WAVES PRODUCED BY A PRESSURE SYSTEM MOVING WITH AN ACCELERATION OVER THE SURFACE OF DEEP WATER

by R. N. BHATTACHARYYA, *Department of Mathematics, Jadavpore University, Jadavpore, Calcutta–32*

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INTRODUCTION

In the study of ship-waves and resistance due to waves the mechanism of the production of waves is often idealized as a pressure system moving on the surface of water. Lord Kelvin (1908) (Mathematical and Physical Papers, Vol. IV) conceived the idea of the initiation and development of canal ship-waves by a sudden commencement and continued motion of a system of surface pressures. Sir T. H. Havelock (1917) advanced a theory of wave resistance of ships in which the ship was replaced by a pressure zone suddenly brought into existence and moving uniformly on the surface of water. In the manner of Poisson's theory of generation of waves by a sudden concentrated pressure distribution on a fixed part of the surface of the liquid, the theory of waves produced by the moving pressure distribution was worked out. The wave system produced was analysed and by specialization of the nature of the moving source a resistance formula, quite in good agreement with some observational results, was obtained.

Sir T. H. Havelock's results apply after a steady state has been attained represented by the uniform motion of the ship. In the present paper we have confined our attention to the nonsteady state when the ship is gathering speed through acceleration or is slowing down. The case is of practical importance as variation of speed is often a necessity, though it may not be of long duration.

1. *Wave system for an accelerated pressure zone*

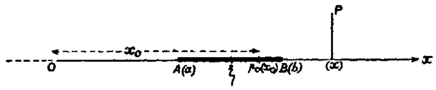

Fig. 1.

Let 0 be a fixed point on the undisturbed free surface. $0x$, $0y$ are taken horizontally, and vertically upwards through 0. Let the surface be disturbed by a pressure distribution $p_0(x_0)$ extending from $x_0 = a$ to $x_0 = b$, delivered at time t_0 and maintained for a period δt_0. This is equivalent to an impulse distribution $p_0 \delta t_0$ extending from 'a' to 'b' at time t_0.

The Fourier double integral representation of such a distribution is

$$f(x) = \frac{1}{\pi}\int_0^\infty dk \int_a^b p_0 \delta t_0 \exp\{ik(x-x_0)\}\, dx_0.$$

The velocity potential and surface elevation at any point x at any time t due to the impulse distribution $f(x)$ are given by

$$\pi\rho\phi = \int_0^\infty \cos\{kV(t-t_0)\} \exp(ky) dk \int_a^b p_0 \delta t_0 \exp\{ik(x-x_0)\}\, dx_0 \quad \cdot\cdot \quad (1.1)$$

and

$$-\pi g\rho y = \int_0^\infty kV \sin\{kV(t-t_0)\}\, dk \int_a^b p_0 \delta t_0 \exp\{ik(x-x_0)\}\, dx_0$$

where

$$V^2 = g/k.$$

This satisfies all the equations of the surface wave motion (of small amplitude) in deep water, viz.

$$\nabla^2\phi = 0\,,\ \text{for } y < 0\,,$$

$$\phi = 0\,,\ \text{for } y = -\infty\,,$$

$$\left.\begin{aligned}\phi_y &= -\partial y/\partial t\\ gy &= \partial\phi/\partial t\end{aligned}\right\}\ \text{on } y = 0.$$

The equations in (1.1) also satisfy the initial conditions, viz.

$$\rho\phi = f(x) \text{ on } y = 0 \text{ at } t = t_0.$$

If we assume the pressure system to have a centre at the point $x = \xi$ and the applied surface pressure at any point to depend only on its horizontal distance α from this centre, then $x_0 = \xi+\alpha$, $dx_0 = d\alpha$ and $p_0(x_0) = H(\alpha)$ say, and the limits of integration for x_0, i.e. α, are $-\gamma$ and γ where $\gamma = b-\xi = \xi-a = \frac{1}{2}(b-a)$.

If the pressure system moves without disintegration it is ξ that changes with time t_0. The surface elevation due to such an initial pressure system, when it has been in motion from time $t_0 = 0$ to $t_0 = t$, is given by

$$-\pi g\rho y = \int_0^\infty kVdk \int_{-\gamma}^{\gamma} H(\alpha) \exp(-ik\alpha)d\alpha \times \int_0^t \sin kV(t-t_0) \exp\{ik(x-\xi)\}dt_0$$

$$= \int_0^\infty \psi(k) kVdk \int_0^t \sin kV(t-t_0) \exp\{ik(x-\xi)\}\, dt_0 \quad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

where

$$\psi(k) = \int_{-\gamma}^{\gamma} H(\alpha) \exp(-ik\alpha)d\alpha$$

$$= \int_{-\infty}^{\infty} H(\alpha) \exp(-ik\alpha)d\alpha \text{ assuming } H(\alpha) = 0 \text{ for } |\alpha| > \gamma$$

$$= 2\int_0^\infty H(\alpha) \cos k\alpha\, d\alpha \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

assuming $H(\alpha)$ to be symmetrical about the centre.

The equations (1.2) and (1.3) represent the wave system for the most general type of motion of the pressure region. We shall study the special case of uniform acceleration of the centre of the pressure system.

