ON THE SOLUTION OF THE EQUATIONS OF INTERNAL BALLISTICS WITH A CUBIC FORM FUNCTION

by N. S. VENKATESAN, *Defence Science Laboratory, New Delhi*

(Communicated by R. S. Varma, F.N.I.)

(*Received January 6; read May 4, 1956*)

1. INTRODUCTION

In the solution of the main problem of internal ballistics, viz. the determination of the maximum pressure and the muzzle velocity, one of the relations called the form function is obtained from considerations of the geometric shape of the propellant grain. It has been shown (H.M.S.O., 1951) that for most of the propellant shapes used, the fraction (z) of the charge burnt up to any instant can be connected to the fraction (f) of the web size (D) remaining unburnt at any instant, by a relation

$$z = (1-f)(1+\theta f) \quad .. \quad .. \quad .. \quad .. \quad (1)$$

with sufficient accuracy where θ is a constant called the 'form factor' and for use in a normal ballistic method $-1 \leqslant \theta \leqslant +1$.

Apart from the fact that there are shapes of propellants, like the sphere and the cube where equation (1) does not hold, the equation itself has been derived under the assumption that the length of the propellant grain is very large compared to its web size. Hence when a spherical or cubical grain is used or when we are interested in studying the effect of broken grains (shortened length) we have to reconsider the form function.

2. THE FORM FUNCTION

(a) In the case of a cylindrical grain (cord) of diameter D and length L the (z, f) relation is given by

$$z = (1-f)\left(1+f+\frac{D}{L}f^2\right) \quad .. \quad .. \quad .. \quad .. \quad (2)$$

(b) In the case of the tube of web size D and length L the (z, f) relation is given by

$$z = (1-f)\left(1+\frac{D}{L}f\right) \quad .. \quad .. \quad .. \quad .. \quad (3)$$

(c) For cubes and spheres the form function is

$$z = 1-f^3$$
$$= (1-f)(1+f+f^2) \quad .. \quad .. \quad .. \quad .. \quad (4)$$

Hence it is seen that for all the propellant shapes that one comes across in practice the form function can be taken as

$$z = (1-f)(1+\theta f+\theta_1 f^2) \quad .. \quad .. \quad .. \quad .. \quad (5)$$

where θ and θ_1 are constants.

Such a form function is useful in (*a*) cases where highly regressive shapes such as spheres and cubes are used (*b*) studying the effect of the breakage of propellant grains on the ballistics. Further if one tries to reduce a composite charge to a single equivalent charge following the method of Corner (1950) we see that it is sometimes necessary to consider a form function of the type of equation (2).

3. Limitations on θ and θ_1

Just as in the orthodox case we found that $-1 \leqslant \theta \leqslant +1$ (so as to conform to physical facts), we shall find certain restrictions imposed on θ and θ_1 if z must lie between 0 and 1 for values of f between 1 and 0. $0 \leqslant z \leqslant 1$ in the interval (0, 1) for f.

The conditions are

(*a*) $\theta^2+\theta\theta_1+\theta_1{}^2-3\theta_1 < 0$

or (*b*) $\theta^2+\theta\theta_1+\theta_1{}^2-3\theta_1 > 0$ and (i) $\theta_1 > 0$; $\theta+2\theta_1 < 0$ and $1+\theta+\theta_1 \geqslant 0$

or (ii) $1 \geqslant \theta > \theta_1 > 0$

or (iii) $\theta_1 < 0$; $\theta \leqslant 1$ and $1+\theta+\theta_1 \geqslant 0$

or (*c*) $\theta^2+\theta\theta_1+\theta_1{}^2-3\theta_1 = 0$ and (i) $\theta \geqslant \theta_1 > 0$

or (ii) $\theta+2\theta_1 \geqslant 0$

or (*d*) $\theta_1 = 0$; $-1 \leqslant \theta \leqslant 1$

4. Equations of Internal Ballistics

The four fundamental equations of Internal Ballistics are

$$z = \zeta(\xi - Bz) + \frac{\gamma-1}{2M}\eta^2 \qquad (6)$$

$$M\zeta = \eta\frac{d\eta}{d\xi} \qquad (7)$$

$$\zeta = -\eta\frac{df}{d\xi} \qquad (8)$$

$$z = (1-f)(1+\theta f+\theta_1 f^2) \qquad (9)$$

where

$$\left.\begin{aligned} B &= \frac{C}{Al}\left(b-\frac{1}{\delta}\right) \\ \xi &= 1+\frac{x}{l} \\ \eta &= \frac{AD}{F\beta C}v \\ \zeta &= \frac{Al}{FC}p \\ \text{and}\quad M &= \frac{A^2D^2}{F\beta^2 C\omega_1} \end{aligned}\right\} \qquad (10)$$

