ON THE ZEROS OF A CLASS OF POLYNOMIALS

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S. K. Singh has proved (Singh, 1953) the following two theorems :—

Theorem A. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial with real or complex coefficients such that for* $n \geq 2$

$$|a_0| + |a_1| + \ldots + |a_{n-1}| \leq n\,|a_n|, \qquad \ldots \quad (1)$$

$$2^{\frac{n+1}{2}} r^n (n+1) \left|\frac{a_n}{a_0}\right| < 1, \qquad \ldots \quad (2)$$

$$r \geq \tfrac{1}{2}. \qquad \ldots \quad (3)$$

Then at least one zero of $P(z)$ *lies outside the circle* $|z| = r$.

Theorem B. *If in* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$

$$\operatorname*{Min}_{\nu=0}^{n}\{\,|a_\nu|\,\} \geq 1, \qquad \ldots \quad (4)$$

$$\operatorname*{Max}_{\nu=0}^{n-1}\{\,|a_\nu|\,\} \geq |a_n|, \qquad \ldots \quad (5)$$

then

$$n\left(\frac{R}{K}\right) \leq \frac{2 \log \{(n+1)\,|a_n|\,R^n\}}{\log K}, \ (K > 1)$$

where $n(x)$ *is the number of zeros of* $P(z)$ *for* $|z| \leq x$ *and*

$$R = \operatorname{Max}\left\{\left|\frac{a_{n-1}}{a_n}\right|, \left|\frac{a_{n-2}}{a_n}\right|^{\frac{1}{2}}, \left|\frac{a_{n-3}}{a_n}\right|^{\frac{1}{3}}, \ldots\right\}.$$

The object of this note is to improve the results of Singh. We prove—

Theorem 1. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial with real or complex coefficients and let the conditions of Theorem A be satisfied, then the least number of zeros of* $P(z)$ *lying outside the circle* $|z| = r$ *is* $\frac{n+1}{2}$ *or* $\frac{n}{2}+1$ *according as n is an odd or an even positive integer.*

THEOREM 2. *If the conditions of Theorem B are satisfied, then*

$$n\left(\frac{R}{K}\right) \leq \frac{\log\{(n+1)|a_n|R^n\}}{\log K}, \textit{ for every } K > 1.$$

Proof of Theorem 1. Let $R \geq 1$.

$$\max_{|z|=R}\left|\frac{P(z)}{a_0}\right| \leq \left|\frac{a_n}{a_0}\right|R^n + \left|\frac{a_{n-1}}{a_0}\right|R^{n-1} + \ldots + \left|\frac{a_1}{a_0}\right|R + 1$$

$$\leq \frac{R^n}{|a_0|}\{|a_n| + |a_{n-1}| + \ldots + |a_0|\}$$

$$\leq (n+1)R^n\left|\frac{a_n}{a_0}\right|.$$

Now put $R = 2r$, $r \geq \frac{1}{2}$. Then

$$\max_{|z|=R}\left|\frac{P(z)}{a_0}\right| \leq (n+1)\left|\frac{a_n}{a_0}\right| 2^n r^n$$

$$< 2^{\frac{n-1}{2}}.$$

Applying Jensen's theorem (Titchmarsh, 1939) to the function $\frac{P(z)}{a_0}$, we get

$$n(r)\log 2 \leq \int_r^{2r}\frac{n(x)}{x}dx \leq \int_0^{2r}\frac{n(x)}{x}dx = \frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{P(2re^{i\theta})}{a_0}\right|d\theta - \log\left|\frac{P(0)}{a_0}\right|$$

$$\leq \frac{n-1}{2}\log 2.$$

Consequently

$$n(r) \leq \frac{n-1}{2}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

If $n = 2m$ is an even positive integer, then

$$n(r) \leq \frac{2m-1}{2} = m - \frac{1}{2}.$$

But $n(r)$ is an integer and therefore

$$n(r) \leq m-1.$$

Hence the number of zeros of $P(z)$ lying outside the circle $|z| = r$ cannot be less than $m+1 = \frac{n}{2}+1$.

Whereas if $n = 2m+1$ is an odd positive integer, then from (6)

$$n(r) \leq m$$

and the number of zeros of $P(z)$ lying outside the circle $|z| = r$ cannot be less than $m+1 = \frac{n+1}{2}$.

Proof of Theorem 2. Under the conditions stated $|a_n z^n|$ is the maximum term of the polynomial for $|z| \geq R$. So

$$\underset{|z|=R}{\text{Max}} |P(z)| < (n+1)|a_n|R^n.$$

Applying Jensen's theorem to the function $P(z)$ we get

$$n\left(\frac{R}{K}\right) \log K \leq \int_{\frac{R}{K}}^{R} \frac{n(x)}{x} dx \leq \int_0^R \frac{n(x)}{x} dx = \frac{1}{2\pi}\int_0^{2\pi} \log|P(Re^{i\phi})|\, d\phi - \log|P(0)|$$

$$\leq \log\{(n+1)|a_n|R^n\} - \log|a_0|$$

$$\leq \log\{(n+1)|a_n|R^n\}$$

and the theorem follows.

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References

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