If the pressure system be supposed to have started initially with its centre at the origin of co-ordinates and moved with uniform velocity c, then

$$\xi = ct_0 = -c\,(t-t_0)+ct.$$

Substituting in (1.2) and making the transformation $t-t_0 = \tau$ we get

$$-\pi g\rho y = \int_0^\infty kV\psi(k)dk \int_0^t \sin kV\tau \exp\left\{ik(\bar{\omega}+c\tau)\right\}d\tau \quad \ldots \quad (1.4)$$

where $\tilde{\omega} = x - ct$ = horizontal distance of P from the centre of the moving pressure system at time t. This t is not subject to the time integration in the above. Assuming a damping factor $e^{-\mu\tau}$ where ultimately we shall make $\mu \to 0$, we get

$$-\pi g\rho y = \int_0^\infty \psi(k) k V dk \int_0^t \sin kV\tau \exp\left[-\mu\tau + ik(\tilde{\omega} + c\tau)\right] d\tau \quad .. \quad (1.5)$$

This result is identical with Sir Havelock's result given in his paper referred to above.

If the pressure system starts from the origin with a velocity c and moves with a uniform acceleration f, then

$$\begin{aligned} \xi &= ct_0 + \tfrac{1}{2}ft_0^2 \\ &= ct + \tfrac{1}{2}ft^2 - c\tau - \tfrac{1}{2}f\tau(2t-\tau). \quad .. \quad .. \quad (1.6) \end{aligned}$$

Substituting in (1.1), making the transformation $t - t_0 = \tau$ and introducing the damping factor $e^{-\mu\tau}$, we have in the same way as before

$$-\pi g\rho y = \int_0^\infty kV\psi(k) \exp(ik\tilde{\omega}) dk$$

$$\times \int_0^t \sin kV\tau \exp\left[-\mu\tau + ik\left\{c\tau + \tfrac{1}{2}f\tau(2\tau - \tau)\right\}\right] d\tau \quad .. \quad .. \quad (1.7)$$

where

$$\tilde{\omega} = x - ct - \tfrac{1}{2}ft^2$$

= horizontal distance of P from the centre of the moving system at any time t. The integral on the right hand side of (1.7) gives the surface elevation of the waves produced by the uniformly accelerated motion of the surface pressure system.

2. *Integration of* (1.7)

The integral in (1.7) contains the time-integral

$$\int_0^t \sin kV\tau \exp\left[-\mu\tau + ik\left\{c\tau + \tfrac{1}{2}f\tau(2t-\tau)\right\}\right] d\tau$$

$$= \frac{1}{2i}(I_+ - I_-) \quad .. \quad .. \quad .. \quad .. \quad (2.1)$$

obtained after replacing $\sin kV\tau$ by its exponential value where

$$I_+ = \int_0^t \exp\left[-\mu + ik(v+V)\right]\tau \exp\left(-\tfrac{1}{2}ikf\tau^2\right) d\tau \quad .. \quad .. \quad (2.2)$$

with $v = c + ft$ = the instantaneous velocity of the moving pressure system at any time t; and I_- is obtained from I_+ by changing the sign of V.

Integrating (2.1) by parts we have

$$I_+ = -\frac{1}{-\mu + ik(v+V)} + \frac{\exp\left[-\mu t + ik\left\{(v+V)t - \tfrac{1}{2}ft^2\right\}\right]}{-\mu + ik(v+V)}$$

$$+ \frac{ikf}{-\mu + ik(v+V)} \int_0^t \tau e^{-\mu\tau} \exp\left\{ik\left[(v+V)\tau - \tfrac{1}{2}f\tau^2\right]\right\} d\tau. \quad .. \quad (2.3)$$

The integral in (2.3) is of the form

$$\int_0^t \phi(u) e^{if(u)}\, du \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

where

$$\phi(u) = u \exp(-\mu u),\, f(u) = k\,\{(v+V)u - \tfrac{1}{2}fu^2\}$$

By using Kelvin's method of group approximation (Lamb, 1932, Art. 241) to evaluate the integral we obtain

$$\int_0^t \phi(u) e^{if(u)}\, du = \frac{\sqrt{\pi}.\phi(\alpha)}{\sqrt{|\tfrac{1}{2}f''(\alpha)|}} \exp\left[i\left\{f(\alpha) \pm \frac{\pi}{4}\right\}\right] \quad \ldots \quad \ldots \quad (2.5)$$

where α is a root of $f'(u) = 0$ and where the upper or the lower sign is to be taken in the integral according as $f''(\alpha)$ is $+Ve$ or $-Ve$. The integral (2.4) will contribute nothing to I_+ for solving $f'(\alpha) = o$, α is found to be equal to $(v+V)/f = t+(c+V)/f$, which being $>t$ lies outside the range of integration. For the integral corresponding to (2.4) in I_-

$$\phi(u) = u \exp(-\mu u)$$

and

$$f(u) = k\left[(v-V)u - \tfrac{1}{2}fu^2\right] \qquad \ldots \quad \ldots \quad \ldots \quad (2.6)$$

In this case $\alpha = (v-V)/f = t+(c-V)/f < t$, if $k < g/c^2$.

The integral in I_- will be $= 0$, if $k > g/c^2$, and will be given by Kelvin's formula (2.5) for $k < g/c^2$.