4.1. Initial conditions are $z = z_0$; $p = p_0$; $x = 0$; $v = 0$; $f = f_0$.

Hence from (6)

$$z_0 = \zeta_0/1+B\zeta_0. \quad \ldots \quad (11)$$

5.1. *Solution up to 'all-burnt'.*

From (7) and (8)

$$\eta = M(f_0-f) \quad \ldots \quad (12)$$

Substituting (7), (9) and (12) in (6) and simplifying we get

$$M^2\eta \frac{d\eta}{d\xi}(\xi-Bz) = (a-\eta)(b+\eta)(c-\theta_1\eta) \quad \ldots \quad (13)$$

where

$$\left.\begin{aligned} abc &= M^3 z_0 \\ -bc+ac-\theta_1 ab &= M^2[1-\theta+2(\theta-\theta_1)f_0+3\theta_1 f_0^2] \\ -c-\theta_1 a+\theta_1 b &= -M(\theta'-\theta_1+3\theta_1 f_0) \\ \text{and} \quad \theta' &= \theta+\tfrac{1}{2}(\gamma-1)M \end{aligned}\right\} \quad \ldots \quad (14)$$

Since z is a cubic in f, theoretically there will be three values of f corresponding to a given z. However since this is a physical case there will be only one admissible value of f corresponding to a given z. Hence for given z, we choose that value of f which lies between 0 and 1.

Equation (13) can be written as

$$\frac{d\xi}{d\eta} - \frac{M^2\xi\eta}{(a-\eta)(b+\eta)(c-\theta_1\eta)} = \frac{-BM^2\eta z}{(a-\eta)(b+\eta)(c-\theta_1\eta)} \quad \ldots \quad (15)$$

Hence

$$\xi(a-\eta)^{\frac{M^2a}{(a+b)(c-\theta_1 a)}}(b+\eta)^{\frac{M^2b}{(a+b)(c+\theta_1 b)}}(c-\theta_1\eta)^{\frac{M^2c}{(\theta_1 a-c)(\theta_1 b+c)}}$$

$$-a^{\frac{M^2a}{(a+b)(c-\theta_1 a)}}\, b^{\frac{M^2b}{(a+b)(c+\theta_1 b)}}\, c^{\frac{M^2c}{(\theta_1 a-c)(\theta_1 b+c)}}$$

$$= -\frac{B}{M} I \quad \ldots \quad (16)$$

where

$$I = \int_0^\eta \left[M^3 z_0 + M^2\eta\left\{1-\theta+2(\theta-\theta_1)f_0+3\theta_1 f_0^2\right\} - M\eta^2(\theta-\theta_1+3\theta_1 f_0)+\theta_1\eta^3\right]$$

$$\times\, \eta(a-\eta)^{\frac{M^2a}{(a+b)(c-\theta_1 a)}-1}(b+\eta)^{\frac{M^2b}{(a+b)(c+\theta_1 b)}-1}$$

$$\times\, (c-\theta_1\eta)^{\frac{M^2c}{(\theta_1 a-c)(\theta_1 b+c)}-1}\, d\eta \quad \ldots \quad (17)$$

5.11. *Maximum pressure.*

Now from equation (7)

$$\frac{1}{\zeta} = \frac{M}{\eta}\frac{d\xi}{d\eta}$$

i.e.

$$\frac{1}{\zeta} = \frac{M^2\xi - B\left[M^3 z_0 + M^2\eta\left\{1-\theta+2(\theta-\theta_1)f_0+3\theta_1 f_0^2\right\} - M\eta^2(\theta-\theta_1+3\theta_1 f_0)+\theta_1\eta^3\right]}{(a-\eta)(b+\eta)(c-\theta_1\eta)} \quad \ldots \quad (18)$$

