UNIQUENESS OF MAXIMUM PRESSURE IN THE GENERAL THEORY OF COMPOSITE CHARGES

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1. Introduction

The general theory of composite charges has been discussed by Clemmow (1920, 1951) and Corner (1950) by reducing a composite charge to an equivalent charge when the composite charge consists of two component charges with their form-functions in the standard form $z = (1-f)\ (1+\theta f)$ and having either same composition (Clemmow) or with different compositions, but adopting, for the equivalent charge, weighted averages of the parameters defining the composition of the component charges (Corner). Their equivalent-charge method has recently been generalised by the present author [Kapur (1956)] to a composite charge consisting of n component charges with different compositions and possibly with cubic form-functions. Venkatesan and Patni (1953) gave an alternative direct method for dealing with the general theory of composite charges by extending the Hunt-Hinds' system of Internal Ballistics to the case of a composite charge consisting of two component charges.

One important problem in the general theory of composite charges, especially from the point of view of designing of guns, is to find out the maximum pressure due to a composite charge and to find out the stage in which it occurs. In particular, it is important to know whether there will be a unique maximum pressure or whether secondary pressure-maxima can arise. In the present paper, the Hunt-Hinds' method has been generalised to a composite charge consisting of n component charges to show that, in general, a unique maximum pressure exists. Rules have also been given to determine the stage in which this maximum pressure will occur.

2. The equations of internal ballistics for a composite charge consisting of n component charges

Neglecting co-volume correction terms and assuming—

(i) $\gamma_1 = \gamma_2 = \gamma_3 = \cdots = \gamma_n = \gamma$;
(ii) a linear law of burning;
(iii) standard form-function for the component charges; the equations of Internal Ballistics are:

The Energy Equation :

$$F_1 C_1 z_1 + F_2 C_2 z_2 + \cdots + F_n C_n z_n = Ap\ (x+l) + \tfrac{1}{2}\ (\gamma-1)\ w_1 v^2, \quad \ldots \quad (1)$$

where

$$w_1 = 1{\cdot}05w + \frac{C_1}{3} + \frac{C_2}{3} + \cdots + \frac{C_n}{3} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

The Equation of Motion:

$$w_1 \frac{dv}{dt} = w_1 v \frac{dv}{dx} = Ap \qquad \ldots \quad \ldots \quad \ldots \quad (3)$$

The Form-Functions:

$$z_i = (1-f_i)\,(1+\theta_i f_i)\ [i = 1, 2, 3, \ldots, n] \qquad \ldots \quad \ldots \quad (4)$$

The Rate-of-Burning Equations:

$$D_i \frac{df_i}{dt} = -\beta_i p\ [i = 1, 2, 3, \ldots, n] \qquad \ldots \quad \ldots \quad (5)$$

Also without loss of generality, we can assume

$$\frac{D_1}{\beta_1} \leqslant \frac{D_2}{\beta_2} \leqslant \frac{D_3}{\beta_3} \leqslant \cdots \leqslant \frac{D_n}{\beta_n} \qquad \ldots \quad \ldots \quad \ldots \quad (6)$$

or

$$\beta_1' \geqslant \beta_2' \geqslant \beta_3' \geqslant \cdots \geqslant \beta_n' \qquad \ldots \quad \ldots \quad \ldots \quad (7)$$

where

$$\beta_i' = \frac{\beta_i}{D_i} \qquad [i = 1, 2, 3, \ldots, n] \qquad \ldots \quad \ldots \quad \ldots \quad (8)$$

Equation (5) can be written as

$$\frac{D_1}{\beta_1}\frac{df_1}{dt} = \frac{D_2}{\beta_2}\frac{df_2}{dt} = \cdots = \frac{D_n}{\beta_n}\frac{df_n}{dt} = -p.$$

Integrating and remembering that initially

$$f_1 = 1,\ f_2 = 1, \ldots, f_n = 1,$$

we get, on using (8)

$$\frac{1-f_1}{\beta_1'} = \frac{1-f_2}{\beta_2'} = \cdots = \frac{1-f_n}{\beta_n'} \qquad \ldots \quad \ldots \quad \ldots \quad (9)$$

From (7) and (9)

$$f_1 \leqslant f_2 \leqslant f_3 \leqslant \cdots \leqslant f_n,$$

so that the first charge burns out first, then the second charge and so on. The final charge to be burnt out is the nth.

If

$$\beta_1' = \beta_2' = \cdots = \beta_n'$$

then

$$f_1 = f_2 = \cdots = f_n$$

and all the charges burn out simultaneously.

In general, however, there would be n distinct stages of burning. In the rth stage, the rth, $(r+1)$th, ..., nth charges will be burning; the first $(r-1)$ charges having been burnt out earlier.

Assuming that the shot starts moving in the first stage of burning, let

$$f_1 = f_{10},\ f_2 = f_{20}, \ldots, f_n = f_{n0}$$

at the instant when it just begins to move.

Then from (9)

$$\frac{1-f_{10}}{\beta_1'} = \frac{1-f_{20}}{\beta_2'} = \cdots = \frac{1-f_{n0}}{\beta_n'} \qquad \ldots \quad \ldots \quad (10)$$

Now from (3) and (5)

$$\frac{w_1}{D_i}\frac{dv}{df_i} = -\frac{A}{\beta_i}.$$

Integrating

$$v = \frac{A}{\beta_i' w_1}(f_{i0} - f_i) \qquad [i = 1, 2, \ldots n] \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

Let $v_1, v_2, \ldots, v_n$ denote the velocities at the ends of the various stages of burning. At the end of the rth stage, $f_r = 0$.

Therefore from (11)

$$v_r = \frac{A}{\beta_r' w_1} f_{r0} \qquad [r = 1, 2, \ldots n] \quad \ldots \quad \ldots \quad (12)$$

During the rth stage of burning, the first $(r-1)$ charges having been already burnt out

$$z_1 = z_2 = \cdots = z_{r-1} = 1$$

so that (1) gives

$$Ap(x+l) + \tfrac{1}{2}(\gamma-1)w_1 v^2 = \sum_{i=1}^{r-1} F_i C_i + \sum_{i=r}^{n} F_i C_i z_i.$$