The usual restrictions on the use of Kelvin's formula are satisfied in the present case with $\phi(u)$ and $f(u)$ given by (2.6) and $\psi(k)$ of the form $k^n \exp(-\alpha k)$, $n \geqslant 0$. For instance, from (2.5) and (2.6) the following results are easily obtained

$$\alpha = (v-V)/f,\, f(\alpha) = k(v-V)^2/2f$$

$$f''(\alpha) = -kf,\, \phi(\alpha) = \frac{1}{f}(v-V)\exp\{-\mu(v-V)/f\}$$

Then we have

$$\int_0^t \phi(u) e^{if(u)}\, du = \left(\frac{2\pi}{kf}\right)^{\frac{1}{2}} \frac{1}{f}(v-V) \exp\{-\mu(v-V)/f\} \times \exp\left[i\left\{k(v-V)^2/2f - \frac{\pi}{4}\right\}\right]$$

Substituting for I_+ and I_- (omitting the contribution of the integral to I_+) we get the following results for the three parts. The 1st part of

$$\frac{1}{2i}(I_+ - I_-) = kV/\{(-\mu+ikv)^2 + k^2V^2\}$$

The 2nd part of

$$\frac{1}{2i}(I_+ - I_-)$$

$$= -\frac{1}{2k}\exp\{ik(vt - \tfrac{1}{2}ft^2)\}\left\{\frac{\exp(ikVt)}{v+V} - \frac{\exp(-ikVt)}{v-V}\right\}$$

in which we have put $\mu = 0$.

The 3rd part of

$$\frac{1}{2i}(I_+ - I_-)$$

$$= i\left(\frac{\pi}{2kf}\right)^{\frac{1}{2}} \exp\left[i\left\{k(v^2+V^2)/2f - \frac{\pi}{4}\right\}\right] \exp(-ikvV/f)$$

with $\mu = 0$, for $k < g/c^2$ and $= 0$,

for $k > g/c^2$.

Now from (1.7) and (2.1) we have

$$y = y_1 + y_2 + y_3$$

where (subject to the conditions of approximation)

$$-\pi g\rho y_1 = \int_0^\infty \frac{k^2V^2\psi(k)\exp(ik\tilde{\omega})}{(-\mu+ikv)^2+k^2V^2}\,dk,$$

$$2\pi g\rho y_2 = \int_0^\infty V\psi(k)\exp(ikX)\left\{\frac{\exp(ikVt)}{v+V} - \frac{\exp(-ikVt)}{v-V}\right\}dk$$

where $X = \tilde{\omega} + vt - \frac{1}{2}ft^2 = \tilde{\omega} + ct + \frac{1}{2}ft^2$

is the distance of P from the starting point 0,

and

$$-\pi g\rho y_3 = i\left(\frac{\pi g}{2f}\right)^{\frac{1}{2}} \exp(-i\pi/4) \int_0^\infty \psi(k)\exp\left[ik\left\{\tilde{\omega} + \frac{v^2+V^2}{2f}\right\}\exp(-ikvV/f)\right]dk. \quad \text{.. (2.7)}$$

In y_1, μ can be put $= 0$, only after the integration has been performed for otherwise the integral does not converge. As there is no such difficulty in case of y_2 or y_3 in these cases μ has been put $= 0$, even before integration.

With acceleration $f = 0$, y_1, y_2 will be the same as in Sir Havelock's paper. There seems to be some difficulty with the third integral which is expected to vanish with f, which however is not evident from the form of (2.7). This is due to the fact that Kelvin's group approximation method cannot be applied for very small values of f. With a very small f

$$\phi(\alpha) = \frac{1}{f}(v-V)\exp\{-\mu(v-V)/f\}$$

becomes very large. Further for the validity of Kelvin's formula $f'''(\alpha)/|f''(\alpha)|^{3/2}$ should be small. In the present case, $f'''(\alpha)=0, f''(\alpha) = -kf$. Obviously the above method fails when $f \to 0$ except when k is very large, in which again we are not interested as with $\psi(k) = k^n e^{-\alpha k}$, $(\alpha > 0)$ the contribution of large values of k to the surface elevation in (1.7) is negligible.
But one sees that the third term of the equation (2.3) $\to 0$, as $f \to 0$. Hence $y_3 \to 0$, as $f \to 0$ expected.

3. *Analysis of y_1*

$$-\pi g\rho y_1 = \int_0^\infty \frac{k^2V^2\psi(k)\,\exp(ik\tilde{\omega})}{(-\mu+ikv)^2+k^2V^2}\,dk$$

After some reduction we get

$$\pi g\rho y_1 = k_0\int_0^\infty \frac{k\psi(k)\,\exp(ik\tilde{\omega})dk}{(k-\alpha)(k-\beta)} \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

where $\alpha, \beta = k_0-2i\mu\, k_0v/g,\ -\mu^2/g$

and $v^2 = g/k_0$, k_0 depending on t through $v = c+ft$

and $V^2 = g/k$.

The poles of the integrand are on the negative real axis and in the fourth quadrant.

