VIBRATIONS OF A CIRCULAR CYLINDER OF TRANSVERSELY ISOTROPIC MATERIAL

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Introduction

Because of greater mathematical difficulties in solving problems of stress distribution in anisotropic elastic bodies, these problems have received much less attention than the corresponding problems of isotropic bodies. The solution of problems of torsion and flexure of anisotropic bodies and the plane problems of orthotropic plates are numerous no doubt, but the number of solutions of three dimensional problems of elastic bodies possessing anisotropy of any kind is few. Abandoning general anisotropy and assuming that the material possesses an axis of elastic symmetry, Michell (1900) has given the solution of the problem of a semi-infinite solid under the action of prescribed surface tractions on the plane boundary. This type of anisotropy is possessed by crystals of hexagonal system and has been called ' transverse isotropy ' by Love (1944, p. 160). Further progress in this connection has been made by Lekhnitsky (1940, 1950), Elliott (1948, 1949), Moisil (1950), Eubanks and Sternberg (1954) and others. Some problems of equilibrium of bodies possessing cylindrical and spherical aeolotropy have also been solved by St. Venant (1865), Carrier (1943) and others.

Although some progress has been made in solving problems of equilibrium of anisotropic materials, very little work has been done on problems of vibration of such bodies. The problem of vibration of a spherical shell of a material possessing spherical aeolotropy and of a cylindrical shell of a material possessing cylindrical aeolotropy have been solved by the present author and are being published elsewhere. In the present paper, the problems of vibration of a circular cylinder of a material possessing ' transverse isotropy ' have been considered. The axis of the cylinder is taken to be parallel to the axis of elastic symmetry of the material. Complete solutions of the problems of longitudinal, transverse and torsional vibrations of a cylinder have been obtained in terms of Bessel's Functions.

Fundamental Equations

Introducing cylindrical co-ordinates r, θ, z with the z-axis parallel to the axis of the cylinder and the axis of symmetry of the material, the strain-energy function of a transversely isotropic material, which contains five elastic constants, can be written as (Love, 1944, p. 160)

$$W = \tfrac{1}{2}c_{11}(e_{rr}^2+e_{\theta\theta}^2)+\tfrac{1}{2}c_{33}e_{zz}^2+c_{13}(e_{rr}+e_{\theta\theta})e_{zz}+c_{12}e_{rr}e_{\theta\theta}+\tfrac{1}{2}c_{44}(e_{\theta z}^2+e_{rz}^2)+\tfrac{1}{2}c_{66}e_{r\theta}^2 \quad (1)$$

where

$$c_{12} = c_{11} - 2c_{66} \qquad \ldots \quad (2)$$

The stress components are given by

$$\left.\begin{aligned} \widehat{rr} &= c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz} \\ \widehat{\theta\theta} &= c_{12}e_{rr} + c_{11}e_{\theta\theta} + c_{13}e_{zz} \\ \widehat{zz} &= c_{13}e_{rr} + c_{13}e_{\theta\theta} + c_{33}e_{zz} \\ \widehat{\theta z} &= c_{44}e_{\theta z},\ \widehat{zr} = c_{44}e_{zr},\ \widehat{r\theta} = c_{66}e_{r\theta} \end{aligned}\right\} \qquad \ldots \quad (3)$$

If u, v, w are the components of displacement in the directions r, θ, z the components of strain are given by (Love, 1944, p. 56)

$$\left.\begin{aligned} &e_{rr} = \frac{\partial u}{\partial r};\quad e_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r};\quad e_{zz} = \frac{\partial w}{\partial z}; \\ &e_{\theta z} = \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z};\quad e_{zr} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}; \\ &e_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}; \end{aligned}\right\} \qquad \ldots \quad (4)$$