Using (3), (4) and (11)

$$w_1 v \frac{dv}{dx}(x+l) + \tfrac{1}{2}(\gamma-1)w_1 v^2$$

$$= \sum_{i=1}^{r-1} F_i C_i + \sum_{i=r}^{n} F_i C_i (1-f_i)(1+\theta_i f_i)$$

$$= \sum_{i=1}^{r-1} F_i C_i + \sum_{i=r}^{n} F_i C_i \left(1 - f_{i0} + \frac{\beta_i' w_1 v}{A}\right)\left(1 + \theta_i f_{i0} - \theta_i \frac{\beta_i' w_1 v}{A}\right)$$

$$= \sum_{i=1}^{r-1} F_i C_i + \sum_{i=r}^{n} F_i C_i (1-f_{i0})(1+\theta_i f_{i0}) + \frac{w_1 v}{A}\sum_{i=r}^{n} \beta_i' F_i C_i (1-\theta_i+2\theta_i f_{i0})$$

$$- \frac{v^2 w_1^2}{A^2}\sum_{i=r}^{n} F_i C_i \beta_i'^2 \theta_i \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (13A)$$

or

$$(x+l)\, v \frac{dv}{dx} = K_r (a_r - v)(b_r + v) \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

where

$$K_r a_r b_r = \frac{\sum_{i=1}^{r-1} F_i C_i + \sum_{i=r}^{n} F_i C_i z_{i0}}{w_1} \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

$$K_r (a_r - b_r) = \frac{1}{A}\sum_{i=r}^{n} F_i C_i \beta_i' (1-\theta_i+2\theta_i f_{i0}) \quad \ldots \quad \ldots \quad (15)$$

and $$K_r = \frac{w_1}{A^2}\sum_{i=r}^{n} F_i C_i \beta_i'^2 \theta_i + \tfrac{1}{2}(\gamma-1) \qquad \ldots \qquad (16)$$

Since $0 \leqslant z_{i0} \leqslant 1$, from (14) $K_r a_r b_r \geqslant 0$.
Since $-1 \leqslant \theta_i \leqslant 1$, we can show from (15) that $K_r(a_r-b_r) \geqslant 0$.
For degressive shapes, for which $0 < \theta_i < 1$,

$$1-\theta_i > 0.$$

Therefore $$1-\theta_i+2\theta_i f_{i0} > 0 \text{ as } 0 \leqslant f_{i0} \leqslant 1.$$

For constant-burning surface $\theta_i = 0$.

Therefore $$1-\theta_i+2\theta_i f_{i0} = 1 > 0.$$

For progressive-burning surfaces $-1 \leqslant \theta_i < 0$;

therefore $$\text{if } \theta_i = -\psi_i, \qquad 0 < \psi_i \leqslant 1.$$

Therefore $$\begin{aligned} 1-\theta_i+2\theta_i f_{i0} &= 1+\psi_i-2\psi_i f_{i0} \\ &= (1-\psi_i f_{i0})+\psi_i(1-f_{i0}) \geqslant 0. \end{aligned}$$

$$1-\theta_i+2\theta_i f_{i0} = 0 \text{ only when } f_{i0} = 1,\ \theta_i = -1.$$

Therefore except in the exceptional case when

$$f_{10} = f_{20} = \cdots = f_{n0} = 1$$
$$\theta_1 = \theta_2 = \cdots = \theta_n = -1$$
$$K_r(a_r-b_r) > 0.$$

Again from (16), for degressive and constant-burning surfaces $[0 \leqslant \theta_i \leqslant 1]$

$$K_r > 0.$$

But for progressively burning surfaces $(-1 \leqslant \theta_i < 0)$ K_r can be negative. Finally, from (3) and (13)

$$p = \frac{w_1}{A}\frac{K_r(a_r-v)(b_r+v)}{(x+l)} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (17)$$

This determines the pressure during the rth stage of burning.

3. Uniqueness of maximum pressure

Differentiating (17) and using (3) and (13), we get

$$\begin{aligned} \frac{dp}{dt} &= \frac{dp}{dv}\frac{dv}{dt} \\ &= \frac{w_1 K_r}{A}\left\{\frac{a_r-b_r-2v}{x+l} - \frac{(a_r-v)(b_r+v)}{(x+l)^2}\,\frac{v}{v\dfrac{dv}{dx}}\right\}\frac{Ap}{w_1} \\ &= \frac{p}{x+l}\left\{K_r(a_r-b_r)-v(2K_r+1)\right\} \qquad \ldots \qquad (18) \end{aligned}$$

For investigating the maxima and minima of p, we have to examine the sign of $\frac{dp}{dt}$ and this sign, from (18), depends on the sign of the expression

$$K_r(a_r-b_r)-v(2K_r+1),$$

since $\frac{p}{x+l}$ is essentially positive.

Now we have already proved in section 2 that $K_r(a_r-b_r)$ is positive, while $2K_r+1$ can be positive or negative. From (18) we note that if $\frac{dp}{dt}$ is positive at the beginning of any stage, it can be negative at the end of that stage ($2K_r+1 > 0$ is a necessary but not a sufficient condition for this); but if $\frac{dp}{dt}$ is negative in the beginning of any stage (in which case $(2K_r+1)$ cannot be negative for this stage), it will remain negative throughout this stage as v steadily increases.

Let p_r denote the pressure and x_r denote the shot-travel at the end of the rth stage, then $\frac{dp}{dt}$ decreases at this instant if

$$\frac{p_r}{x_r+l}\{K_r(a_r-b_r)-v_r(2K_r+1)\} \geqslant \frac{p_r}{x_r+l}\{K_{r+1}(a_{r+1}-b_{r+1})-v_r(2K_{r+1}+1)\}$$

i.e., if $$K_r(a_r-b_r)-K_{r+1}(a_{r+1}-b_{r+1}) \geqslant 2v_r(K_r-K_{r+1})$$

i.e., if $$\frac{1}{A}F_rC_r\beta_r'(1-\theta_r+2\theta_rf_{r0}) \geqslant \frac{2A}{\beta_r'w_1}f_{r0}\frac{w_1}{A^2}F_rC_r\beta_r'^2\theta_r$$

i.e., if $$1-\theta_r+2\theta_rf_{r0} \geqslant 2\theta_rf_{r0}$$

i.e., if $$1-\theta_r \geqslant 0,$$

which is true, since $-1 \leqslant \theta_r \leqslant 1$.

Therefore, if $\theta_r = 1$, $\frac{dp}{dt}$ is continuous at the end of the rth stage. If $\theta_r < 1$, $\frac{dp}{dt}$ is discontinuous at this point and receives a negative increment, equal in magnitude to

$$\frac{p_rF_rC_r\beta_r'(1-\theta_r)}{x_r+l}.$$

During the first stage, from (18)

$$\frac{dp}{dt}=\frac{p}{x+l}[K_1(a_1-b_1)-v(2K_1+1)].$$

Initially when $v = 0$, $\frac{dp}{dt}$ is positive.

At the end of this stage, three cases can arise :—

Case I: $\frac{dp}{dt}$ is negative, in which case the maximum pressure occurs in the first stage.

