NOTE ON NON-SIMPLE K^*-SPACES

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1. K^*-spaces, both simple and non-simple, have been studied in great details by Ruse (1951) and Walker (1950). This short note is concerned with non-simple K^*-spaces. A non-flat Riemannian space is a K^*-space if, for some non-zero vector k_i, its Riemann Christoffel tensor satisfies either

$$R_{hijk,\,l} = R_{hijk}k_l \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1.1)$$

or

$$R_{hijk}k_l + R_{hikl}k_j + R_{hilj}k_k = 0 \text{ and } R_{hijk,\,l} = 0, \qquad \ldots \quad \ldots \qquad (1.2)$$

where comma denotes covariant derivative with respect to Levi-Civita parallelism in the space. Since the K^*-space is supposed to be non-simple, k_i is a null vector which is taken to be a gradient, $k_i = \partial k/\partial x^i$. Walker, in his paper referred to above, has shown that k_i satisfies an equation of the form

$$k_{i,\,j} = \phi k_i k_j, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1.3)$$

where ϕ is a function of k. It follows from Walker's paper that (1.3) remains unaltered when the right-hand side of (1.1) is changed into its negative. We shall therefore replace the equation (1.1) by

$$R_{hijk,\,l} = \pm R_{hijk}k_l, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1.4)$$

where either the upper or the lower sign is to be chosen. In this note an attempt has been made to replace the equations (1.4) and (1.2) of a non-simple K^*-space by replacing the curvature tensor of the space by the sum of curvature tensors of suitable conformal spaces.

2. Let g_{ij} be the metric tensor of the non-simple K^*-space under consideration and σ any function of the co-ordinates. Consider in the space two symmetric tensors $e^{2\sigma}g_{ij}$ and $e^{-2\sigma}g_{ij}$ which may be regarded as the metric tensors of two conformal spaces, and let the curvature tensors of these conformal spaces be denoted by σR^t_{ijk} and $\bar{\sigma}R^t_{ijk}$ respectively. Further let

$$\sigma R_{hijk} = g_{ht}\sigma R^t_{ijk}, \quad \bar{\sigma}R_{hijk} = g_{ht}\bar{\sigma}R^t_{ijk}. \qquad \ldots \quad \ldots \qquad (2.1)$$

Obviously, these covariant tensors have all the properties of indices of R_{hijk}. Then, writing $\sigma_i = \partial\sigma/\partial x^i$, we have (Eisenhart, 1926)

$$\tfrac{1}{2}\,(\sigma R_{hijk} + \bar{\sigma}R_{hijk}) - R_{hijk} =$$

$$= g_{hj}\sigma_i\sigma_k + g_{ik}\sigma_h\sigma_j - g_{hk}\sigma_i\sigma_j - g_{ij}\sigma_h\sigma_k + (g_{hk}g_{ij} - g_{hj}g_{ik})\sigma_t\sigma^t. \qquad \ldots \qquad (2.2)$$

Suppose now we choose σ defined by

$$\sigma = A\int \exp\,(k/2 - \int\phi\,dk)\,dk, \qquad \ldots \quad \ldots \quad \ldots \qquad (2.3)$$

where $A \neq 0$ is any arbitrary constant and ϕ is given by (1.3). Then

$$\sigma_i = Ak_i \exp\,(k/2 - \int\phi\,dk). \qquad \ldots \quad \ldots \quad \ldots \qquad (2.3a)$$

By (1.3) $$\sigma_{i,j} = \tfrac{1}{2}Ak_ik_j \exp(k/2-\int\phi\, dk). \qquad (2.3b)$$

Therefore $$\sigma_{i,j}\sigma_k = \sigma_i\sigma_j\sigma_k/2A \exp(k/2-\int\phi\, dk). \qquad (2.3c)$$

Since k_i is a null vector, so σ_i is also a null vector. Therefore, applying (2.3c) and (2.3a), it follows from (2.2) that

$$\tfrac{1}{2}(\sigma R_{hijk}+\bar{\sigma}R_{hijk}),_l - R_{hijk,l} = [\tfrac{1}{2}(\sigma R_{hijk}+\bar{\sigma}R_{hijk})-R_{hijk}]k_l.$$

Similarly, if we suppose σ to be defined by

$$\sigma = A\int \exp(-k/2-\int\phi\, dk)\, dk \qquad (2.4)$$

and proceed as above we find that

$$\tfrac{1}{2}(\sigma R_{hijk}+\bar{\sigma}R_{hijk}),_l - R_{hijk,l} = -[\tfrac{1}{2}(\sigma R_{hijk}+\bar{\sigma}R_{hijk})-R_{hijk}]k_l.$$

Combining these two results we see that the equations (1.4) of the non-simple recurrent space can be replaced by the equations

$$(\sigma R_{hijk}+\bar{\sigma}R_{hijk}),_l = \pm(\sigma R_{hijk}+\bar{\sigma}R_{hijk})k_l, \qquad (2.5)$$

where the upper and the lower sign in (2.5) correspond respectively to the upper and the lower sign in (1.4) and σR_{hijk}, $\bar{\sigma}R_{hijk}$ are defined by (2.1) in which σ is given by (2.3) or (2.4) according as the upper or the lower sign is chosen.

3. In order to obtain equations which can replace (1.2), let us suppose that it is possible to specialise the function ϕ, satisfying (1.3), as follows. Let ϕ be defined by

$$\sigma = \tfrac{1}{2}\int \phi dk, \qquad (3.1)$$

where σ is given by (2.3) or (2.4) according as we choose the upper or the lower sign in (1.4). This will, of course, leave the form of the equations (2.5) unaltered.