The stress equations of motion are (Love, 1944, p. 90)

$$\left.\begin{aligned} &\frac{\partial \widehat{rr}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{r\theta}}{\partial \theta} + \frac{\partial \widehat{rz}}{\partial z} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} = \rho \ddot{u} \\ &\frac{\partial \widehat{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{\theta\theta}}{\partial \theta} + \frac{\partial \widehat{\theta z}}{\partial z} + \frac{2\widehat{r\theta}}{r} = \rho \ddot{v} \\ &\frac{\partial \widehat{rz}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{\theta z}}{\partial \theta} + \frac{\partial \widehat{zz}}{\partial z} + \frac{\widehat{rz}}{r} = \rho \ddot{w} \end{aligned}\right\} \qquad \ldots \quad (5)$$

For vibrations of an infinite circular cylinder, we seek solutions of the equations (5) in the form

$$u = U_1 e^{i(\gamma z + pt)},\ v = V_1 e^{i(\gamma z + pt)},\ w = W_1 e^{i(\gamma z + pt)} \qquad \ldots \quad (6)$$

in which U_1, V_1, W_1 are functions of r and θ only.

Longitudinal Vibrations

We can obtain a solution of the equations (5) in which V_1 vanishes and U_1, W_1 are functions of r alone. We take

$$u = Ue^{i(\gamma z + pt)},\ v = 0,\ w = We^{i(\gamma z + pt)} \qquad \ldots \quad (7)$$

where U and W are functions of r alone.

From (1), (3), (4) and (7) we get

$$\left.\begin{aligned}
\widehat{rr} &= \left[c_{11}\frac{\partial U}{\partial r}+c_{12}\frac{U}{r}+c_{13}i\gamma W\right] e^{i(\gamma z+pt)}\\
\widehat{\theta\theta} &= \left[c_{12}\frac{\partial U}{\partial r}+c_{11}\frac{U}{r}+c_{13}i\gamma W\right] e^{i(\gamma z+pt)}\\
\widehat{zz} &= \left[c_{13}\frac{\partial U}{\partial r}+c_{13}\frac{U}{r}+c_{33}i\gamma W\right] e^{i(\gamma z+pt)}\\
\widehat{rz} &= c_{44}\left[\frac{\partial W}{\partial r}+i\gamma U\right] e^{i(\gamma z+pt)}\\
\widehat{\theta z} &= \widehat{r\theta} = 0
\end{aligned}\right\} \quad \ldots \quad \ldots \quad (8)$$

Substituting from (8) in the equations (5), we see that the second equation is identically satisfied and the remaining two equations reduce to

$$c_{11}\left(\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}-\frac{U}{r^2}\right)+(\rho p^2-c_{44}\gamma^2)U+i\gamma(c_{13}+c_{44})\frac{\partial W}{\partial r}=0 \quad \ldots \quad (9)$$

$$c_{44}\left(\frac{\partial^2 W}{\partial r^2}+\frac{1}{r}\frac{\partial W}{\partial r}\right)+(\rho p^2-c_{33}\gamma^2)W+i\gamma(c_{13}+c_{44})\left(\frac{\partial U}{\partial r}+\frac{U}{r}\right)=0 \quad \ldots \quad (10)$$

As a trial solution, let us assume

$$U = AJ_1(kr),\ W = BJ_0(kr) \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

where J_0 and J_1 are Bessel's Functions of the arguments concerned of orders zero and one respectively.

Substituting from (11) in (9) and (10) and using the equations satisfied by $J_0(kr)$ and $J_1(kr)$ and the recurrence formulae

$$\left.\begin{aligned}
J_n'(z) &= J_{n-1}(z)-\frac{n}{z}J_n(z)\\
J_n'(z) &= \frac{n}{z}J_n(z)-J_{n+1}(z)
\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

we get

$$A[c_{11}k^2-(\rho p^2-c_{44}\gamma^2)]+i\gamma Bk(c_{13}+c_{44})=0 \quad \ldots \quad \ldots \quad (13)$$

$$B[c_{44}k^2-(\rho p^2-c_{33}\gamma^2)]-i\gamma Ak(c_{13}+c_{44})=0 \quad \ldots \quad \ldots \quad (14)$$

Eliminating A and B from (13) and (14) we see that in order that (11) may satisfy (9) and (10) k must be root of the equation

$$[c_{11}k^2-(\rho p^2-c_{44}\gamma^2)]\,[c_{44}k^2-(\rho p^2-c_{33}\gamma^2)]-\gamma^2k^2(c_{13}+c_{44})^2=0 \quad \ldots \quad (15)$$