For $\tilde{\omega} > 0$, integrating over the contour of the first quadrant of an infinite circle we get

$$\int_0^\infty \frac{k\psi(k)\,\exp(ik\tilde{\omega})}{(k-\alpha)(k-\beta)}dk \to \int_0^\infty \frac{\psi(ik)}{k+ik_0}\,\exp(-\tilde{\omega}k)dk = I_1, \text{ say, as } \mu \to 0. \quad (3.2)$$

For $\tilde{\omega} < 0$, integrating along the contour of the fourth quadrant of an infinite circle we get

$$\int_0^\infty \frac{k\psi(k)\,\exp(ik\tilde{\omega})}{(k-\alpha)(k-\beta)}dk \to -2\pi i\psi(k_0)\,\exp(ik_0\tilde{\omega})+I_2 \text{ as } \mu \to 0 \quad \ldots \quad (3.3)$$

where

$$I_2 = \int_0^\infty \frac{\psi(ik)}{k-ik_0}\,\exp(\tilde{\omega}k)dk.$$

Then from (3.1)

$$\pi g\rho y_1 = k_0 I_1, \quad \text{for} \quad \tilde{\omega} > 0$$

and

$$= -2\pi i\psi(k_0)\,\exp(ik_0\tilde{\omega})+k_0I_2, \text{ for } \tilde{\omega} < 0.$$

For the actual surface elevation of the waves we are to take the real parts of the above results (since y the sum of y_1, y_2 and y_3 is entirely real). We should note that our integrated value of y has been obtained after an approximation. Since $\psi(ik)$ is an even function it appears from (3.2) and (3.3) that if the real part of $k_0I_1 = f(\tilde{\omega})$ then the real part of $k_0I_2 = f(-\tilde{\omega})$. Further $f(\tilde{\omega})$ is small for large values of $\tilde{\omega}$. So taking the real parts we have

$$\pi g\rho y_1 = f(\tilde{\omega}), \quad \text{for } \tilde{\omega} > 0 \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

and

$$= 2\pi k_0\psi(k_0)\sin k_0\tilde{\omega}+f(-\tilde{\omega}), \text{ for } \tilde{\omega} < 0.$$

This represents an infinite train of harmonic waves following in the rear of the travelling pressure system together with a disturbance symmetrical fore and aft which becomes negligible at large distance from the travelling system. The wave velocity is the same as the *instantaneous velocity* of the travelling disturbance. We shall henceforth consider the wave part of y_1. These results are similar to the well-known results for the uniform motion of the pressure system, given by Sir Havelock before. In the present case, however, it may be noted that the wave length $2\pi/k_0 = \dfrac{2\pi v^2}{g}$ increases with time, i.e. with $v(= c+ft)$.

4. *Analysis of y_2*

From (2.7) we have

$$2\pi g\rho y_2 = \int_0^\infty V\psi(k)\exp(ikX)\left\{\frac{\exp(ikVt)}{v+V} - \frac{\exp(-ikVt)}{v-V}\right\}dk$$
$$= A-B$$

where

$$A = \int_0^\infty [V\psi(k)/(v+V)]\exp[ik(X+Vt)]\,dk.$$

$$B = \int_0^\infty [V\psi(k)/(v-V)]\exp[ik(X-Vt)]\,dk$$

Replacing v and V by $(g/k_0)^{\frac{1}{2}}$ and $(g/k)^{\frac{1}{2}}$ respectively and making the transformation $k = u^2$ we have

$$A = 2(k_0)^{\frac{1}{2}}\int_0^\infty \phi(u)\,e^{if(u)}\,du \quad \ldots \quad (4.1)$$

with

$$\phi(u) = u\psi(u^2)/[u+(k_0)^{\frac{1}{2}}]$$
$$f(u) = Xu^2+pu$$

and

$$p = (g)^{\frac{1}{2}}t.$$

By Kelvin's formula (2.5) the contribution to (4.1) comes from α (provided it lies within the range of integration) where α is a root of

$$f'(u) \equiv 2Xu+p = 0$$

i.e. $\alpha = -p/2X$, which is positive, if $X < 0$.
Thus $A = 0$, for $X > 0$, and for $X < 0$, by Kelvin's formula

$$A = 2.\frac{(\pi k_0)^{\frac{1}{2}}\phi(\alpha)}{|\frac{1}{2}f''(\alpha)|^{\frac{1}{2}}}\exp[i(f(\alpha)\pm\pi/4)] \quad \ldots \quad (4.2)$$

where the upper or the lower sign is to be taken according as $f''(\alpha)$ is $+Ve$ or $-Ve$.

The following results are easily obtained

$$f''(\alpha) = 2X \qquad \text{(in the present case)}$$
$$f(\alpha) = -p^2/4X$$
$$\phi(\alpha) = [vt\psi(p^2/4X^2)]/(vt-2X)$$

Then from (4.2), we have

$$A = 2[(\pi g)^{\frac{1}{2}}t/\{|X|^{\frac{1}{2}}(vt-2X)\}]\psi(p^2/4X^2)\times\exp[-i(\pi/4+p^2/4X)],\ \text{for } X<0$$
$$\text{and} = 0,\ \text{for } X>0 \quad \ldots \quad (4.3)$$

Proceeding similarly with B we get

$$B = 2[(\pi g)^{\frac{1}{2}} t/\{X^{\frac{1}{2}}(vt-2X)\}]\,\psi(p^2/4X^2)\,.\ \exp[i(\pi/4-p^2/4X)] \text{ for } X>0$$
$$\text{and } = 0, \text{ for } X<0 \quad \ldots \quad \ldots \quad \ldots \quad (4.4)$$