At the position of maximum pressure $d\zeta = 0$. Hence differentiating (18) and putting $d\zeta = 0$ we get if we denote values at the maximum pressure by the suffix '1'.

$$M^3 \frac{d\xi_1}{d\eta_1} = \frac{M^3\xi_1 \left[3\theta_1\eta_1^2 - 2\eta_1(\theta_1 a + c - \theta_1 b) + ac - bc - \theta_1 ab\right]}{(a-\eta_1)(b+\eta_1)(c-\theta_1\eta_1)}$$

$$+B\left[M^2\left\{1-\theta+2(\theta-\theta_1)f_0+3\theta_1 f_0^2\right\} - 2M\eta_1(\theta-\theta_1+3\theta_1 f_0) + 3\theta_1\eta_1^2\right]$$

$$-\frac{B\left[M^3 z_0 + M^2\eta_1\left\{1-\theta+2(\theta-\theta_1)f_0+3\theta_1 f_0^2\right\} - M\eta_1^2(\theta-\theta_1+3\theta_1 f_0) + \theta_1\eta_1^3\right] \times \left[3\theta_1\eta_1^2 - 2\eta_1(\theta_1 a + c - \theta_1 b) + ac - bc - \theta_1 ab\right]}{(a-\eta_1)(b+\eta_1)(c-\theta_1\eta_1)} \qquad (19)$$

But from equation (15)

$$\frac{d\xi_1}{d\eta_1} = \frac{M^2\xi_1\eta_1 - BM^2\eta_1 z_1}{(a-\eta_1)(b+\eta_1)(c-\theta_1\eta_1)} \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

Substituting (20) in (19) and simplifying we get

$$M\xi_1\left[3\theta_1\eta_1^2 - 2M\eta_1\left\{\theta' - \theta_1 + 3\theta_1 f_0 + \frac{M}{2}\right\} + M^2\left\{1-\theta+2(\theta-\theta_1)f_0+3\theta_1 f_0^2\right\}\right]$$

$$= -B\eta_1\left[M^3 z_0\gamma + M^2\left\{1-\theta+2(\theta-\theta_1)f_0+3\theta_1 f_0^2\right\}\left(\frac{\gamma+1}{2}\right)\eta_1 - M(\theta-\theta_1+3\theta_1 f_0)\eta_1^2 - \frac{\theta_1(\gamma-1)}{2}\eta_1^3\right] \quad \ldots \quad (21)$$

Equations (16) and (21) give ξ_1 and η_1 and from equation (18) we can get ζ_1.

5.12. *'All-burnt'*.

Denoting by suffix '2' the values at 'all-burnt' we have

$$\eta_2 = Mf_0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

As has already been remarked we choose that value of f_0 which is admissible. Hence substituting this value of η_2 in (16) and (18) we obtain ξ_2 and ζ_2 respectively.

5.2. *After 'all-burnt'*.

After 'all-burnt' $z = 1$. Hence by differentiating equation (6) we get

$$d\zeta(\xi - B) + \gamma\zeta d\xi = 0$$

i.e. $$\zeta(\xi-B)^\gamma = \text{constant} = \Phi \text{ (say)}.$$

Now equation (6) with $z = 1$ gives

$$1 = \zeta(\xi - B) + \frac{\gamma-1}{2M}\eta^2$$

i.e. $$\zeta(\xi-B)^\gamma = (\xi-B)^{\gamma-1}\left[1 - \frac{\gamma-1}{2M}\eta^2\right] = \Phi \quad \ldots \quad \ldots \quad (23)$$

where Φ is obtained by putting the values at all-burnt, viz.

$$\Phi = \zeta_2(\xi_2-B)^{\gamma} = (\xi_2-B)^{\gamma-1}\left[1-\frac{\gamma-1}{2}Mf_0^2\right] \qquad \ldots \quad \ldots \quad (24)$$

the value of ξ_2 being obtainable from equation (16) with $\eta_2 = Mf_0$.