If k^2 and h^2 be the values of k^2 as obtained from (15), the solutions of (9) and (10) may be written as

$$U = AJ_1(kr)+CJ_1(hr) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

$$W = A\left\{\frac{\rho p^2-c_{44}\gamma^2-c_{11}k^2}{ik\gamma(c_{13}+c_{44})}\right\}J_0(kr)+C\left\{\frac{\rho p^2-c_{44}\gamma^2-c_{11}h^2}{ih\gamma(c_{13}+c_{44})}\right\}J_0(hr) \quad \ldots \quad (17)$$

If the boundary of the cylinder be $r = a$, the stresses $\widehat{rr}$, $\widehat{r\theta}$ and $\widehat{rz}$ vanish when $r = a$, so that

$$Af(k)+Cf(h) = 0 \quad \ldots \quad (18)$$

and

$$Ag(k)+Cg(h) = 0 \quad \ldots \quad (19)$$

where

$$xf(x) = (c_{11}c_{44}x^2+c_{13}\rho p^2-c_{13}c_{44}\gamma^2)aJ_0(xa) + (c_{12}-c_{11})(c_{13}+c_{44})xJ_1(xa)$$

$$g(x) = (c_{11}x^2-\rho p^2-c_{13}\gamma^2)J_1(xa).$$

Eliminating A and C from (18) and (19) we get the frequency equation as

$$f(k)g(h)-f(h)g(k) = 0 \quad \ldots \quad (20)$$

When the radius of the cylinder is small we may approximate to the frequency by expanding the Bessel's Functions. On putting

$$J_0(ha) = 1-\frac{1}{4}h^2a^2+\frac{1}{64}h^4a^4,\ J_1(ka) = \frac{1}{2}\left(ka-\frac{1}{8}k^3a^3\right)$$

and omitting terms of order a^2 we find a first approximation to the value of p from

$$c_{11}h^2k^2L-2c_{13}(\rho p^2-c_{44}\gamma^2)\{(\rho p^2+c_{13}\gamma^2)-c_{11}(h^2+k^2)\} = 0 \quad \ldots \quad (21)$$

where

$$L = c_{12}c_{13}+c_{12}c_{44}+c_{11}c_{44}-c_{11}c_{13}$$

and k^2 and h^2 are the values of k^2 as obtained from (15).

Substituting the values of k^2 and h^2 we get the approximate value of p as

$$p = \gamma\left(\frac{M}{\rho N}\right)^{\frac{1}{2}} \quad \ldots \quad (22)$$

where

$$M = c_{12}c_{13}c_{33}+c_{12}c_{44}c_{33}+c_{11}c_{33}c_{44}-2c_{13}^3-2c_{13}^2c_{44}+c_{11}c_{13}c_{33}$$
$$= (c_{13}+c_{44})(c_{12}c_{33}+c_{11}c_{33}-2c_{13}^2)$$

and

$$N = c_{12}c_{13}+c_{12}c_{44}+c_{11}c_{44}+c_{11}c_{13} = (c_{13}+c_{44})(c_{11}+c_{12})$$

When we retain terms in a^2 we obtain as a second approximation

$$p = \gamma\left(\frac{M}{\rho N}\right)^{\frac{1}{2}}\left\{1-\frac{1}{4}\gamma^2a^2\frac{c_{13}^2}{(c_{11}+c_{12})^2}\right\} \quad \ldots \quad (23)$$

If E be the Young's modulus for the direction of the axis of the cylinder and σ be Poisson's ratio for contractions parallel to r, then as in Love, p. 161,

$$E = \frac{M}{N} \text{ and } \sigma = \frac{c_{13}}{c_{11}c_{12}}.$$

The above expression is then seen to be identical with that given by Chree (1899), as given by Love, p. 290.

Transverse Vibrations

Another solution of the equations (5) can be obtained in which V_1 is proportional to $\sin\theta$ and U_1 and W_1 are proportional to $\cos\theta$.