Hence

$$\pi g \rho y_2 = A = [(\pi g)^{\frac{1}{2}} t\psi(p^2/4X^2)/\{|X|^{\frac{1}{2}}(2|X|+vt)\}]\cos(\pi/4-p^2/4|X|) \text{ for } X<0$$

and

$$= -B = [(\pi g)^{\frac{1}{2}} t\psi(p^2/4X^2)/\{X^{\frac{1}{2}}(2X-vt)\}]\cos(\pi/4-p^2/4X) \text{ for } X>0 \quad (4.5)$$

5. *Analysis of* y_3

From (2.7) we have

$$-\pi g \rho y_3 = i(\pi g/2f)^{\frac{1}{2}} \exp(-i\pi/4)\int_0^\infty \psi(k)\exp\left\{ik\left(\tilde{\omega}+\frac{v^2+V^2}{2f}\right)\right\}\exp(-ikvV/f)\,dk$$

$$= i\left(\frac{\pi g}{2f}\right)^{\frac{1}{2}} \exp\left[i\left(\frac{g}{2f}-\frac{\pi}{4}\right)\right]\int_0^\infty \psi(k)\exp[i(kX_1-p_1k^{\frac{1}{2}})]\,dk$$

using $V^2 = g/k$, where $X_1 = \tilde{\omega}+v^2/2f$, and $p_1 = g^{\frac{1}{2}}v/f$.

Making the transformation $k = u^2$, we have

$$-\pi g\rho y_3 = 2i\left(\frac{\pi g}{2f}\right)^{\frac{1}{2}}\exp\left[i\left(\frac{g}{2f}-\frac{\pi}{4}\right)\right]\int^\infty \phi(u)e^{if(u)}\,du\,, \quad \ldots \quad (5.1)$$

writing $\phi(u) = u\psi(u^2)$ and $f(u) = X_1u^2-p_1u$.

In this case the contribution comes from α (if it lies within the range of integration) where α is a root of

$$f'(u) \equiv 2X_1u-p_1 = 0,$$

so that

$$\alpha = p_1/2X_1, \text{ which is } +Ve \text{ if } X_1>0.$$

Thus $$y_3 = 0, \text{ if } X_1<0.$$

$$\text{Further, } f''(\alpha) = 2X_1(+Ve \text{ in the present case}),$$

$$f(\alpha) = -p_1{}^2/4X_1, \text{ and } \phi(\alpha) = (p_1/2X_1)\psi(p_1{}^2/4X_1{}^2)$$

Applying Kelvin's formula to (5.1) and taking the real part only we get

$$\rho y_3 = [t\psi(p_1{}^2/4X_1{}^2)/(2f)^{1/2}X_1{}^{3/2}]\sin(g/2f-p_1{}^2/4X_1), \text{ for } X_1>0$$
$$\text{and } = 0, \quad \text{for } X_1<0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.2)$$

This component of the wave does not exist to the left of the starting point, for a motion of the pressure system starting from rest ($c = 0$) for

$$X_1 = X = \text{distance from the starting point.}$$

It may also be noted that $p_1 = p$, for $c = 0$. For $c \neq 0$, X_1 will still be the distance measured from a point in the rear of the starting point from which a motion from rest with acceleration f will give the velocity c at the starting point.

6. *Interpretation of the three components*

We thus see that the surface elevation of the wave generated by a symmetrical pressure system starting with an initial velocity c and moving in a straight line with a uniform acceleration f is given by

$$y = y_1+y_2+y_3$$

where y_1, y_2 and y_3 are given by (2.7), (3.4), (4.5) and (5.2).

The general interpretation of y_1 is well known from Havelock's work. It represents a train of waves (mostly simple harmonic) following in the wake of the pressure system. For uniform velocity of the pressure system, as in Sir Havelock's treatment, y_1 is not dependent on the time and represents a pattern fixed relatively to the system. In our case y_1, directly dependent on the instantaneous velocity, involves time implicitly. The dependence of the amplitude on time has been shown in an example worked out in the next section. y_2 represents a disturbance extending on either side of the starting point and becomes small at large distance from that point. In the neighbourhood of the starting point the fluctuations of y_2 become more and more rapid. y_3 represents a disturbance extending only to the right (direction of motion) of the starting point and diminishing at large distance from that point. This component also will have some rapidly fluctuating part near the starting point.

The above shows, however, only a general behaviour of the three components. Their character in detail will depend on the choice of the function $\psi(k)$. The success of the theory will depend on the possibility of choice of $\psi(k)$ in such a plausible manner as to yield conclusion in agreement with observational results. With $\psi(k) = Ae^{-rk}$ and $A =$ Const. (6.1), the amplitude of y_1 will attain a maximum at $v = (2gr)^{\frac{1}{2}}$ and then fall off. Similar maximum might be attained by the amplitudes of y_2 and y_3 also. This undesirable characteristic, however, can be avoided by taking A proportional to v^2, as has been considered more probable by Sir T. H. Havelock. In this case the amplitudes will show no maximum at any definite velocity but will go on increasing with time as the velocity increases.

