ON A TYPE OF TENSOR IN A RIEMANNIAN SPACE

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1. Let F_{ijkl} be a tensor in a Riemannian space V_n, satisfying the identities

$$F_{ijkl}+F_{ijlk} = 0,\ F_{ijkl}-F_{klij} = 0 \qquad \text{(1.1)}$$

Then

$$F_{ijkl}+F_{jikl} = 0 \qquad \text{(1.2)}$$

Now, for a V_2 and V_3 the indices i, j, k, l are not distinct. Hence, in these cases

$$F_{ijkl}+F_{iklj}+F_{iljk} = 0 \qquad \text{(1.3)}$$

As in a V_2 and V_3, F_{ijkl} satisfies all the identities of the Riemann tensor R_{ijkl}, it can be shown (Hlavaty', 1953) that in these cases F_{ijkl} can be expressed in the form

$$F_{ijkl} = g_{ik}a_{jl}+g_{jl}a_{ik}-g_{il}a_{jk}-g_{jk}a_{il} \qquad \text{(1.4)}$$

where a_{ij} is a symmetric tensor.

Let $F_{ij} = g^{mn}F_{imnj}$ and $F = g^{ij}F_{ij}$. Then for a V_2 and V_3 the values of a_{ij} are given by

$$a_{ij} = -\frac{F}{4}g_{ij} \text{ and } a_{ij} = -F_{ij}+\frac{1}{4}F_{lm}g^{lm}g_{ij} \text{ respectively.}$$

Evidently, a tensor F_{ijkl} satisfying (1.1) can be built out of an arbitrary affine connection of a V_n. For, let Γ^t_{ij} be an arbitrary affine connection and K_{ijkl} be the corresponding covariant curvature tensor. That is

$$K_{ijkl} = g_{ti}\left[\frac{\partial\Gamma^t_{jl}}{\partial x^k}-\frac{\partial\Gamma^t_{jk}}{\partial x^l}+\Gamma^t_{sk}\Gamma^s_{jl}-\Gamma^t_{sl}\Gamma^s_{jk}\right] \qquad \text{(1.5)}$$

Put

$$F_{ijkl} = \tfrac{1}{4}(K_{ijkl}+K_{jilk}+K_{klij}+K_{lkji}) \qquad \text{(1.6)}$$

Then the tensor F_{ijkl}, as defined by (1.6), satisfies the identities (1.1) and therefore also (1.2).

2. In this section we consider F_{ijkl} for a V_2.

From (1.4) we have

$$\begin{aligned} F_{ijkl} &= -\frac{F}{4}(g_{ik}g_{jl}+g_{jl}g_{ik}-g_{il}g_{jk}-g_{jk}g_{il}) \\ &= -\frac{F}{2}(g_{ik}g_{jl}-g_{il}g_{jk}) \\ &= -\frac{F}{2}g_{ijkl}, \text{ say} \qquad \text{(2.1)} \end{aligned}$$

Since $R_{ijkl} = -\frac{R}{2} g_{ijkl}$, it follows from (2.1) that

$$F_{ijkl} = \sigma R_{ijkl} \text{ where } \sigma = \frac{F}{R} \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

Therefore
$$F_{il} = \sigma R_{il} = \sigma \frac{R}{2} g_{il} = \frac{F}{2} g_{il} \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

Taking covariant derivatives of both sides of (2.2) with respect to the Christoffel connection, we have

$$F_{ijkl,\,p} = \frac{\partial \sigma}{\partial x_p} R_{ijkl} + \sigma R_{ijkl,\,p} \quad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

where a comma (,) followed by indices denotes covariant differentiation.
Now, a space V_n is defined to be Cartan-symmetric if and only if

$$R_{ijkl,\,p} = 0.$$

Hence from (2.4) and (2.3) we have the following theorems:

Theorem 1.—If a V_2 of non-zero scalar curvature R admits a tensor F_{ijkl} satisfying (1.1) and for which the invariant $F = g^{ij} F_{ij}$ is not zero, then the necessary and sufficient condition that the V_2 may be Cartan-symmetric is that

$$F_{ijkl,\,p} = \frac{\partial \sigma}{\partial x_p} R_{ijkl} \text{ where } \sigma = \frac{F}{R}.$$

Theorem 2.—If a V_2 admits a tensor F_{ijkl} satisfying (1.1) then the V_2 is homogeneous with respect to the tensor F_{ij}.