Case II: $\frac{dp}{dt}$ is positive in which case the maximum pressure does not occur in the first stage.

Case III: $\frac{dp}{dt} = 0$, in which case the maximum pressure occurs at the end of the first stage, since at the beginning of the second stage, $\frac{dp}{dt}$ becomes negative, receiving a non-positive increment. (If $\theta_1 = 1$, $\frac{dp}{dt} = 0$ at the beginning of the second stage, but for this, $2K_2+1$ has to be positive and $\frac{dp}{dt}$ becomes negative immediately afterwards. Thus even in this case maximum pressure occurs at the end of the first stage.)

In the *first* case, once $\frac{dp}{dt}$ has become negative at the end of the first stage, it will remain negative throughout all the subsequent stages of burning. In the *second* case, where $\frac{dp}{dt}$ is positive at the end of the first stage, it may become negative or zero at the beginning of the second stage in which case the maximum pressure will occur at the end of the first stage. If, however, it remains positive, the maximum pressure does not occur in or at the end of the first stage. In this case we have to use the arguments used above for the first stage to the second and other subsequent stages. In the *third* case, $\frac{dp}{dt}$ has to become negative either at the beginning of the second stage or immediately afterwards and then it will remain negative throughout all the subsequent stages of burning.

Thus we see that once $\frac{dp}{dt}$ has become negative in any stage or at the end of any stage, it cannot become positive again, and therefore $\frac{dp}{dt}$ can change sign from positive to negative only once. Hence the existence of a unique maximum pressure is established.

Now we shall deduce the conditions for the maximum pressure to occur in the rth stage of burning or for it to occur at the end of this stage.

The condition that the maximum pressure should occur in the r*th stage of burning* is that $\frac{dp}{dt}$ should be positive at the beginning of this stage and negative at the end of it. This gives from (18)

$$K_r(a_r - b_r) > v_{r-1}(2K_r+1) \quad \text{.. .. .. (20}a\text{)}$$

and

$$K_r(a_r - b_r) < v_r(2K_r+1) \quad \text{.. .. .. .. (20}b\text{)}$$

The condition that the maximum pressure should occur at the end of the r*th stage is that—*

either (i) $\frac{dp}{dt}$ is positive at the end of the rth stage and negative at the beginning of the $(r+1)$th ;

or (ii) $\frac{dp}{dt}$ is zero at the end of the rth stage;

or (iii) $\frac{dp}{dt}$ is zero at the beginning of the $(r+1)$th stage.

For these, conditions are respectively as follows:—

Either (i) $$K_r(a_r - b_r) > v_r(2K_r+1) \quad \text{.. .. .. .. (21}a\text{)}$$

$$\text{and } K_{r+1}(a_{r+1} - b_{r+1}) < v_r(2K_{r+1}+1) \quad \text{.. .. .. (21}b\text{)}$$

or (ii) $$K_r(a_r - b_r) = v_r(2K_r + 1) \quad \ldots \quad (21c)$$

or (iii) $$K_{r+1}(a_{r+1} - b_{r+1}) = v_r(2K_{r+1} + 1) \quad \ldots \quad (21d)$$

The condition that the maximum pressure should occur at all-burnt is that $\frac{dp}{dt}$ should be $\geqslant 0$ at this instant and the condition for this is

$$K_n(a_n - b_n) \geqslant v_n(2K_n + 1) \quad \ldots \quad (22)$$

In conditions (20), (21), (22) the values of v_r, K_r and $K_r(a_r - b_r)$ can be substituted from (12), (16) and (15) respectively.

4. Particular case $n = 2$

We shall now consider the case of a composite charge consisting of two component charges, on account of its importance in practical use. This case has been treated by Venkatesan and Patni (1953) and the conditions obtained by them have been used later on in their discussion of the power law of burning by Patni (1955) in the case when θ_1, θ_2 have general values and by Aggarwal and Mehta (1955) in the case when $\theta_1 = \theta_2 = 0$. It would be of interest to compare the conditions arrived at as a result of the present investigation with those arrived at by these authors.

4.1. *The condition that the maximum pressure occurs in the first stage is* from (20),

$$K_1(a_1 - b_1) > v_0(2K_1 + 1) \quad \ldots \quad (23a)$$

and

$$K_1(a_1 - b_1) < v_1(2K_1 + 1) \quad \ldots \quad (23b)$$

(23*a*) is obviously satisfied, since

$$v_0 = 0 \text{ and } K_1(a_1 - b_1) > 0.$$

On substituting for v_1 from (12) in (23*b*), it becomes

$$f_{10} > \frac{\beta_1' w_1}{A} \frac{K_1(a_1 - b_1)}{2K_1 + 1} \quad \ldots \quad (24)$$

This is the same as the condition [30*D*] obtained by Venkatesan and Patni. But there they state another condition as well, viz.

$$f_{20} > \frac{\beta_2' w_1}{A} \frac{K_1(a_1 - b_1)}{2K_1 + 1} \quad \ldots \quad (25)$$

We shall show that (25) is superfluous in as much as it is implied by (24).

From (10)

$$\beta_1' - \beta_2' = \beta_1' f_{20} - \beta_2' f_{10} \quad \ldots \quad (26)$$

As we are considering two distinct stages of burning

$$\beta_1' > \beta_2'$$

therefore

$$\beta_1' f_{20} > \beta_2' f_{10}$$

or

$$\frac{f_{20}}{\beta_2'} > \frac{f_{10}}{\beta_1'} \quad \ldots \quad (27)$$

Therefore if (24) is satisfied, (25) is automatically satisfied.

4.2. *The condition that the maximum pressure should occur at the end of the first stage is—*

Either (i) $K_1(a_1-b_1) > v_1(2K_1+1)$

and $K_2(a_2-b_2) < v_1(2K_2+1)$

or (ii) $K_1(a_1-b_1) = v_1(2K_1+1)$

or (iii) $K_2(a_2-b_2) = v_1(2K_2+1)$.

Substituting the value of v_1 from (12), these conditions reduce to:

Either (i)
$$f_{10} < \frac{\beta_1' w_1}{A} \frac{K_1(a_1-b_1)}{2K_1+1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28a)$$

$$\text{and } f_{10} > \frac{\beta_1' w_1}{A} \frac{K_2(a_2-b_2)}{2K_2+1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28b)$$

or (ii)
$$f_{10} = \frac{\beta_1' w_1}{A} \frac{K_1(a_1-b_1)}{2K_1+1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28c)$$

or (iii)
$$f_{10} = \frac{\beta_1' w_1}{A} \frac{K_2(a_2-b_2)}{2K_2+1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28d)$$

The possibility that maximum pressure can occur at the end of the first stage has not been considered at all in the earlier papers and so no corresponding conditions exist there.