We shall now work out numerically a particular case and compare the contributions of y_1, y_2, y_3 at a point 500 feet behind and fixed relative to the centre of the moving pressure system for

$$r = 1, c = 0, f = \tfrac{1}{4} ft/sec^2, \tilde{\omega} = -500 ft$$

and

$$\psi(k) = v^2e^{-k}, k_0 = g/v^2.$$

In this case the regular part of y_1 is given by

$$\pi g\rho y_1 = 2\pi g \exp(-g/v^2) \sin(\tilde{\omega}g/v^2)$$

$$= -2\pi g \exp(-512/t^2) \sin(256000/t^2)$$

or

$$\rho y_1 = -2 \exp(-512/t^2) \sin(256000/t^2) \quad \ldots \quad \ldots \quad (6.2)$$

The graph of this y_1 lies between the amplitude curves represented by

$$\rho y_1 = \pm Q_1$$

where

$$Q_1 = 2 \exp(-512/t^2)$$

Q_1 is tabulated against time from 0 to 120 seconds.

TABLE 1

Showing Q_1 between 0 and 120 seconds

0	5	10	15	20	25	30	35	40	45	50	55	60
		0·02	0·02	0·56	0·88	1·14	1·36	1·46	1·55	1·66	1·69	1·72
65	70	75	80	85	90	95	100	105	110	115	120	
1·77	1·81	1·83	1·85	1·87	1·88	1·89	1·90	1·91	1·92	1·93	1·93	

The graph of Q_1 is shown in Fig. 2. The fluctuations of y_1 given by the argument of the sine function are however abnormally large for the earlier seconds and grow less and less rapid with time and ultimately y_1 will cease to fluctuate. For instance, there are 20 complete periods between 30 and 40 seconds, 9 between 40 and 50 seconds and this dwindles to 1 between 80 and 90 seconds. A table for the values of y_1 between 90 and 120 seconds is given below. The graph of Q_1 up to 90 seconds (Table 1) and y_1 in full beyond that (Table 2) is shown in Fig. 2.

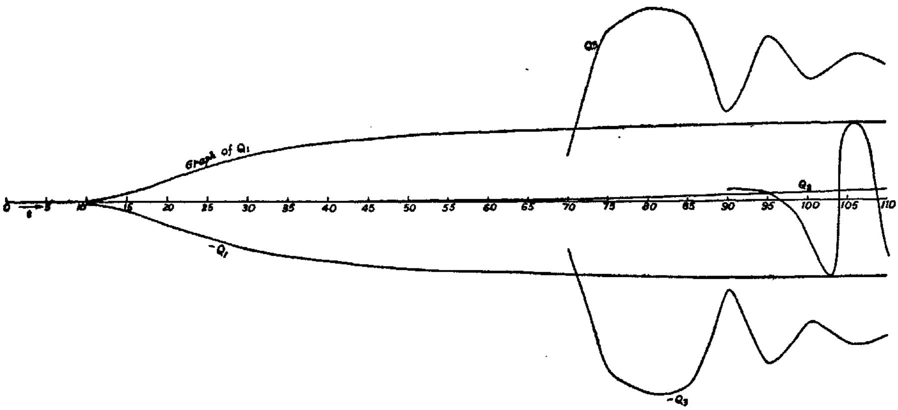

FIG. 2

TABLE 2

Showing y_1 between 90 and 120 seconds

90	95	100	105	110	115	120
+0·32	+0·19	—0·89	+1·81	—1·40	—0·97	+1·68

At 500 feet behind and fixed relative to the centre, we note, as above, that when the system starts, that point is behind the starting point. For $c = 0$ that point crosses the starting point at time given by

$$\tfrac{1}{2}ft^2 = 500$$

that is, $t = 62$ seconds nearly.

y_2 is given by the 1st of the two relations in (4.5) before 62 seconds and by the 2nd after that.

If Q_2 be the amplitude of ρy_2 we have, for $X < 0$

$$Q_2 = [t\psi(p^2/4X^2)]/[(\pi g\,|\,X\,|)^{\frac{1}{2}}\,(2\,|\,X\,|+vt)]$$
$$= [t^3 \exp\,(-gt^2/4X^2)]/[16000(\pi g\,|\,X\,|)^{\frac{1}{2}}]$$

obtained by substituting for ψ and the constants.

TABLE 3

Showing Q_2 between 0 and 60 seconds

0	5	10	15	20	25	30	35	40	45	50	55	60
0	0·000	0·000	0·001	0·002	0·005	0·008	0·013	0·020	0·028	0·032	0·018	0·001

For $t > 62$, y_2 will be given by the 2nd formula in (4.5), i.e.

$$\rho y_2 = -Q_2 \cos\,(\pi/4 - gt^2/4X)$$

where

$$Q_2 = [t^3 \exp\,(-gt^2/4X^2)]/[16000(\pi g X)^{\frac{1}{2}}]$$

TABLE 4

Showing Q_2 between 70 and 120 seconds

70	75	80	85	90	95	100	105	110	115	120
0·009	0·06	0·10	0·13	0·16	0·18	0·20	0·22	0·24	0·26	0·31

The graph of Q_2 is shown in Fig. 2. From the values of Q_2 it is evident that the y_2— waves behind the starting point are of smaller amplitude than those in front. But, at any rate, Q_2, though rising towards the front, is very much smaller than Q_1 at corresponding times. y_2 may be regarded as a sort of deviation from the more regular wave pattern represented by y_1. The y_2— component, however, might be more significant at a point much nearer the centre of the moving system.