Let now the co-ordinate system be so chosen that the line element of V_2 is given by

$$ds^2 = 2\lambda dx\, dy.$$

We shall call this system of co-ordinates the 'S' system. Let R be the scalar curvature of V_2. Then

$$R = g^{ij} R_{ij} = g^{12} R_{12} + g^{21} R_{21} = 2g^{12} R_{12} = \frac{2}{\lambda} R_{12} \quad \ldots \quad \ldots \quad (2.5)$$

Again,
$$R_{12} = g^{mn} R_{1mn2} = g^{21} R_{1212} = \frac{1}{\lambda} R_{1212} \quad \ldots \quad \ldots \quad \ldots \quad (2.6)$$

But in the 'S' system

$$R_{1212} = \frac{1}{2}\left(\frac{\partial^2 \lambda}{\partial x \partial y} + \frac{\partial^2 \lambda}{\partial x \partial y}\right) - \lambda \cdot \frac{1}{\lambda} \cdot \frac{\partial \lambda}{\partial x} \cdot \frac{1}{\lambda} \frac{\partial \lambda}{\partial y} = \frac{\partial^2 \lambda}{\partial x \partial y} - \frac{1}{\lambda} \frac{\partial \lambda}{\partial x} \frac{\partial \lambda}{\partial y} \quad \ldots \quad (2.7)$$

Hence from (2.5), (2.6) and (2.7) we have

$$R = \frac{2}{\lambda} \cdot \frac{1}{\lambda} \left[\frac{\partial^2 \lambda}{\partial x \partial y} - \frac{1}{\lambda} \frac{\partial \lambda}{\partial x} \frac{\partial \lambda}{\partial y}\right] = \frac{2}{\lambda} \left[\frac{1}{\lambda} \frac{\partial^2 \lambda}{\partial x \partial y} - \frac{1}{\lambda^2} \frac{\partial \lambda}{\partial x} \frac{\partial \lambda}{\partial y}\right] = \frac{2}{\lambda} \frac{\partial^2 \log \lambda}{\partial x \partial y} \quad \ldots \quad (2.8)$$

Now, if $\Delta_2 f$ be a differential parameter of the second order (Eisenhart, 1926)

then
$$\Delta_2 f = g^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial f}{\partial x^k} \left\{ {k \atop ij} \right\}\right).$$

In this case

$$\Delta_2 f = 2g^{12} \frac{\partial^2 f}{\partial x \partial y} = \frac{2}{\lambda} \frac{\partial^2 f}{\partial x \partial y} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.9)$$

Therefore

$$\Delta_2 \log \sigma = \frac{2}{\lambda} \frac{\partial^2 \log \sigma}{\partial x \partial y} \quad \ldots \quad (2.10)$$

Let V'_2 be the space whose fundamental tensor is F_{ij}. Then by (2.3) V'_2 is conformal to V_2. Denote the scalar curvature of V'_2 by $\bar{R}$.

Then $$\sigma \bar{R} = R + 2\Delta_2 \log \sqrt{\sigma} = R + \Delta_2 \log \sigma \text{ (Eisenhart, 1926)} \quad \ldots \quad (2.11)$$

Therefore by (2.8) and (2.10) we have

$$\sigma \bar{R} = \frac{2}{\lambda} \frac{\partial^2 \log \lambda}{\partial x \partial y} + \frac{2}{\lambda} \frac{\partial^2 \log \sigma}{\partial x \partial y} = \frac{2}{\lambda} \frac{\partial^2 \log \lambda\sigma}{\partial x \partial y} \quad \ldots \quad (2.12)$$

If $\bar{R} = 0$, we have from (2.12)

$$\frac{\partial^2 \log \lambda\sigma}{\partial x \, \partial y} = 0$$

whence $$\lambda\sigma = AB \quad \ldots \quad (2.13)$$

where A is a function of x only and B is a function of y only. Hence by (2.13)

$$F_{12} = \sigma g_{12} = \lambda\sigma = AB \quad \ldots \quad (2.14)$$

Conversely, if $F_{12} = AB$ where A is a function of x only and B is a function of y only, then $\dfrac{\partial^2 \log \lambda\sigma}{\partial x \, \partial y} = 0$ and therefore $\bar{R} = 0$ if $\sigma \neq 0$.