4.3. *The condition that the maximum pressure occurs in the second stage is—*

$$K_2(a_2-b_2) > v_1(2K_2+1)$$

$$\text{and } K_2(a_2-b_2) < v_2(2K_2+1).$$

Substituting from (12), these give

$$f_{10} < \frac{\beta_1' w_1}{A} \frac{K_2(a_2-b_2)}{2K_2+1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29a)$$

$$\text{and } f_{20} > \frac{\beta_2' w_1}{A} \frac{K_2(a_2-b_2)}{2K_2+1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29b)$$

(29*b*) is the same as the second condition [31*D*] of Venkatesan and Patni. But instead of (29*a*) they give (28*d*).

We note that if (28*d*) is satisfied, then in virtue of (27), (29*b*) is automatically satisfied, so their two conditions [31*D*] are just equivalent to one, viz. (28*d*). But this we have seen is the condition for the maximum pressure to occur at the end of the first stage of burning and not for the maximum pressure to occur in the second stage. For this we should use (29*a*) and (29*b*).

It appears, as explained below, that the discrepancy has arisen on account of taking $f_1 = 0$ at the instant when $\frac{dp}{dt}$ vanishes in the second stage.

From (11) by putting $i = 1$, we get

$$v = \frac{A}{\beta_1' w_1}(f_{10} - f_1).$$

When $f_1 = 0$, it gives the velocity

$$v = \frac{A}{\beta_1' w_1} f_{10} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

and from (18) $\frac{dp}{dt}$ will vanish in the second stage when

$$v = \frac{K_2(a_2 - b_2)}{2K_2 + 1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

Venkatesan and Patni obtain their first condition [31D] by substituting the value of v from (30) in (31). But (30) gives the velocity, only at the beginning of the second stage. Therefore their method could yield only the condition that the maximum pressure should occur at the beginning of the second stage, which, of course, is the same as the end of the first stage. Actually for maximum pressure to occur in the second stage v obtained from (30) should be less than v obtained from (31). This gives us our condition (29a).

4.4. *The condition that the maximum pressure occurs at all-burnt is—*

$$K_2(a_2 - b_2) \geqslant v_2(2K_2 + 1),$$

i.e.

$$f_{20} \leqslant \frac{\beta_2' w_1}{A} \frac{K_2(a_2 - b_2)}{2K_2 + 1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

The case has been discussed by Venkatesan and Patni, only for the simultaneous burning out of the propellants and not for the general case. In the particular case, their conditions [32] are

$$f_{10} = \frac{\beta_1' w_1}{A} \frac{K(a-b)}{2K+1}$$

$$f_{20} = \frac{\beta_2' w_1}{A} \frac{K(a-b)}{2K+1}$$

where

$$\frac{K_1(a_1 - b_1)}{2K_1 + 1} = \frac{K_2(a_2 - b_2)}{2K_2 + 1} = \frac{K(a-b)}{2K+1} \text{ (say)},$$

since there is only one stage of burning. We note that

(i) their two conditions are identical, since for simultaneous burning $\beta_1' = \beta_2'$ and therefore from (10) $f_{10} = f_{20}$;

(ii) their conditions imply that $\frac{dp}{dt}$ should become zero just at the position of all-burnt. But it is possible for the pressure to be still increasing at all-burnt, specially if the component charges have progressively burning surfaces. Therefore our condition is that, at all-burnt, $\frac{dp}{dt} > 0$, which explains the inequality sign in (32).

4.5. We may summarise our results as follows:—

Condition	*Maximum Pressure occurs*
(i) If $f_{10} > \frac{\beta_1' w_1}{A} \frac{K_1(a_1-b_1)}{2K_1+1}$	in the first stage when both the charges are burning.
(ii) If $f_{10} = \frac{\beta_1' w_1}{A} \frac{K_1(a_1-b_1)}{2K_1+1}$	at the end of the first stage, when the first charge is burnt out.
(iii) If $f_{10} < \frac{\beta_1' w_1}{A} \frac{K_1(a_1-b_1)}{2K_1+1}$.	
Three sub-cases arise:	
(*a*) If $f_{10} \geqslant \frac{\beta_1' w_1}{A} \frac{K_2(a_2-b_2)}{2K_2+1}$	at the end of the first stage.
(*b*) If $f_{10} < \frac{\beta_1' w_1}{A} \frac{K_2(a_2-b_2)}{2K_2+1}$, but $f_{20} > \frac{\beta_2' w_1}{A} \frac{K_2(a_2-b_2)}{2K_2+1}$	in the second stage.
(*c*) If $f_{20} \leqslant \frac{\beta_2' w_1}{A} \frac{K_2(a_2-b_2)}{2K_2+1}$	at all-burnt.

The table also illustrates the uniqueness of the maximum pressure.

5. Some further results

5.1. *Shot-start pressure*

We have expressed the conditions for the occurrence of the maximum pressure in the various stages of burning in terms of $f_{10}, f_{20}, \ldots, f_{n0}$. But in practice, we are given shot-start pressure p_0 and therefore $f_{10}, f_{20}, \ldots, f_{n0}$ have to be expressed in terms of p_0. Now from (10)

$$\frac{1-f_{10}}{\beta_1'} = \frac{1-f_{20}}{\beta_2'} = \cdots = \frac{1-f_{n0}}{\beta_n'} = \frac{1-f_0}{\beta'} = R \text{ (say)} \qquad .. \quad (33)$$

Let us assume that the shot starts in the first stage of burning. From (17) during the first stage

$$p = \frac{w_1}{A} \frac{K_1(a_1-v)\ (b_1+v)}{x+l}.$$

Initially when shot starts

$$v = 0, p = p_0, x = 0.$$

Therefore
$$p_0 = \frac{w_1}{A}\frac{K_1 a_1 b_1}{l}$$
$$= \frac{w_1}{A}\frac{\sum_{i=1}^{n} F_i C_i z_{i0}}{w_1 l}$$
$$= \frac{1}{Al}\sum_{i=1}^{n} F_i C_i (1-f_{i0})\,(1+\theta_i f_{i0})$$
$$= \frac{1}{Al}\sum_{i=1}^{n} F_i C_i R\beta_i'\,[1+\theta_i - \theta_i R\beta_i']$$

or

$$p_0 = \frac{R}{Al}\sum_{i=1}^{n} F_i C_i \beta_i'\,(1+\theta_i) - \frac{R^2}{Al}\sum_{i=1}^{n} F_i C_i \theta_i \beta_i'^2 \quad \ldots \quad \ldots \quad (34)$$

This is a quadratic to determine R. When all the propellants are tubular, it reduces to a first degree equation, so that R is uniquely determined. Even in the general case, since corresponding to each value of z, there is only one permissible value of f; only one of the values of R will be permissible. Knowing R in terms of p_0 from (34) we can find $f_{10}, f_{20}, \ldots, f_{n0}$ from (33).

