral below exists

$$\sum_{n=0}^{\infty} (-y)^n P_{2n}(x) \frac{(2n)!}{n!} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} \cos(2xz\sqrt{y})\, A(2iz\sqrt{y})\, dy.$$

Substituting the expression for $P_{2n}(x)$ from (2), when $\mu_{2r+1} = 0, r = 0, 1, 2, \ldots$, in the left hand side and changing the order of summation we get

$$\sum_{n=0}^{\infty} (-y)^n P_{2n}(x) \frac{(2n)!}{n!}$$

$$= \int_a^b \sum_{l=0}^{\infty} \frac{(a_1, 2l) \ldots (a_r, 2l)}{(b_1, 2l) \ldots (b_r, 2l)} \sum_{n=l}^{\infty} \frac{x^{2n-2l}(-y)^n (2n)!}{n!\,(2n-2l)!} \frac{t^{2l}}{(2l)!} d\beta(t)$$

$$= \int_a^b \sum_{l=0}^{\infty} \frac{(a_1, 2l) \ldots (a_r, 2l)}{(b_1, 2l) \ldots (b_r, 2l)} \frac{t^{2l}(-4y)^l}{(2l)!\sqrt{y}} \sum_{n=0}^{\infty} \frac{x^{2n}(-y)^n 2^{2n} \Gamma(n+l+\frac{1}{2})}{(2n)!} d\beta(t)$$

wherein we have used the duplication formula for Gamma function. The change of the order of summation and integration in the above process is justified because the series involved are absolutely and uniformly convergent. Now, putting

$$\Gamma(l+n+\tfrac{1}{2}) = \int_0^{\infty} e^{-z} z^{n+l-\frac{1}{2}} dz$$

and again changing the order of summation and integration, which can be easily justified, we get

$$\sum_{n=0}^{\infty} (-y)^n P_{2n}(x) \frac{(2n)!}{n!}$$

$$= \frac{1}{\sqrt{\pi}} \int_a^b \left[\int_0^{\infty} \sum_{l=0}^{\infty} \frac{(a_1, 2l) \ldots (a_r, 2l)}{(b_1, 2l) \ldots (b_r, 2l)} \frac{(2it\sqrt{yz})^{2l}}{(2l)!} \sum_{n=0}^{\infty} \frac{(2ix\sqrt{yz})^{2n}}{2n!} \frac{e^{-z}\, dz}{\sqrt{z}} \right] d\beta(t)$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} \cos(2xz\sqrt{y})\, A(2iz\sqrt{y})\, dz$$

whenever the final integral exists.

As a particular case we see that if $P_{2n}(x)$ is Hermite polynomial

$$\sum_{n=0}^{\infty} \frac{H_{2n}(x)}{n!} \frac{(-y)^n}{n!} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-(1-y)z^2} \cos(2xz\sqrt{y})dz$$

$$= \frac{e^{-yx^2/(1-y)}}{\sqrt{1-y}} \quad y < 1.$$

By a similar procedure we can also prove the following result:

Result 2.

$$\sum_{n=0}^{\infty} (-y)^{n+\frac{1}{2}} \frac{P_{2n+1}(x)\,(2n+1)!}{n!} = i \int_0^{\infty} e^{-z} A(2i\sqrt{yz}) \sin(2x\sqrt{yz})\, dz$$

whenever the integral on the right hand side exists.

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Summary

With the help of integral representation for Appell polynomials, another proof of the fact that the only orthogonal sub-set of Appell set of polynomials is Hermite polynomial has been given. Some infinite series involving these polynomials when the generating function is an even function have also been summed.

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