Next, we suppose that $\bar{R}$ = a positive constant $= l^2$, say.

Then $$\sigma l^2 = \frac{2}{\lambda} \frac{\partial^2 \log \lambda\sigma}{\partial x \partial y}$$

or $$\frac{\partial^2 \log \lambda\sigma}{\partial x \partial y} - \frac{l^2}{2} (\lambda\sigma) = 0$$

whence $$\lambda\sigma = -\frac{4}{l^2} \left[\frac{A'B'}{(1+AB)^2} \right] \quad \ldots \quad (2.15)$$

where A', B' denote derivatives of A and B with respect to x and y respectively.

Hence $$F_{12} = \lambda\sigma = -\frac{4}{l^2} \left[\frac{A'B'}{(1+AB)^2} \right] \quad \ldots \quad (2.16)$$

Conversely, if F_{12} has the value given by (2.16), then

$$\sigma \bar{R} = l^2 \sigma \text{ and therefore } \bar{R} = l^2 \text{ if } \sigma \neq 0.$$

Hence we have the following theorem:

Theorem 3.—If a V_2 of non-zero scalar curvature admits a tensor F_{ijkl} satisfying (1.1) where the invariant F is not zero and V'_2 be the space whose fundamental tensor is F_{ij}, then the necessary and sufficient condition, that the scalar curvature of V'_2 may be zero, or a positive quantity l^2 is that the only non-zero component of F_{ij} is given by (2.14) and (2.16) respectively in the 'S' system.

It can be verified that in the 'S' system the only non-zero components of the Christoffel connection $\left\{ {t \atop ij} \right\}$ are $\left\{ {1 \atop 11} \right\}$ and $\left\{ {2 \atop 22} \right\}$ and in that co-ordinate system

$$R_{1212} = g_{12}\frac{\partial}{\partial y}\left\{ {1 \atop 11} \right\} = g_{12}\frac{\partial}{\partial x}\left\{ {2 \atop 22} \right\}.$$

We now enquire as to what form the tensor F_{ijkl}, as defined by (1.6), will assume in a co-ordinate system if the only non-zero components of an arbitrary affine connection Γ^t_{ij}, out of which F_{ijkl} is built, are Γ^1_{11} and Γ^2_{22} in that system.

As the only non-zero components of Γ^t_{ij} are Γ^1_{11} and Γ^2_{22}, so

$$K_{1212} = g_{12}\frac{\partial \Gamma^2_{22}}{\partial x^1} \text{ and } K_{2121} = g_{12}\frac{\partial \Gamma^1_{11}}{\partial x^2}.$$

Hence from (1.6)

$$F_{1212} = \tfrac{1}{2} g_{12}\left[\frac{\partial \Gamma^2_{22}}{\partial x^1} + \frac{\partial \Gamma^1_{11}}{\partial x^2}\right] \quad \ldots \quad \ldots \quad \ldots \quad (2.17)$$

We have therefore the following theorem:

Theorem 4.—If there exists a co-ordinate system in a V_2 in which the only non-zero components of an arbitrary affine connection Γ^t_{ij} are Γ^1_{11} and Γ^2_{22}, then in that co-ordinate system the only non-zero components of F_{ijkl}, as defined by (1.6), is given by (2.17).

Further, if the tensor $S_{ij} = K^t_{tij}$ be a zero tensor, then

$$F_{1212} = g_{12}\frac{\partial \Gamma^1_{11}}{\partial x^2} = g_{12}\frac{\partial \Gamma^2_{22}}{\partial x^1}.$$

If, in particular,

$$\Gamma^t_{ij} = \left\{ {t \atop ij} \right\}$$

then

$$R_{1212} = g_{12}\frac{\partial}{\partial x^2}\left\{ {1 \atop 11} \right\} = g_{12}\frac{\partial}{\partial x^1}\left\{ {2 \atop 22} \right\}.$$

Now, we find the components Γ^1_{11} and Γ^2_{22} in the 'S' system when the scalar curvature of the space V'_2 is a non-negative constant.