If the value of f_{10} comes out to be negative, this will mean that the shot does not start in the first stage.

If the shot starts in the rth stage,

$$p_0 = \frac{w_1}{A}\frac{K_r a_r b_r}{l}$$
$$= \frac{w_1}{A}\frac{\sum_{i=1}^{r-1} F_i C_i + \sum_{i=r}^{n} F_i C_i z_{i0}}{w_1 l}$$
$$= \frac{1}{Al}\left[\sum_{i=1}^{r-1} F_i C_i + \sum_{i=r}^{n} F_i C_i (1-f_{i0})\,(1+\theta_i f_{i0})\right]$$
$$= \frac{1}{Al}\sum_{i=1}^{r-1} F_i C_i + \frac{R}{Al}\sum_{i=r}^{n} F_i C_i \beta_i'\,(1+\theta_i) - \frac{R^2}{Al}\sum_{i=r}^{n} F_i C_i \beta_i'^2 \theta_i \quad \ldots \quad (35)$$

The first value of r, for which f_{r0} does not come out to be negative, will determine the stage during which the shot starts. Of course, in that case $f_{10} = f_{20} = f_{30} = \cdots = f_{\overline{r-1}0} = 0$ and $f_{r0}, f_{\overline{r+1}0}, \ldots, f_{n0}$ are determined from (33) after determining R from (35).

5.2. *Effect of dependence of* $f_{10}, f_{20}, \ldots, f_{n0}$

As (10) or (11) show, $f_{10}, f_{20}, \ldots f_{n0}$ are not independent. Therefore all the conditions obtained about maximum pressure can be expressed in terms of only one of these, say f_{10} or of f_0.

As a particular example, Venkatesan and Patni obtained the condition for charge C_1 to be burnt out first as

$$\beta_1' f_{20} > \beta_2' f_{10}.$$

But since

$$\frac{1-f_{10}}{\beta_1'} = \frac{1-f_{20}}{\beta_2'}$$

the condition reduces to

$$\beta_1' > \beta_2'$$

which is the condition we obtained and their condition

$$\beta_1' f_{20} = \beta_2' f_{10}$$

for the simultaneous burning out of the two component charges reduces to

$$\beta_1' = \beta_2'.$$

5.3. *The case when some or all component charges burn out simultaneously*

If some or all of the quantities $\beta_1', \beta_2', \ldots \beta_n'$ are equal, the corresponding charges burn out simultaneously.

If $\beta_1' = \beta_2' = \cdots = \beta_n'$, all the charges burn out simultaneously and there is only one distinct stage of burning.

If p of these become equal, the number of distinct stages of burning is $n-p+1$.

If p of these become equal to one quantity and q of these become equal to some other quantity, the number of distinct stages of burning is $n-p-q+2$.

In the above discussion regarding maximum pressure, we have discussed the case of n distinct stages of burning.

The necessity of discussion of other cases does not arise in view of the following result proved by the present author [Kapur (1956)]:—

If
$$\frac{D_1}{\beta_1} = \frac{D_2}{\beta_2} = \cdots = \frac{D_p}{\beta_p}$$

then the composite charge consisting of these p charges behaves as a single charge with mass

$$C_1 + C_2 + \cdots + C_p$$

force constant

$$\frac{C_1F_1 + C_2F_2 + \cdots + C_pF_p}{C_1 + C_2 + \cdots + C_p}.$$

and form factor

$$\frac{F_1C_1\theta_1 + F_2C_2\theta_2 + \cdots + F_pC_p\theta_p}{F_1C_1 + F_2C_2 + \cdots + F_pC_p}.$$

Consequently in all our previous discussions we replace the charges with equal values of $\frac{D}{\beta}$ by a single charge and all our previous results will hold.

5.4. *All-burnt position*

From (12), the velocity at all-burnt v_B is given by

$$v_B = v_n = \frac{A}{\beta_n' w_1} f_{n0} \quad \ldots \ldots \ldots \ldots \quad (36)$$

Integrating (13), we get

$$\frac{x+l}{x_{r-1}+l} = \left[\left(\frac{a_r-v_{r-1}}{a_r-v}\right)^{a_r}\left(\frac{b_r+v_{r-1}}{b_r+v}\right)^{b_r}\right]^{\frac{1}{K_r(a_r+b_r)}} \quad \ldots \quad \ldots \quad (37)$$

so that the shot travel x_r up to the end of the rth stage is given by

$$\frac{x_r+l}{x_{r-1}+l} = \left[\left(\frac{a_r-v_{r-1}}{a_r-v_r}\right)^{a_r}\left(\frac{b_r+v_{r-1}}{b_r+v_r}\right)^{b_r}\right]^{\frac{1}{K_r(a_r+b_r)}} \quad \ldots \quad \ldots \quad (38)$$

and the shot travel x_B up to all-burnt is given by

$$\frac{x_B+l}{l} = \prod_{r=1}^{n}\left[\left(\frac{a_r-v_{r-1}}{a_r-v_r}\right)^{a_r}\left(\frac{b_r+v_{r-1}}{b_r+v_r}\right)^{b_r}\right]^{\frac{1}{K_r(a_r+b_r)}} \quad \ldots \quad \ldots \quad (39)$$

and from (17), the pressure at all-burnt is given by

$$p_B = \frac{w_1}{A}K_n\frac{(a_n-v_n)(b_n+v_n)}{x_B+l} \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

5.5. *Muzzle Velocity*

After all-burnt, $z_1 = z_2 = \cdots\cdots = z_n = 1$.

Therefore (1) becomes

$$\sum_{i=1}^{m} F_iC_i = Ap(x+l)+\tfrac{1}{2}(\gamma-1)w_1v^2$$

$$= w_1v\frac{dv}{dx}(x+l)+\tfrac{1}{2}(\gamma-1)w_1v^2$$

or

$$\frac{vdv}{\sum_{i=1}^{n}\frac{F_iC_i}{w_i}-\tfrac{1}{2}(\gamma-1)v^2} = \frac{dx}{x+l}$$

or

$$\frac{vdv}{A^2-B^2v^2} = \frac{dx}{x+l}$$

where

$$A^2 = \frac{\sum_{i=1}^{n} F_iC_i}{w_1}; \quad B^2 = \tfrac{1}{2}(\gamma-1).$$

Integrating and remembering that $v = v_B$, when $x = x_B$,

$$\frac{A^2-B^2v^2}{A^2-B^2v_B^2} = \left(\frac{x+l}{x_B+l}\right)^{-(\gamma-1)}$$

If X denotes the shot travel up to muzzle and V is the muzzle velocity

$$\frac{A^2-B^2V^2}{A^2-B^2v_B^2}=\left(\frac{X+l}{x_B+l}\right)^{-(\gamma-1)} \qquad \cdots \quad (41)$$

(41) determines the muzzle velocity.