Taking

$$u = U\cos\theta e^{i(\gamma z+pt)},\ v = V\sin\theta e^{i(\gamma z+pt)},\ w = W\cos\theta e^{i(\gamma z+pt)} \quad \ldots \quad (24)$$

where U, V, W are functions of r alone. We have from (1), (3), (4) and (24)

$$\left.\begin{aligned}
\widehat{rr} &= \left[c_{11}\frac{\partial U}{\partial r}+c_{12}\left(\frac{V}{r}+\frac{U}{r}\right)+c_{13}i\gamma W\right]\cos\theta\,.\,e^{i(\gamma z+pt)}\\
\widehat{\theta\theta} &= \left[c_{12}\frac{\partial U}{\partial r}+c_{11}\left(\frac{V}{r}+\frac{U}{r}\right)+c_{13}i\gamma W\right]\cos\theta\,.\,e^{i(\gamma z+pt)}\\
\widehat{zz} &= \left[c_{13}\frac{\partial U}{\partial r}+c_{13}\left(\frac{V}{r}+\frac{U}{r}\right)+c_{33}i\gamma W\right]\cos\theta\,.\,e^{i(\gamma z+pt)}\\
\widehat{\theta z} &= c_{44}\left(i\gamma V-\frac{W}{r}\right)\sin\theta\,.\,e^{i(\gamma z+pt)}\\
\widehat{zr} &= c_{44}\left(i\gamma U+\frac{\partial W}{\partial r}\right)\cos\theta\,.\,e^{i(\gamma z+pt)}\\
\widehat{r\theta} &= c_{66}\left(\frac{\partial V}{\partial r}-\frac{V}{r}-\frac{U}{r}\right)\sin\theta.\,e^{i(\gamma z+pt)}
\end{aligned}\right\} \quad \ldots \quad \ldots \quad (25)$$

Substituting from (25) in the equations (5), we get

$$c_{11}\left(\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}\right)-(c_{11}+c_{66})\frac{U}{r^2}+(\rho p^2-c_{44}\gamma^2)U$$
$$+(c_{12}+c_{66})\frac{1}{r}\frac{\partial V}{\partial r}-(c_{11}+c_{66})\frac{V}{r^2}+(c_{13}+c_{44})i\gamma\frac{\partial W}{\partial r}=0 \quad \ldots \quad (26)$$

$$c_{66}\left(\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}\right)-(c_{11}+c_{66})\frac{V}{r^2}+(\rho p^2-c_{44}\gamma^2)V$$
$$-(c_{12}+c_{66})\frac{1}{r}\frac{\partial U}{\partial r}-(c_{11}+c_{66})\frac{U}{r^2}-(c_{13}+c_{44})i\gamma\frac{W}{r}=0 \quad \ldots \quad \ldots \quad (27)$$

$$c_{44}\left(\frac{\partial^2 W}{\partial r^2}+\frac{1}{r}\frac{\partial W}{\partial r}-\frac{W}{r^2}\right)+(\rho p^2-c_{33}\gamma^2)W$$
$$+i\gamma(c_{13}+c_{44})\left(\frac{\partial U}{\partial r}+\frac{U}{r}+\frac{V}{r}\right)=0 \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

For a first type of solutions of these equations, let us assume that

$$U = A\frac{\partial}{\partial r}J_1(kr),\ V = \frac{B}{r}J_1(kr),\ W = CJ_1(kr) \quad \ldots \quad \ldots \quad (29)$$

Substituting in (26), (27) and (28) and using (2) and the relations

$$\frac{d^2}{dr^2}J_1(kr)+\frac{1}{r}\frac{d}{dr}J_1(kr)+\left(k^2-\frac{1}{r^2}\right)J_1(kr)=0$$

and

$$\frac{d}{dr}J_1(kr) = kJ_0(kr)-\frac{1}{r}J_1(kr)$$

and simplifying, we get

$$[\{A(c_{11}k^2+c_{44}\gamma^2-\rho p^2)-i\gamma C(c_{13}+c_{44})\}r^2+(c_{66}-3c_{11})(A+B)]J_1(kr)$$
$$+[\{A(\rho p^2-c_{44}\gamma^2-c_{11}k^2)+i\gamma C(c_{13}+c_{44})\}r^2+(c_{11}-c_{66})(A+B)]krJ_0(kr)=0 \quad (30)$$