From (2.17) we have

$$g^{12}F_{1212} = \tfrac{1}{2} g^{21}g_{12}\left[\frac{\partial \Gamma^2_{22}}{\partial x^1} + \frac{\partial \Gamma^1_{11}}{\partial x^2}\right]$$

or

$$F_{21} = \tfrac{1}{2}\left[\frac{\partial \Gamma^2_{22}}{\partial x} + \frac{\partial \Gamma^1_{11}}{\partial y}\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.18)$$

But by (2.14) $\qquad F_{12} = AB$ when $\bar{R} = 0$.

Hence (2.18) can be written as

$$\frac{\partial \Gamma^2_{22}}{\partial x} + \frac{\partial \Gamma^1_{11}}{\partial y} = 2AB \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.19)$$

If the tensor $S_{ij} = 0, \quad \frac{\partial \Gamma^2_{22}}{\partial x} = \frac{\partial \Gamma^1_{11}}{\partial y}$.

Therefore (2.19) reduces to

$$\frac{\partial \Gamma^1_{11}}{\partial y} = \frac{\partial \Gamma^2_{22}}{\partial x} = AB \cdot$$

Hence

$$\left.\begin{aligned} \Gamma^1_{11} &= A \int B dy + A_1 \\ \text{and} \quad \Gamma^2_{22} &= B \int A dx + B_1 \end{aligned}\right\} \quad \ldots \quad (2.20)$$

where A_1 is a function of x and B_1 is a function of y.

Again, if $\bar{R} = l^2, \quad F_{12} = -\frac{4}{l^2}\left[\frac{A'B'}{(1+AB)^2}\right]$.

Hence from (2.18) we have

$$\frac{\partial \Gamma^2_{22}}{\partial x} + \frac{\partial \Gamma^1_{11}}{\partial y} = -\frac{8}{l^2}\left[\frac{A'B'}{(1+AB)^2}\right] \quad \ldots \quad (2.21)$$

Now, if S_{ij} be a zero tensor, (2.21) reduces to

$$\frac{\partial \Gamma^1_{11}}{\partial y} = \frac{\partial \Gamma^2_{22}}{\partial x} = -\frac{4}{l^2}\frac{A'B'}{(1+AB)^2}.$$

Hence

$$\left.\begin{aligned} \Gamma^1_{11} &= -\frac{4}{l^2}\int \frac{A'B'}{(1+AB)^2} dy + A_1 = -\frac{4}{l^2}\frac{A'B}{1+AB} + A_1 \\ \text{and} \quad \Gamma^2_{22} &= -\frac{4}{l^2}\int \frac{A'B'}{(1+AB)^2} dx + B_1 = -\frac{4}{l^2}\frac{AB'}{1+AB} + B_1 \end{aligned}\right\} \quad \ldots \quad (2.22)$$

where A_1 is a function of x and B_1 is a function of y.

Thus we have the following theorem:

Theorem 5.—If in the co-ordinate system 'S' the only non-zero components of an arbitrary affine connection Γ^t_{ij} be Γ^1_{11} and Γ^2_{22}, V'_2 be the space whose fundamental tensor is F_{ij} and S_{ij} be a zero tensor, then the components Γ^1_{11} and Γ^2_{22} are given by (2.20) or (2.22) according as the scalar curvature of V'_2 is zero or a positive constant l^2 respectively.

If, in particular,

$$\Gamma^t_{ij} = \left\{ {t \atop ij} \right\}, \quad S_{ij} = R^t_{tij} = 0.$$

Hence $\left\{ {1 \atop 11} \right\}$ and $\left\{ {2 \atop 22} \right\}$ must be of the form (2.20) and (2.22) when the scalar curvature of V'_2 is zero or a positive constant l^2.