5.6. *Maximum Pressure*

From the results of section 3, we can determine the stage in which the maximum pressure occurs. Let this be the rth stage, then from (18), the velocity v_M at the instant when the pressure is maximum is given by

$$v_M=\frac{K_r(a_r-b_r)}{2K_r+1} \qquad \cdots \quad (42)$$

(37) then determines the shot travel x_M till the maximum pressure is attained:

$$\frac{x_M+l}{x_{r-1}+l}=\left[\left(\frac{a_r-v_{r-1}}{a_r-v_M}\right)^{a_r}\left(\frac{b_r+v_{r-1}}{b_r+v_M}\right)^{b_r}\right]^{\frac{1}{K_r(a_r+b_r)}} \qquad \cdots \quad (43)$$

Substituting from (42) and (43) in (17) we determine the maximum pressure

$$p_{\max}=\frac{w_1}{A}\frac{K_r(a_r-v_M)(b_r+v_M)}{x_M+l} \qquad \cdots \quad (44)$$

6. Effect of co-volume correction terms on the uniqueness of maximum pressure

In this section, we investigate the effect of taking co-volume correction terms in the energy equation on the uniqueness of the maximum pressure. Taking these, the energy equation (1) becomes

$$\sum_{i=1}^{n}F_iC_iz_i=p\left\{A(x+l)-C_1z_1\left(b_1-\frac{1}{\delta_1}\right)-C_2z_2\left(b_2-\frac{1}{\delta_2}\right)-\cdots-C_nz_n\left(b_n-\frac{1}{\delta_n}\right)\right\}+\tfrac{1}{2}(\gamma-1)w_1v^2 \quad (45)$$

During the rth stage of burning, since $z_1=z_2=\cdots=z_{r-1}=1$, (45) becomes

$$\sum_{i=1}^{r-1}F_iC_i+\sum_{i=r}^{n}F_iC_iz_i=p\left\{A(x+l)-\sum_{i=1}^{r-1}C_i\left(b_i-\frac{1}{\delta_i}\right)-\sum_{i=r}^{n}C_iz_i\left(b_i-\frac{1}{\delta_i}\right)\right\}+\tfrac{1}{2}(\gamma-1)w_1v^2 \quad (46)$$

Substituting $z_i=(1-f_i)(1+\theta_if_i)$, using (3) and substituting for F_i from (11) in terms of v, (46) gives

$$(x+l)v\frac{dv}{dx}-\frac{w_1v}{A}\frac{dv}{dx}\left\{K_r'(a_r'-v)(b_r'+v)\right\}=K_r(a_r-v)(b_r+v) \qquad \cdots \quad (47)$$

where K_r', a_r', b_r' are obtained from K_r, a_r, b_r by putting $\gamma=1$ and replacing F_i by $b_i-\dfrac{1}{\delta_i}$.

Using (3) again, we get

$$p=\frac{K_r w_1}{A}\frac{(a_r-v)(b_r+v)}{(x+l)-\frac{w_1K_r'}{A}(a_r'-v)(b_r'+v)} \qquad \ldots \quad \ldots \quad \ldots \quad (48)$$

Therefore $$\frac{dp}{dt}=\frac{dp}{dv}\frac{dv}{dt}$$

$$=\frac{Ap}{w_1}\frac{K_r w_1}{A}\frac{1}{\left[(x+l)-\frac{w_1K_r'}{A}(a_r'-v)(b_r'+v)\right]}\times$$

$$\times\left[a_r-b_r-2v-\frac{v}{K_r}+\frac{K_r'w_1}{A}\frac{(a_r-v)(b_r+v)(a_r'-b_r'-2v)}{(x+l)-\frac{w_1K_r'}{A}(a_r'-v)(b_r'+v)}\right]$$

$$=\frac{p}{\left[(x+l)-\frac{w_1K_r'}{A}(a_r'-v)(b_r'+v)\right]}\begin{bmatrix}K_r(a_r-b_r)-v(2K_r+1)\\ +pK_r'(a_r'-b_r'-2v)\end{bmatrix} \qquad \ldots \quad (49)$$

Now the expression

$$A(x+l)-w_1K_r'(a_r'-v)(b_r'+v)$$

is always positive, since it corresponds to the free space available behind the shot. Therefore the sign of $\frac{dp}{dt}$ depends on the sign of the expression

$$K_r(a_r-b_r)-v(2K_r+1)+pK_r'(a_r'-b_r'-2v) \quad \ldots \quad \ldots \quad \ldots \quad (\mathrm{I})$$

or $$K_r(a_r-b_r-2v)+pK_r'(a_r'-b_r'-2v)-v \quad \ldots \quad \ldots \quad \ldots \quad (\mathrm{II})$$

Now, from (15) and (16) we have

$$K_r(a_r-b_r)=\frac{1}{A}\sum_{i=r}^{n} F_iC_i\beta_r'(1-\theta_i+2\theta_i f_{i0})$$

$$K_r=\frac{w_1}{A^2}\sum_{i=r}^{n} F_iC_i\beta_i'^2\theta_i+\tfrac{1}{2}(\gamma-1)$$

$$K_r'(a_r'-b_r')=\frac{1}{A}\sum_{i=r}^{n}\left(b_i-\frac{1}{\delta_i}\right)C_i\beta_r'(1-\theta_i+2\theta_i f_{i0})$$

$$K_r'=\frac{w_1}{A^2}\sum_{i=r}^{n}\left(b_i-\frac{1}{\delta_i}\right)C_i\beta_i'^2\theta_i.$$

Now for all propellants, whose form-function can be represented by

$$z=(1-f)(1+\theta f),$$

the possible range of θ is restricted. θ cannot be less than -1, for then z would be negative for f near unity. Nor can θ be greater than unity, for then

z would be greater than unity for some accessible range of f. Actually for most of the gun propellants θ lies between $\theta = 0$ (tube) and $\theta = 1$ (cord). The most progressive form used in guns is the multitube for which θ can be taken as slightly negative of the order of $-0{\cdot}1$ (Corner, p. 34). The still more progressive forms with θ lying between $-0{\cdot}1$ and -1 can be obtained by inhibiting of burning by coating certain surfaces of the propellant. But this technique, while common in rockets, is rarely used for gun propellants. Thus for all gun propellants we can take $-0{\cdot}1 \leqslant \theta \leqslant 1$.