6. Approximate Solution when B is small

In a number of systems of Internal Ballistics the co-volume b is taken equal to the specific volume, in which case $B = \frac{C}{Al}\left(b-\frac{1}{\delta}\right) = 0$. In the present section B is not neglected but is treated as a small quantity and as a correction term. Hence second and higher powers of B are neglected.

6.1. *Up to 'all-burnt'.*

ξ is given in terms of η by equations (16) and (17). Since B is a small quantity, we are justified in making the following assumptions in evaluating the integral I, as it is multiplied by B.

(1) The shot-start pressure is negligible.

(2) $\theta_1 = 0$.

Under such conditions we have

$$a = \frac{M(1+\theta)}{\theta}$$

$$b = 0$$

$$c = M\theta'.$$

Denoting by I' the approximate value of I and integrating by parts we have

$$I' = \left[\frac{1}{M\theta'}\right]^{\frac{M}{\theta'}}\left[\frac{M(1+\theta)}{\theta}\right]^{\frac{M}{\theta'}} MK \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

so that

$$K = \frac{1}{M}\left[1-\frac{\theta'\eta}{M(1+\theta)}\right]^{\frac{M}{\theta'}}\left[\frac{\theta\eta^2}{M+2\theta'} - \frac{M(1+\theta)(M+2\theta'-2\theta)(\eta+1+\theta)}{(M+\theta')(M+2\theta')} + \frac{(1+\theta)^2(M+2\theta'-2\theta)}{(M+\theta')(M+2\theta')}\right] \quad \ldots \quad (26)$$

Equation (16) gives

$$\xi = \left(1-\frac{\eta}{a}\right)^{-\frac{M^2a}{(a+b)(c-\theta_1 a)}}\left(1+\frac{\eta}{b}\right)^{-\frac{M^2b}{(a+b)(c+\theta_1 b)}}\left(1-\frac{\theta_1\eta}{c}\right)^{\frac{M^2c}{(c-\theta_1 a)(c+\theta_1 b)}}$$

$$\times\left[1-\frac{B}{M}I'a^{-\frac{M^2a}{(a+b)(c-\theta_1 a)}}b^{-\frac{M^2b}{(a+b)(c+\theta_1 b)}}c^{-\frac{M^2c}{(\theta_1 a-c)(\theta_1 b+c)}}\right].$$

Since B is small we can write

$$a = \frac{M}{\theta}(1+\theta')$$

$$b = 0$$

$$c = M\theta'$$

in terms multiplying I' so that

$$\xi = \left(1-\frac{\eta}{a}\right)^{-\frac{M^2a}{(a+b)(c-\theta_1 a)}}\left(1+\frac{\eta}{b}\right)^{-\frac{M^2b}{(a+b)(c+\theta_1 b)}}\left(1-\frac{\theta_1\eta}{c}\right)^{-\frac{M^2c}{(\theta_1 a-c)(\theta_1 b+c)}}[1-BK].$$

If we denote by ξ' the value of ξ when B is zero,

$$\xi' = \left(1-\frac{\eta}{a}\right)^{-\frac{M^2a}{(a+b)(c-\theta_1 a)}}\left(1+\frac{\eta}{b}\right)^{-\frac{M^2b}{(a+b)(c+\theta_1 b)}}$$

$$\times\left(1-\frac{\theta_1\eta}{c}\right)^{-\frac{M^2c}{(\theta_1 a-c)(\theta_1 b+c)}} \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

and the next approximation to ξ is given by

$$\xi = \xi'(1-BK) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

6.11. *Maximum pressure.*

An exact value of the maximum pressure can theoretically be obtained from (16), (17), (18) and (21). However when B is small the expressions can be very much simplified.

If ξ' and ζ' are the values when $B = 0$ then equation (6) gives

$$\zeta(\xi-Bz) = \zeta'\xi'$$

which using (28) at the point of maximum pressure gives

$$\zeta_1 = \zeta_1'\left\{1+B\left(K_1+\frac{z_1}{\xi_1'}\right)\right\}$$

if we neglect B^2 and higher terms.

This can be worked out to sufficient accuracy if we make the usual assumption in the terms multiplying B.