3. In this section we consider F_{ijkl} for a V_3.
From (1.4) we have

$$F_{ijkl} = g_{ik}[-F_{jl}+\tfrac{1}{4}F_{pm}g^{pm}g_{jl}]+g_{jl}[-F_{ik}+\tfrac{1}{4}F_{pm}g^{pm}g_{ik}]-g_{il}[-F_{jk}+\tfrac{1}{4}F_{pm}g^{pm}g_{jk}]$$
$$-g_{jk}[-F_{il}+\tfrac{1}{4}F_{pm}g^{pm}g_{il}]$$

or
$$F_{ijkl}+(F_{jl}g_{ik}+F_{ik}g_{jl}-F_{jk}g_{il}-F_{il}g_{jk}) = \tfrac{1}{2}F(g_{ik}g_{jl}-g_{jk}g_{il}) \quad .. \quad (3.1)$$

If the V_3 is homogeneous with respect to F_{ij}, i.e. $F_{ij} = \sigma g_{ij}$, then from (3.1) we have

$$F_{ijkl} = -\frac{F}{6}(g_{jl}g_{ik} - g_{jk}g_{il}) = -\frac{F}{6}g_{ijkl}.$$

Conversely, if
$$F_{ijkl} = -\frac{F}{6}g_{ijkl}$$

then
$$g^{jk}F_{ijkl} = -\frac{F}{6}g^{jk}g_{ijkl}$$

or
$$F_{il} = -\frac{F}{6}(g_{il} - 3g_{il})$$

whence
$$F_{il} = \frac{F}{3}g_{il}.$$

Hence we have the following theorem:

Theorem 6.—A necessary and sufficient condition that a V_3 may be homogeneous with respect to the tensor F_{ij} is that

$$F_{ijkl} = -\frac{F}{6}g_{ijkl}.$$

Let K_{ijkl} be the covariant curvature tensor corresponding to an arbitrary affine connection Γ^t_{ij} in a V_3. Now, let there exist a co-ordinate system in which the only non-zero components of Γ^t_{ij} are Γ^1_{11}, Γ^1_{13} and Γ^3_{21}. Then it can be verified that in that co-ordinate system the components of K_{ijkl} will have the following values:

$$K_{1212} = -g_{13}\frac{\partial\Gamma^3_{21}}{\partial x^2}, \quad K_{2121} = g_{12}\frac{\partial\Gamma^1_{11}}{\partial x^2}, \quad K_{1313} = 0$$

$$K_{3131} = g_{13}\left(\frac{\partial\Gamma^1_{11}}{\partial x^3} - \frac{\partial\Gamma^1_{13}}{\partial x^1}\right), \; K_{2323} = 0, \; K_{3232} = 0$$

$$K_{1323} = 0, \; K_{2312} = 0, \; K_{3232} = 0$$

$$K_{1213} = -g_{13}\frac{\partial\Gamma^3_{21}}{\partial x^3}, \quad K_{2131} = g_{12}\left(\frac{\partial\Gamma^1_{11}}{\partial x^3} - \frac{\partial\Gamma^1_{13}}{\partial x^1}\right), \; K_{1312} = 0$$

$$K_{3121} = g_{13}\frac{\partial\Gamma^1_{11}}{\partial x^2}, \quad K_{1223} = 0, \quad K_{2132} = -g_{12}\frac{\partial\Gamma^1_{13}}{\partial x^2}$$

$$K_{3132} = -g_{13}\frac{\partial\Gamma^1_{13}}{\partial x^2}, \quad K_{2313} = 0, \quad K_{3231} = g_{33}\frac{\partial\Gamma^3_{21}}{\partial x^3}.$$

If F_{ijkl} be now defined, as in (1.6), then

$$F_{1212} = \frac{1}{2}\left(g_{12}\frac{\partial \Gamma^1_{11}}{\partial x^2} - g_{13}\frac{\partial \Gamma^3_{21}}{\partial x^2}\right)$$

$$F_{1313} = \frac{1}{2}g_{13}\left(\frac{\partial \Gamma^1_{11}}{\partial x^3} - \frac{\partial \Gamma^1_{13}}{\partial x^1}\right)$$

$$F_{2323} = 0$$

$$F_{1223} = \frac{1}{4}\left(g_{33}\frac{\partial \Gamma^3_{21}}{\partial x^2} - g_{12}\frac{\partial \Gamma^1_{13}}{\partial x^2}\right)$$

$$F_{1213} = \frac{1}{4}\left[g_{12}\left(\frac{\partial \Gamma^1_{11}}{\partial x^3} - \frac{\partial \Gamma^1_{13}}{\partial x^1}\right) + g_{13}\left(\frac{\partial \Gamma^1_{11}}{\partial x^2} - \frac{\partial \Gamma^3_{21}}{\partial x^3}\right)\right]$$

$$F_{1323} = \frac{1}{4}\left(g_{33}\frac{\partial \Gamma^3_{21}}{\partial x^3} - g_{13}\frac{\partial \Gamma^1_{13}}{\partial x^2}\right).$$