$$[\{B(\rho p^2-c_{44}\gamma^2-c_{66}k^2)+A(c_{11}-c_{66})k^2-i\gamma C(c_{13}+c_{44})\}r^2$$
$$+(3c_{66}-c_{11})(A+B)]J_1(kr)-2c_{66}(A+B)krJ_0(kr)=0 \quad \ldots \quad \ldots \quad (31)$$

$$[C(\rho p^2-c_{33}\gamma^2-c_{44}k^2)-i\gamma A(c_{13}+c_{44})k^2]r^2+i\gamma(c_{13}+c_{44})(A+B)=0 \quad \ldots \quad (32)$$

These equations are satisfied if

$$A+B=0 \quad \ldots \quad (33)$$

$$A(\rho p^2-c_{44}\gamma^2-c_{11}k^2)+i\gamma C(c_{13}+c_{44})=0 \quad \ldots \quad (34)$$

$$C(\rho p^2-c_{33}\gamma^2-c_{44}k^2)-i\gamma A(c_{13}+c_{44})k^2=0 \quad \ldots \quad (35)$$

Eliminating A and C from (34) and (35) we get a quadratic equation for k^2, viz.

$$c_{44}c_{11}k^4-k^2\{\rho p^2(c_{11}+c_{44})-\gamma^2(c_{11}c_{33}-c_{13}{}^2-2c_{13}c_{44})\}$$
$$+(\rho p^2-c_{33}\gamma^2)(\rho p^2-c_{44}\gamma^2)=0 \quad \ldots \quad (36)$$

Let k^2 and k'^2 be the roots of this equation. Then corresponding to these roots the two values of C/A are $i\gamma\alpha(k)$ and $i\gamma\alpha(k')$, where

$$\alpha(x)=\frac{(c_{13}+c_{44})x^2}{\rho p^2-c_{33}\gamma^2-c_{44}x^2} \quad \ldots \quad (37)$$

For a second type of solution of the equations (26), (27) and (28) let us assume that

$$U=\frac{E}{r}J_1(hr),\ V=F\frac{\partial}{\partial r}J_1(hr),\ W=0 \quad \ldots \quad (38)$$

Substituting these in the above equations and using (2) and the relations

$$\frac{d^2}{dr^2}J_1(hr)+\frac{1}{r}\frac{d}{dr}J_1(hr)+\left(h^2-\frac{1}{r^2}\right)J_1(hr)=0$$

and

$$\frac{d}{dr}J_1(hr)=hJ_0(hr)-\frac{1}{r}J_1(hr)$$

and simplifying, we get

$$-[\{E(\rho p^2-c_{44}\gamma^2-c_{11}h^2)-F(c_{11}-c_{66})h^2\}\,r^2+(3c_{11}-c_{66})(E+F)]J_1(hr)$$
$$-2c_{11}(E+F)hrJ_0(hr)=0 \quad \ldots \quad (39)$$

$$[F(\rho p^2-c_{44}\gamma^2-c_{66}h^2)r^2+(3c_{66}-c_{11})(E+F)]J_1(hr)-[F(\rho p^2-c_{44}\gamma^2-c_{66}h^2)r^2$$
$$+(c_{66}-c_{11})(E+F)]hrJ_0(hr)=0 \quad \ldots \quad (40)$$

$$E+F=0 \quad \ldots \quad (41)$$

Substituting from (41) in (39) and (40) we see that these are satisfied if

$$h^2=\frac{\rho p^2-c_{44}\gamma^2}{c_{66}} \quad \ldots \quad (42)$$

Combining the two types of solution we write

$$\left.\begin{aligned} U &= A\frac{\partial}{\partial r}J_1(kr)+B\frac{\partial}{\partial r}J_1(k'r)+\frac{C}{r}J_1(hr) \\ V &= -\frac{A}{r}J_1(kr)-\frac{B}{r}J_1(k'r)-C\frac{\partial}{\partial r}J_1(hr) \\ W &= i\gamma A\alpha(k)J_1(kr)+i\gamma B\alpha(k')J_1(k'r) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (43)$$

where $\alpha(x)$ is given by (37), h^2 by (42) and k^2, k'^2 are the roots of the equation (36).