After some simplification this leads us to

$$\left.\begin{aligned} &\zeta_1 = \zeta_1'\left[1+B\left\{\zeta_1'+\frac{(\theta+1)^2\gamma\lambda}{M}\right\}\right] \\ &\text{where} \\ &\lambda = \frac{1}{2n+1}\left[\frac{1}{n+1}-\frac{3n+2}{(2n+1)^2}\left(\frac{n+1}{2n+1}\right)^{\frac{1}{n}}\right] \\ &\text{and} \quad n = \frac{\theta'}{M}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (29)$$

It is shown (H.M.S.O., 1951) that an approximate value for λ is given by

$$\lambda = 0{\cdot}26/(n+1)(2n+1) \quad \ldots \quad (30)$$

6.12. *'All-burnt'.*

Equation (28) gives

$$\xi_2 = \xi_2'(1-BK_2)$$

where

$$\xi_2' = \left(1-\frac{\eta_2}{a}\right)^{-\frac{M^2a}{(a+b)(c-\theta_1 a)}}\left(1+\frac{\eta_2}{b}\right)^{-\frac{M^2b}{(a+b)(c+\theta_1 b)}}\left(1-\frac{\theta_1\eta_2}{c}\right)^{-\frac{M^2c}{(\theta_1 a-c)(\theta_1 b+c)}} \quad (31)$$

and

$$\eta_2 = Mf_0.$$

In the evaluation of K_2 we put $\eta_2 = M$ in equation (26).

6.2. *After 'all-burnt'.*

ζ and η are given by equation (23).

$$\zeta(\xi-B)^{\gamma} = (\xi-B)^{\gamma-1}[1-(\gamma-1)\eta^2/2M] = \Phi$$

where

$$\Phi = (\xi_2-B)^{\gamma-1}\left[1-\frac{(\gamma-1)\eta_2^2}{2M}\right]$$

$$= \xi_2'^{\gamma-1}\left[1-\frac{(\gamma-1)\eta_2^2}{2M}\right]\left[1-B\left(\phi_2+\frac{1}{\xi_2'}\right)\right]^{\gamma-1}$$

Using (28)

As B is treated small.

$$\Phi = \Phi'[1-(\gamma-1)Bc'] \quad \ldots \quad (32)$$

where

$$\Phi' = \xi_2'^{\gamma-1}[1-(\gamma-1)\eta_2^2/2M] \quad \ldots \quad (33)$$

and

$$c' = K_2+\frac{1}{\xi_2'} \quad \ldots \quad (34)$$

ξ_2' and η_2 being given by equations (31) and (22) respectively. The pressure and velocity after all-burnt are therefore given, Φ being known, by the equation

$$\zeta = \Phi(\xi-B)^{-\gamma} \quad \ldots \quad (35)$$

$$\eta^2 = \frac{2M}{\gamma-1}[1-\Phi(\xi-B)^{1-\gamma}] \quad \ldots \quad (36)$$

The muzzle velocity η_3 is given by

$$\eta_3^2 = \frac{2M}{\gamma-1}[1-\Phi(\xi_3-B)^{1-\gamma}] \quad \ldots \quad (37)$$

While making the calculations use can be made of the table for c' given by G.M. II (1945). Also it is shown (H.M.S.O., 1951) that c' can be given sufficiently accurately by the formula

$$c' = \frac{(1+\theta)^2+2{\cdot}5M\zeta_0}{\{(1+\theta)+M(0{\cdot}28-0{\cdot}086\,\theta)\}^2}. \quad \ldots \quad (38)$$

The author is deeply thankful to Dr. D. S. Kothari, F.N.I., Scientific Adviser, Ministry of Defence, and Dr. R. S. Varma, F.N.I., for their keen interest and kind encouragement in this investigation.

Summary

The equations of the Internal Ballistics are solved with a form function of the type $z = (1-f)(1+\theta f+\theta_1 f^2)$ and including the co-volume correction term B, following the Hunt-Hinds system.

Bibliography

Corner, J. (1950). Theory of the Internal Ballistics of Guns, New York.
H.M. Stationery Office, London (1951). Internal Ballistics.
Military College of Science (1945). A System of Internal Ballistics, G.M. II.

Issued October 25, 1956.