Let us further suppose that the only non-zero components of g_{ij} are g_{12} and g_{33}. It can be verified from above that g_{12}, g_{13}, g_{33} cannot vanish simultaneously. Then

$$F_{1212} = \tfrac{1}{2}g_{12}\frac{\partial \Gamma^1_{11}}{\partial x^2}, \quad F_{1313} = 0, \quad F_{2323} = 0$$

$$F_{1223} = \tfrac{1}{4}\left(g_{33}\frac{\partial \Gamma^3_{21}}{\partial x^2} - g_{12}\frac{\partial \Gamma^1_{13}}{\partial x^2}\right), \; F_{1213} = \tfrac{1}{4}g_{12}\left(\frac{\partial \Gamma^1_{11}}{\partial x^3} - \frac{\partial \Gamma^1_{13}}{\partial x^1}\right), \; F_{1323} = \tfrac{1}{4}g_{33}\frac{\partial \Gamma^3_{21}}{\partial x^3}.$$

Therefore by theorem 6 the V_3 is homogeneous with respect to F_{ij} if and only if

$$\tfrac{1}{2}g_{12}\frac{\partial \Gamma^1_{11}}{\partial x^2} = -\frac{F}{6}g^2_{12}$$

$$\tfrac{1}{4}\left(g_{33}\frac{\partial \Gamma^3_{21}}{\partial x^2} - g_{12}\frac{\partial \Gamma^1_{13}}{\partial x^2}\right) = 0, \quad \tfrac{1}{4}g_{12}\left(\frac{\partial \Gamma^1_{11}}{\partial x^3} - \frac{\partial \Gamma^1_{13}}{\partial x^1}\right) = 0$$

$$\tfrac{1}{4}g_{33}\frac{\partial \Gamma^3_{21}}{\partial x^3} = -\frac{F}{6}g_{12}g_{33},$$

i.e. if and only if

$$\left.\begin{aligned} g_{33}\frac{\partial \Gamma^3_{21}}{\partial x^2} - g_{12}\frac{\partial \Gamma^1_{13}}{\partial x^2} &= 0 \\ \frac{\partial \Gamma^1_{11}}{\partial x^3} - \frac{\partial \Gamma^1_{13}}{\partial x^1} &= 0 \\ 2\frac{\partial \Gamma^1_{11}}{\partial x^2} + \frac{\partial \Gamma^3_{21}}{\partial x^3} &= 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

Hence we have the following theorem:

Theorem 7.—If there exists a co-ordinate system in a V_3 in which the only non-zero components of an arbitrary affine connection Γ^t_{ij} are Γ^1_{11}, Γ^1_{13}, Γ^3_{21} and in which the only non-zero components of g_{ij} are g_{12} and g_{33}, then the V_3 is homogeneous with respect to F_{ij}, where F_{ijkl} is defined by (1.6), if and only if the non-zero components of Γ^t_{ij} and g_{ij} satisfy the equations (3.2).

As an example, if there exists a co-ordinate system in which the non-zero components of an arbitrary affine connection Γ^t_{ij} in a V_3 are given by

$$\Gamma^1_{11} = -x_2,\ \Gamma^3_{21} = 2x_3,\ \Gamma^1_{13} = f(x_3)$$

and if in that co-ordinate system the non-zero components g_{ij} of the V_3 are given by

$$g_{12} = e^{-x_1 x_2},\ g_{33} = \phi(x_1, x_2, x_3)$$

then the V_3 is homogeneous with respect to F_{ij} where F_{ijkl} is defined by (1.6).

4. In this section we shall consider F_{ijkl} for a V_4.

Let $^\circ F_{ijkl} = \frac{1}{4}\epsilon_{ijmn}\epsilon_{klpq}F^{mnpq}$

where, as usual, $F^{mnpq} = g^{im}g^{jn}g^{pk}g^{ql}F_{ijkl}$

and the alternating tensor ϵ_{ijkl} (4 indices) is of components

$\pm\sqrt{g}$, 0, g being $= |g_{ij}|$.