As regards the y_3— wave (which does not exist for $X_1 < 0$) it may be considered to represent the exclusive effect (at least the direct effect) of the acceleration on the surface elevation. It comes into existence only after the point to which we are fixing our attention (e.g. a point 500 ft. behind the centre) has crossed the starting point (with $c = 0$), i.e. after about 62 seconds. If Q_3 be the amplitude of ρy_3 then

$$Q_3/Q_2 = 8000(\pi)^{\frac{1}{2}}/X = 14180/X, \text{ nearly}$$

where

$$X = \tilde{\omega} + \tfrac{1}{8}\,t^2$$

This shows that at a point close to the centre of the moving system the acceleration effect Q_3 will be much more pronounced than Q_2.

TABLE 5

Showing Q_3 between 70 and 120 seconds

70	75	80	85	90	95	100	105	110	115
1·13	4·19	4·73	4·57	2·20	4·06	3·00	3·55	3·36	3·20

The graph is shown in Fig. 2. The amplitude curve in this case (for $c = 0$) is reached by y_3 as frequently as in the case of y_2. It may further be noticed that y_2 and y_3 maintain a constant phase difference between them. *The curve for Q_3 heaves up several times unlike those for Q_1 and Q_2.*

Termination of the accelerated phase of the motion

The accelerated phase of the motion cannot be continued long. In a moving system like the ship, as the resistance gradually approaches the engine thrust a uniform velocity will be reached. At this stage the acceleration can no longer be maintained, the y_3— component will cease to exist, for this term exists only as long as the acceleration f exists. From then on the pressure system moves with constant velocity v and y_1, given by

$$\pi g\rho y_1 = f(\tilde{\omega}),\ \tilde{\omega} > 0$$

$$= 2\pi k_0\,\psi(k_0)\sin k_0\tilde{\omega} + f(-\tilde{\omega}),\ \tilde{\omega} < 0$$

then represents a regular wave train of constant length $2\pi/k_0$ or $2\pi v^2/g$ and moving with constant wave velocity v. This agrees evidently with Havelock's result where v is the limiting velocity.

Now X becomes equal to $\tilde{\omega}+vt$ and y_2 is given by

$$\pi g\rho y_2 = -\{[(\pi g)^{\frac{1}{2}}t\psi(p^2/4X^2)]/[\,|X|^{\frac{1}{2}}(2\tilde{\omega}+vt)]\}\cos\left(\frac{\pi}{4}-\frac{p^2}{4|X|}\right),\ \text{for } X<0$$

and

$$= \{[(\pi g)^{\frac{1}{2}}t\psi(p^2/4X^2)]/[X^{\frac{1}{2}}(2\tilde{\omega}+vt)]\}\cos\left(\frac{\pi}{4}-\frac{p^2}{4X}\right),\ \text{for } X>0$$

In either case as $t \to \infty$, $y \to 0$ as $1/t^{\frac{1}{2}}$. It is thus reasonable to consider y_2 as a deviation from the steady state which vanishes with time. Hence only y_1 is ultimately established. We thus get to Sir Havelock's result for $f \to 0$.

Sir Havelock's hypothesis was that a pressure system is suddenly formed and instantaneously set into motion by a jerk with uniform velocity. This hypothesis gives only the components y_1 and y_2. If, however, the velocity v is gradually acquired through uniform acceleration f, then there is an additional component y_3 which represents the major part of the surface elevation and lasts as long as the acceleration lasts.

Wave pattern left behind by the moving pressure system

We have so long studied the wave pattern at a fixed point behind the moving pressure system. This is the wave pattern as will appear to an observer stationed at the centre of the pressure system. We shall now study the wave pattern left behind by the moving pressure system at a distance, say 32 feet, ahead of the starting point. This point is crossed by the centre moving with acceleration $\frac{1}{4}ft/sec^2$ in

16 seconds. Before 16 seconds y_1 is 0 at this point and after 16 seconds, y_1 is given by

$$\rho y_1 = -2 \exp(-512/t^2) \sin 64(1-256/t^2)$$

with the same numerical data as before, for in this case

$$\tilde{\omega} = 32 - \tfrac{1}{2}ft^2 = 32 - t^2/8.$$

This represents a vibration of which the maximum amplitudes are limited by $\pm 2 \exp(-512/t^2)$ after 16 seconds. It may be shown that after 75 seconds y_1 goes through less than half a complete period, changing sign for the last time between 120 and 125 seconds and then gradually tends to the constant value $-2 \sin 64 = -1{\cdot}82$. After about 175 seconds the vibration practically ceases. By this time the centre of the pressure system has travelled through 3,828 feet nearly and is thus about 3,786 feet ahead of the point under consideration. This result shows that even the more or less regular part of the vibration dies out at a certain stage behind the moving pressure system. At a fixed point there will be a definite time, depending on its position, after which no further y_1— wave will arrive there. This represents an important deviation produced by acceleration from Sir Havelock's result for constant velocity. In the case of constant velocity

$$\rho y_1 = 2 \exp(-g/c^2) \sin(g\tilde{\omega}/c^2).$$

If now we put $\tilde{\omega} = -(ct-32)$ corresponding to a point 32 feet ahead of the starting point we get

$$\rho y_1 = -2 \exp(-g/c^2) \sin[g(ct-32)/c^2]$$

which represents a vibration with no decaying factor in time. In our case the vibration practically stops after 75 seconds.