When these forms of U, V, W are substituted in (24) we get a solution of the equations (5). Calculating $u \sin\theta+v\cos\theta$ from (24) and (43) we see that it vanishes when $r=0$ so that the axis of the cylinder vibrates in a plane containing its unstrained position and the line from which θ is measured. Further $w=0$ when $r=0$, so that the displacement of a point of the axis is perpendicular to the axis. The vibrations are therefore of the transverse type.

If the boundary of the cylinder be $r=a$ then $\widehat{rr}$, $\widehat{r\theta}$ and $\widehat{rz}$ vanish when $r=a$. These conditions give after a little simplification

$$\left.\begin{aligned} Af_1(k)+Bf_1(k')-Cc_{33}f_2(h) &= 0 \\ Af_2(k)+Bf_2(k')+Cg_2(h) &= 0 \\ Af_3(k)+Bf_3(k')+CJ_1(ha) &= 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (44)$$

where

$$f_1(x) = [4c_{66}-c_{11}x^2a^2-c_{13}\gamma^2a^2\alpha(x)]J_1(xa)-2c_{66}xaJ_0(xa)$$

$$f_2(x) = 4J_1(xa)-2xaJ_0(xa)$$

$$g_2(x) = (x^2a^2-4)J_1(xa)+2xaJ_0(xa)$$

$$f_3(x) = [\alpha(x)+1][xaJ_0(xa)-J_1(xa)]$$

Eliminating A, B and C between the equations (44) we get the frequency equation as

$$\begin{vmatrix} f_1(k) & f_1(k') & -c_{33}f_2(h) \\ f_2(k) & f_2(k') & g_2(h) \\ f_3(k) & f_3(k') & J_1(ha) \end{vmatrix} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (45)$$

TORSIONAL VIBRATIONS

We seek a third type of solution of the equations (5) and (6) in which U_1, W_1 vanish and V_1 is independent of θ. We take

$$u=0,\ v=Ve^{i(\gamma z+pt)},\ w=0 \quad \ldots \quad \ldots \quad \ldots \quad (46)$$

where V is a function of r alone.

We have from (1), (3), (4) and (46)

$$\widehat{rr}=\widehat{\theta\theta}=\widehat{zz}=0,\ \widehat{\theta z}=i\gamma c_{44}V_e^{i(\gamma z+pt)}$$

$$\widehat{rz}=0,\ \widehat{r\theta}=c_{66}\left(\frac{\partial V}{\partial r}-\frac{V}{r}\right)$$

Substituting in equations (5) we see that the first and the third equations are identically satisfied and the second reduces to

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\left(k^2-\frac{1}{r^2}\right)V = 0 \qquad \ldots \quad (47)$$

where

$$k^2 = \frac{\rho p^2 - c_{44}\gamma^2}{c_{66}} \qquad \ldots \quad (48)$$

A solution of (47), finite when $r = 0$, is

$$V = AJ_1(kr) \qquad \ldots \quad (49)$$

If $r = a$ be the boundary of the cylinder then $\widehat{r\theta} = 0$ when $r = a$. Therefore

$$\frac{\partial}{\partial r}J_1(kr)-\frac{1}{r}J_1(kr) = 0$$

when $r = a$. One root of this equation is $k = 0$.

We have therefore found a simple solution of the equations (5) in the form

$$u = 0,\ v = Bre^{i(\gamma z+pt)},\ w = 0 \qquad \ldots \quad (50)$$

The vibration is therefore of the torsional type, the velocity of wave propagation along the cylinder being $(c_{44}/\rho)^{\frac{1}{2}}$.

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Summary

The paper deals with the vibrations of an infinite solid circular cylinder of a material possessing transverse isotropy, when the axis of the cylinder is parallel to the axis of symmetry of the material. The three types of vibrations, viz. longitudinal, transverse and torsional, have been completely determined.

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