(I) Consider first the case when the upper sign in (1.4) is chosen. From (2.3) and (3.1) we have

$$\phi = 2A \exp(k/2-\int\phi\, dk),$$

or $$\phi'+\phi^2 = \tfrac{1}{2}\phi, \quad \text{where } \phi' = d\phi/dk.$$

The integral of this equation is given by, B being an arbitrary constant,

$$1/\phi = Be^{-k/2}+2. \qquad (3.2)$$

Now it is known (Walker, 1950) that the non-simple K^*-space admits a null parallel vector field τ_i, codirectional with k_i, defined by

$$\tau_i = k_i \exp(-\int\phi\, dk). \qquad (3.3)$$

Substituting (3.2) in (3.3) we find (neglecting constant of integration) that

$$\tau_i = k_i e^{-k/2}\phi, \qquad (3.4)$$

where ϕ is given by (3.2). It is seen from (3.4) that τ_i is a gradient, $\tau_i = \partial\tau/\partial x^i$, where τ is given by (neglecting constant of integration)

$$\tau = \frac{2}{B}\log\phi, \qquad (3.5)$$

where ϕ is given by (3.2) and $B \neq 0$.

With this value of τ, let us take the symmetric tensors $e^{2\tau}g_{ij}$ and $e^{-2\tau}g_{ij}$, which may be regarded as the metric tensors of two conformal spaces, and form the curvature tensors τR^t_{ijk} and $\bar{\tau}R^t_{ijk}$ of these conformal spaces. As in (2.1), let

$$\tau R_{hijk} = g_{ht}\tau R^t_{ijk}, \quad \bar{\tau}R_{hijk} = g_{ht}\bar{\tau}R^t_{ijk}.$$

Then, as in (2.2),

$$\tfrac{1}{2}(\tau R_{hijk}+\bar{\tau}R_{hijk})-R_{hijk} = g_{hj}\tau_i\tau_k+g_{ik}\tau_h\tau_j-g_{hk}\tau_i\tau_j-g_{ij}\tau_h\tau_k. \quad .. \quad (3.6)$$

Since τ_i is proportional to k_i and $\tau_{i,j} = 0$, it follows from (3.6) and (1.2) that

$$\left.\begin{aligned}(\tau R_{hijk}+\bar{\tau}R_{hijk})k_l+(\tau R_{hikl}+\bar{\tau}R_{hikl})k_j+(\tau R_{hilj}+\bar{\tau}R_{hilj})k_k = 0,\\ (\tau R_{hijk}+\bar{\tau}R_{hijk}),_l = 0.\end{aligned}\right\} \quad .. \quad (3.7)$$

Also, it follows from (3.1), (3.2) and (3.5) that

$$\sigma = \tfrac{1}{2}\log(B+2e^{k/2}), \quad \tau = -\frac{2}{B}\log(Be^{-k/2}+2). \quad .. \quad .. \quad (3.8)$$

Thus the equations (1.4) with the upper sign or the equations (1.2) of a non-simple K^*-space can be replaced respectively by (2.5) with the upper sign or (3.7), where σ and τ have the values given in (3.8).

(II) Consider secondly the case when the lower sign in (4.1) is chosen. Proceeding as in case (I) above, we find that in place of (3.2) we have here

$$1/\phi = Be^{k/2}-2. \quad .. \quad .. \quad .. \quad .. \quad (3.9)$$

And therefore in place of (3.4), (3.5) we have respectively

$$\tau_i = -k_i e^{k/2}\phi, \quad \tau = \frac{2}{B}\log\phi \quad .. \quad .. \quad .. \quad (3.10)$$

where ϕ is now given by (3.9) and $B \neq 0$ is an arbitrary constant. It is therefore seen that in this case we obtain analogous results.

Finally, it follows from (2.2) and (3.6) that in either case, with the corresponding values of σ and τ,

$$R = \tfrac{1}{2}(\sigma R+\bar{\sigma}R) = \tfrac{1}{2}(\tau R+\bar{\tau}R). \quad .. \quad .. \quad .. \quad (3.11)$$

And therefore $R = 0$ (which holds for all K^*-spaces other than V_2 and its flat extensions) implies

$$\sigma R+\bar{\sigma}R = \tau R+\bar{\tau}R = 0.$$

4. Let the covariant derivative with respect to Levi-Civita parallelism in the conformal space having the metric tensor $e^{\int\phi dk}g_{ij}$, where ϕ is given by (1.3), be denoted by a solidus. Then

$$k_{i/j} = \frac{\partial k_i}{\partial x^j}-k_s\left[\left\{{s \atop ij}\right\}+\tfrac{1}{2}\phi\delta^s_j k_i+\tfrac{1}{2}\phi\delta^s_i k_j-g_{ij}k^s\right] = k_{i,j}-\phi k_i k_j = 0.$$

Therefore the conformal space $e^{\int\phi dk}g_{ij}$ admits the null parallel vector field k_i, where k_i specifies the non-simple K^*-space g_{ij} defined either by (1.4) or by (1.2) for which (1.3) holds.

Applying this result to the particular cases when ϕ has the value given by (3.2) or (3.9), it is seen that in the first case the space $(B+2e^{k/2})g_{ij}$ admits the null parallel vector field k_i, where k_i specifies the non-simple K^*-space g_{ij} defined either by (2.5) with the upper sign or by (3.7), where σ and τ have the values given in (3.8). Analogously for the second case.

Abstract

This paper replaces the defining equations of a non-simple K^*-space by replacing the curvature tensor of the space by the sum of the curvature tensors of two suitably chosen spaces which are conformal to the K^*-space. The equations for the case where the K^*-space is recurrent have been obtained first and these have been modified later (under certain assumption) for the case where the K^*-space is either recurrent or symmetric.

References

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