We have proved in section 2 that $K_r(a_r - b_r)$ is always positive (except possibly when $\theta_i = -1$ for all i when it can be zero. But in this possibility we are not interested). By using the same argument, we can show that $K'_r(a'_r - b'_r)$ is also always positive, since $b_i - \frac{1}{\delta_i}$ is positive for all gun propellants. If $0 \leqslant \theta_i \leqslant 1$, for all propellants, both K_r and K'_r will also be positive $(K'_r = 0$, if all the components are tubular). If some or all of the component charges have progressively burning surfaces $(\theta_i < 0)$, K_r, K'_r can be positive or negative.
But

$$K_r = \frac{w_1}{A^2} \sum_{i=r}^{n} F_i C_i \beta_i'^2 \theta_i + \tfrac{1}{4}, \text{ taking } \gamma = 1{\cdot}25.$$

Now

$$\frac{w_1 F_i C_i \beta_i'^2}{A^2} = \frac{w_1 F_i C_i \beta_i^2}{A^2 D_i^2}$$

is a dimensionless constant and is the reciprocal of the Central Ballistic Parameter if only the ith component were burning. For most composite charges and guns M is likely to be greater than 0·5 and therefore even if some component charges are moderately progressively burning, K_r will be positive.

Now we discuss the possibility of change of sign in $\frac{dp}{dt}$ as v increases in any stage on the assumption that $K_r > 0$.

As v increases $K_r(a_r - b_r - 2v)$ decreases, but $pK'_r(a'_r - b'_r - 2v)$ may increase or decrease due to increase of p. But the second expression is very small compared to the first as $p\left(b_i - \frac{1}{\delta_i}\right)$ will be very small compared to F_i even when p takes its maximum value. Therefore the two expressions taken together decrease and when we consider the last negative term $-v$ in expression (II), we find that as v increases $\frac{dp}{dt}$ can change sign from positive to negative but not from negative to positive in the same stage.

At the end of the rth stage, $\frac{dp}{dt}$ will decrease if

$$\left[K_r(a_r - b_r) - v(2K_r + 1) + p_r K'_r(a'_r - b'_r - 2v_r)\right]$$
$$\geqslant \left[K_{r+1}(a_{r+1} - b_{r+1}) - v(2K_{r+1} + 1) + p\, K'_{r+1}(a'_{r+1} - b'_{r+1} - 2v_r)\right]$$

i.e., if

$$\left[K_r(a_r - b_r) - K_{r+1}(a_{r+1} - b_{r+1})\right] - 2v_r\left[K_r - K_{r+1}\right]$$
$$+ p_r\left[K'_r(a'_r - b'_r) - K'_{r+1}(a'_{r+1} - b'_{r+1})\right] - 2p_r v_r(K'_r - K'_{r+1}) \geqslant 0,$$

i.e., if

$$\frac{1}{A}F_rC_r\beta_r'(1-\theta_r+2\theta_rf_{r0})-\frac{2A}{\beta_r'w_1}f_{r0}\frac{w_1}{A^2}F_rC_r\beta_r'^2\theta_r$$

$$+p_r\left\{\frac{1}{A}\left(b_r-\frac{1}{\delta r}\right)C_r\beta_r'(1-\theta_r+2\theta_rf_{r0})-\frac{2A}{\beta_r'w_1}f_{r0}\frac{w_1}{A^2}\left(b_r-\frac{1}{\delta_r}\right)(C_r\beta_r'^2\theta_r)\right\}\geqslant 0$$

i.e., if

$$F_r(1-\theta_r)+p_r\left(b_r-\frac{1}{\delta_r}\right)(1-\theta_r)\geqslant 0 \quad \ldots \quad \ldots \quad \ldots \quad (50)$$

If $\theta_r = 1$, there is no sudden change in $\frac{dp}{dt}$ and $\frac{dp}{dt}$ remains continuous at the end of the rth stage even when we take co-volume terms into account.

If $\theta_r < 1$, the condition becomes

$$F_r+p_r\left(b_r-\frac{1}{\delta_r}\right)\geqslant 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (51)$$

For propellants in everyday use $b-\frac{1}{\delta}$ is positive and thus the above condition is always satisfied. Thus once $\frac{dp}{dt}$ becomes negative in or at the end of any stage, it will remain negative throughout afterwards. Consequently $\frac{dp}{dt}$ can change sign only once and the uniqueness of maximum pressure is established even when we take the co-volume terms into account.

The condition that the maximum pressure occurs in the rth stage now is

$$K_r(a_r-b_r-2v_{r-1})+K_r'(a_r'-b_r'-2v_{r-1})p_{r-1} > v_{r-1} \quad \ldots \quad \ldots \quad (52a)$$

and

$$K_r(a_r-b_r-2v_r)+K_r'(a_r'-b_r'-2v_r)p_r < v_r \quad \ldots \quad \ldots \quad (52b)$$

and the condition that it should occur at the end of the rth stage is that

either

$$K_r(a_r-b_r-2v_r)+K_r'(a_r'-b_r'-2v_r)p_r > v_r \quad \ldots \quad \ldots \quad (53a)$$

and

$$K_{r+1}(a_{r+1}-b_{r+1}-2v_r)+K_{r+1}'(a_{r+1}'-b_{r+1}'-2v_r)p_r<v_r \quad \ldots \quad \ldots \quad (53b)$$

or

$$K_r(a_r-b_r-2v_r)+K_r'(a_r'-b_r'-2v_r)p_r = v_r \quad \ldots \quad \ldots \quad \ldots \quad (53c)$$

or

$$K_{r+1}(a_{r+1}-b_{r+1}-2v_r)+K_{r+1}'(a_{r+1}'-b_{r+1}'-2v)p_r = v_r \quad \ldots \quad \ldots \quad (53d)$$

and the condition that the maximum pressure occurs at all-burnt is

$$K_n(a_n-b_n-2v_n)+K_n'(a_n'-b_n'-2v_n)\frac{p_n}{w_1}\geqslant v_n \quad \ldots \quad \ldots \quad (54)$$

If we put K_r', K_{r+1}', K_n' zero (52), (53), (54) reduce to (20), (21) and (22) respectively. But in practice, the application of these conditions is much more difficult, since these involve the calculations of $p_1, p_2, \ldots, p_n$ and from (17) this requires the calculation of $x_1, x_2, \cdots x_n$. To find these quantities, we have to integrate (47), which, on putting $x+l = \xi$, reduces to the linear differential equation of the first order

$$\frac{d\xi}{dv}-\frac{\xi v}{K_r(a_r-v)(b_r+v)} = -\frac{1}{A}\frac{vK_r'(a_r'-v)(b_r'+v)}{K_r(a_r-v)(b_r+v)} \quad \ldots \quad \ldots \quad (55)$$

In fact, if we calculate $p_1, p_2, \dots p_n$, we shall find that these go on increasing up to a certain stage and then begin to decrease and the examination of the two stages neighbouring the maximum of these values will determine the stage in which maximum pressure occurs.