Since

$$\epsilon_{ijmn}\epsilon_{klpq} = \begin{vmatrix} g_{ik} & g_{jk} & g_{mk} & g_{nk} \\ g_{il} & g_{jl} & g_{ml} & g_{nl} \\ g_{ip} & g_{jp} & g_{mp} & g_{np} \\ g_{iq} & g_{jq} & g_{mq} & g_{nq} \end{vmatrix},$$

it is seen by actual calculation that

$$^\circ F_{ijkl} = F_{ijkl} + (g_{ik}F_{jl} + g_{jl}F_{ik} - g_{il}F_{jk} - g_{jk}F_{il}) - \tfrac{1}{2}Fg_{ijkl} \quad \ldots \quad (4.1)$$

Now, if $^\circ F_{ijkl} = F_{ijkl}$

then $g_{ik}F_{jl} + g_{ji}F_{ik} - g_{il}F_{jk} - g_{jk}F_{il} = \frac{1}{2}Fg_{ijkl}$.

Therefore multiplying by g^{jk} and summing for j, k we find

$$F_{il} = \frac{F}{4}g_{il} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

Again, if $\qquad F_{il} = \sigma g_{il}$

then

$$g_{ik}F_{jl} + g_{jl}F_{ik} - g_{il}F_{jk} - g_{jk}F_{il} = \sigma(2g_{ik}g_{jl} - 2g_{jk}g_{il}) = 2\sigma g_{ijkl} = \frac{F}{2}g_{ijkl}.$$

Hence from (4.1) we have

$$^\circ F_{ijkl} = F_{ijkl} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.3)$$

From (4.2) and (4.3) we have the following theorem:

Theorem 8.—A necessary and sufficient condition that a V_4 may be homogeneous with respect to F_{ij} is that

$$°F_{ijkl} = F_{ijkl}.$$

Let us now put

$$A_{ijkl} = \tfrac{1}{2}(F_{ijkl} + °F_{ijkl}) \quad \ldots \quad (4.4)$$

Then

$$A_{il} = g^{jk}A_{ijkl} = \tfrac{1}{2}F_{il} + \tfrac{1}{2}g^{jk}\,°F_{ijkl} \quad \ldots \quad (4.5)$$

But from (4.1)

$$g^{jk}\,°F_{ijkl} = -F_{il} + \frac{F}{2}g_{il} \quad \ldots \quad (4.6)$$

Therefore

$$A_{il} = \frac{F}{4}g_{il} \quad \ldots \quad (4.7)$$

Hence

$$A = g^{il}A_{il} = \frac{F}{4}g^{il}g_{il} = F.$$

Therefore

$$A_{il} = \frac{A}{4}g_{il}.$$

Hence we have the following theorem:

Theorem 9.—If a V_4 admits a tensor F_{ijkl} which satisfies (1.1) and the tensor $A_{ij} = g^{mn}A_{imnj}$, where A_{ijkl} is defined by (4.4), then the V_4 is homogeneous with respect to A_{ij}.

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Abstract

In this paper it has been assumed that there exists in a Riemannian space V_n, with metric tensor g_{ij}, a tensor F_{ijkl} satisfying the identities $F_{ijkl}+F_{ijlk} = 0$, $F_{ijkl}-F_{klij} = 0$ and the properties of such a tensor have been studied in a V_2, V_3 and V_4. Defining a space V_n to be homogeneous with respect to a symmetric tensor a_{ij} if the metric tensor satisfies $a_{ij} = \sigma g_{ij}$, where σ is a function of the co-ordinates, it has been shown that every V_2 is homogeneous with respect to the tensor F_{ij} where $F_{ij} = g^{mn}F_{imnj}$ and conditions both necessary and sufficient for a V_3 and V_4 to be homogeneous with respect to F_{ij} have been obtained. Further, building a tensor of the type F_{ijkl} out of an arbitrary affine connection Γ^t_{ij}, some theorems involving F_{ijkl} and Γ^t_{ij} have been established for a V_2 and V_3.

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