In Fig. 3 the amplitude curve of ρy_1, viz. $\pm 2 \exp(-512/t^2)$, is graphed between 20 and 75 seconds and the actual curve of ρy_1 after 75 seconds.

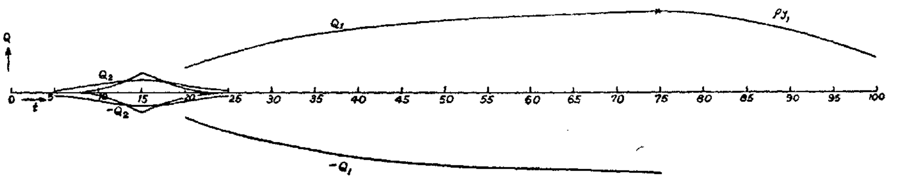

FIG. 3

For $X = 32$, y_2 and y_3 are given by

$$\rho y_2 = \{t^3/[16(2\pi)^{\frac{1}{2}}(256-t^2)]\} \exp(-t^2/128) \cos[(\pi-t^2)/4]$$

and

$$\rho y_3 = (t^3/2048) \exp(-t^2/128) \sin(64-t^2/4).$$

The graphs of the amplitudes of these wave-components are shown along with Q_1 and ρy_1 in Fig. 3. The graph shows ρy_2 and ρy_3 are only small disturbances, before the pressure system passes the fixed point, rising to a maximum just about the time when the centre of the moving system reaches that point. These deviation (ρy_2) and acceleration (ρy_3) effects die out rather quickly after the pressure system has crossed the fixed point, whereafter there develops a regular sinusoidal wave of gradually diminishing frequency which also dies out in the long run as described before.

Wave profile at a fixed time $t = 100$ seconds

With
$$t = 100,\ p = p_1 = gt^2 = 32{\cdot}10^4$$
$$X = X_1 = \tilde{\omega}+ft^2/2 = \tilde{\omega}+(100)^2/8 = \tilde{\omega}+1250$$
we have for $\tilde{\omega} < 0$ and $X = X_1 > 0$ from the formulae of § 6,
$$\rho y_1 = 1{\cdot}9 \sin(\cdot 05\tilde{\omega})$$
$$\rho y_2 = Q_2' \cos[\pi/4 - 8{\cdot}10^4/(\tilde{\omega}+1250)]$$
$$\rho y_3 = Q'_3 \sin[64 - 8{\cdot}10^4/(\tilde{\omega}+1250)]$$
where
$$Q_2' = [15625/2\tilde{\omega}\{\pi(2\tilde{\omega}+2500)\}^{\frac{1}{2}}] \exp\left\{-\frac{32{\cdot}10^4}{(2\tilde{\omega}+2500)^2}\right\}$$
and
$$Q_3' = [10^6/(2\tilde{\omega}+2500)^{\frac{3}{2}}] \exp\left\{-\frac{32{\cdot}10^4}{(2\tilde{\omega}+2500)^2}\right\}$$
ρy_1 represents a regular sinusoidal wave of length $2\pi/{\cdot}05 = 40\pi$ limited by the maximum amplitudes $\pm 1{\cdot}9$. The graphs for the three components are shown in Fig. 4.

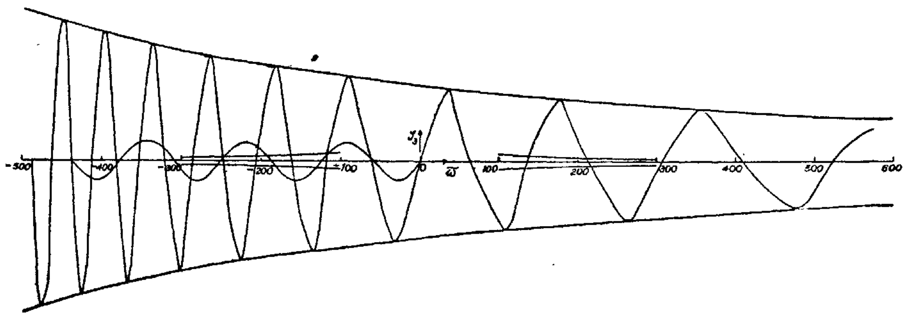

FIG. 4

In conclusion, the author has much pleasure to acknowledge his indebtedness to Prof. N. R. Sen, F.N.I., for many helpful suggestions and encouragement during the preparation of this work.

ABSTRACT

The accelerated motion of the pressure system generates three kinds of disturbances of the free surface:

(1) A more or less regular wave pattern y_1 propagating backwards, the wave number $k_0 = g/v^2$ depending on the instantaneous velocity of the centre of the pressure system. This wave dies out at a certain distance behind the moving pressure system. This represents an important deviation produced by acceleration from Havelock's result for constant velocity which gives a sinusoidal vibration behind the moving system with no decaying factor in time.

(2) A small deviation effect y_2 extending to a small distance on either side of the moving system but very weak behind the *starting point*.

(3) A direct acceleration effect propagating forwards depending on the wave number $k_1 = p/2X = g^{\frac{1}{2}}t/(2\tilde{\omega}+ft^2)$ which attains a maximum value $g^{\frac{1}{2}}/2^{\frac{3}{2}}(\tilde{\omega}f)^{\frac{1}{2}}$. This represents the dominant effect when and where it exists. This component disappears along with the acceleration.

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