Alternatively, since the effect of the co-volume terms is bound to be small, we can determine approximately where the maximum pressure occurs by using (20), (21) and (22) and then use (52), (53), (54) to determine exactly the stage in which maximum pressure occurs.

We have established above the uniqueness of the maximum pressure in the practically important case $K_r > 0$. If, however, $K_r < 0$, $K_r' < 0$ the following cases can arise:—

(i) $2K_r+1 < 0$, in which case, the whole expression (I) is positive and the question of change of sign does not arise. The uniqueness of the maximum pressure holds.

(ii) $2K_r+1 > 0$. In this case, once $\frac{dp}{dt}$ has become negative p is decreasing. Since $K_r'(a_r'-b_r') > 0$, $pK_r'(a_r'-b_r')$ decreases. Also as v is increasing $-v(2K_r+1)$ decreases. The term $-2pK_r'v$ will decrease only if pv decreases. But this is not necessarily true. When p is increasing pv definitely increases. Even when p begins to decrease pv may still increase, at least for some time immediately after the instant of maximum pressure. After some time, however, the decrease in pressure may be sufficient to balance the effect of increase of v. In this case, therefore, the uniqueness of maximum pressure is not definitely established.

(iii) If, however, $2K_r+1+2pK_r'$ is positive even for the maximum value of p that is likely to arise, the expression (I) goes on decreasing and once $\frac{dp}{dt}$ has become negative, it would not become positive again and the maximum pressure would be unique.

7. Assumptions made and a possible physical explanation

We have proved the uniqueness of the maximum pressure under the following assumptions:—

(i) $\gamma_1 = \gamma_2 = \dots = \gamma_n$, i.e. the ratio of the specific heats of the product gases for all the component charges are the same. This is not a serious assumption, as for most of the propellants used in practice, γ_i has practically the same value.

(ii)
$$K_r = \frac{w_1}{A^2}\sum_{i=r}^{n} F_i C_i \beta_i'^2 \theta_i + \tfrac{1}{2}(\gamma-1) > 0 \text{ for all } r.$$

If $\theta_i \geqslant 0$ for all i, this is obviously satisfied. Even if some or all θ_i are negative, but small in magnitude, we have shown above that this condition would be satisfied for ordinary guns and for composite charges likely to be used in guns. If however K_r can be negative, we have shown that the maximum pressure would be unique if we neglect co-volume correction terms. If we take these terms into account, the study of uniqueness will require further investigation. This case which would be not of much interest in gun ballistics may be useful in the study of rocket ballistics, where highly progressive forms obtained by inhibiting certain surfaces are used.

(iii) *Linear Law of Burning.*—

This again is not far from truth for present-day propellants. The discussion of non-linear laws of burning leads to non-linear differential equations which can, in general, be integrated only numerically. It will not, therefore, be easy to establish strictly a similar uniqueness law specially in the case when the rates of burning indices are different. Yet, since the non-linear law can be very closely approximated by the linear law for modern gun propellants, we surmise that the theorem will be true for non-linear laws also.

However, under the comparatively lower pressures in rockets, the pressure-index varies from 0·3 to 0·7 and the uniqueness theorem may fail in this case.

(iv) We have assumed that the form-function for each component charge is of the standard form

$$z = (1-f)\,(1+\theta f).$$

This does not cover such highly degressive forms as spheres and cubes. Almost all form-functions occurring in practice can be represented by taking the cubic form-function

$$z = (1-f)\,(1+\theta f+\psi f^2).$$

For this case (13*A*) becomes

$$Ap(x+l)+\tfrac{1}{2}(\gamma-1)w_1v^2$$
$$=\sum_{i=1}^{r-1} F_iC_i+\sum_{i=r}^{n} F_iC_i(1-f_i)(1+\theta_if_i+\psi_if_i^2)$$

and instead of (18) we get

$$\frac{w_1}{p}\frac{dp}{dt}(x+l) = w_1[K_r(a_r-b_r)-v(2K_r+1)]+\frac{w_1}{A}\sum_{i=r}^{n}F_iC_i\beta_r'\psi_if_{i0}(3f_{i0}-2)$$
$$-\frac{vw_1^2}{A^2}\sum_{i=r}^{n}F_iC_i\psi_i\beta_i'^2(6f_{i0}-2)+\frac{v^2w_1^3}{A^3}\sum_{i=r}^{n}F_iC_i\psi_i\beta_i'^3 \quad \ldots \quad (56)$$

The R.H.S. is a quadratic in v. In any stage, this need not necessarily decrease. It is, therefore, possible that $\frac{dp}{dt}$ may be negative in the beginning of this interval and positive at its end. One of our main arguments for establishing a unique maximum pressure will then break down.

Thus for highly degressive forms which can be represented only by cubic form-function, there is a possibility of the existence of secondary pressure maxima. But this is not a serious restriction on our uniqueness theorem, since such highly degressive forms are rarely used in practice.

A possible physical explanation of the uniqueness of maximum pressure established under very general conditions above appears to be that while the more rapidly burning propellants burn out first and help in building up the pressure to its maximum, the comparatively slower burning propellants, that remain, can arrest the sudden drop in pressure which would have otherwise been caused due to the rapid increase of space behind the shot, but cannot produce an increase in pressure, at least under the assumptions we have made in the present investigation.

It may be noted that secondary pressure maxima have been observed in the case of rockets even for single charges and therefore the question of uniqueness of maximum pressure for composite charges does not arise. The secondary pressure maxima may arise due to (i) non-linear law of burning, (ii) progressively burning surfaces, (iii) irregular burning or (iv) some other causes, and the question is well worth investigating mathematically.

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9. Abstract

Extending Hunt-Hinds' system of Internal Ballistics for a single charge to the case of a composite charge consisting of n components, the existence of a unique maximum pressure is established, assuming a linear law of burning, the standard form-functions for the component charges and neglecting co-volume correction terms. The effect of co-volume correction terms has also been examined and it has been shown that for degressive, for constant-burning and for very moderately progressive burning surfaces, the maximum pressure would still be unique. The conditions for the occurrence of maximum pressure in any stage or at the end of any stage have also been explicitly obtained. The particular case $n = 2$ has been studied in detail and the conditions obtained here have been compared with those obtained by Venkatesan and Patni.

10. References

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