ON NEGATIVE ORDER SUMMABILITY OF A FOURIER SERIES

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1. We suppose that $f(t)$ is a periodic function, with period 2π, and integrable (L) over $(-\pi, \pi)$. Let the Fourier series of $f(t)$ be given by :—

$$\frac{1}{2}a_0+\sum_1^{\infty}(a_n\cos nt+b_n\sin nt) \tag{1.1}$$

$$=\frac{1}{2}A_0+\sum_{n=1}^{\infty}A_n(t).$$

We write

$$\phi(t)=\tfrac{1}{2}\{f(x+t)+f(x-t)-2S\}.$$

Young's criterion (Zygmund, 1952, p. 33) for the convergence, that is summability $(C, 0)$, of a Fourier series has been extended and generalised in various directions by Pollard (1927), Hardy and Littlewood (1928), Bosanquet (1935), and others. Pollard's generalisation of Young's criterion reads as follows :—

Theorem A (Pollard, 1927). If

$$\int_0^t \phi(u)\,du=o(t), \tag{1.2}$$

as $t\to 0$, and

$$\int_0^t |du\,\phi(u)|=O(t), \tag{1.3}$$

$0\leqslant t\leqslant \eta$, the Fourier series of $f(t)$ converges to S at the point $t=x$.

Hardy and Littlewood (1934) proposed the problem whether it is possible to replace (1.2) and (1.3) by

$$\int_0^t \phi(u)\,du=o\left(\frac{t}{\log\frac{1}{t}}\right),$$

as $t\to 0$, and

$$\int_0^t |d\{u^{\Delta}\phi(u)\}|=O(t),$$

$0\leqslant t\leqslant \eta$, for some $\Delta>1$. Later Randels (1935) proved that this is impossible. But Sunouchi adopted a different line of procedure and proved the following theorem.

Theorem B (Sunouchi, 1951). If

$$\int_0^t \phi(u)du = o(t^{\Delta}),\ \Delta > 1, \tag{1.4}$$

as $t \to 0$, and

$$\int_0^t |d\{u^{\Delta}\phi(u)\}| = O(t), \tag{1.5}$$

$0 \leqslant t \leqslant \eta$, then the Fourier series of $f(t)$, at $t = x$ converges to S.

When we proceed to investigate Cesàro summability of a negative order of Fourier series, we find that it no longer remains a local property (Kogbetliantz, 1919). In order to secure summability of negative order we require to strengthen the condition (1.3) by demanding that it should hold not merely for small t, but for all t of $(0, \pi)$. With this idea Young (1918) proved that when $\phi(t) \to 0$, as $t \to 0$ and the condition (1.3) holds for all values of t, then the Fourier series of $f(t)$ is summable $(C, -1+\delta)$, for $\delta > 0$ to S, at the point $t = x$.

Later Hardy and Littlewood generalised this theorem of Young in the same direction in which Pollard had generalised Young's test for summability $(C, 0)$ of Fourier series. They established the following theorem:—

Theorem C (Hardy and Littlewood, 1928). If

$$\int_0^t \phi(u)du = o(t), \tag{1.6}$$

as $t \to 0$, and

$$\int_0^t |d\{u\phi(u)\}| = O(t), \tag{1.7}$$

$0 \leqslant t \leqslant \pi$, the Fourier series of $f(t)$ is summable $(C, -1+\delta)$, for $\delta > 0$ to S, at the point $t = x$.

The object of this paper is to modify and extend the above theorem of Hardy and Littlewood for negative order of summability of Fourier series in directions in which Sunouchi had developed Pollard's generalisation of Young's theorem for summability $(C, 0)$. We establish the following theorem:—

Theorem : *If, there is a* $\Delta > 1$, *such that*

$$\Phi(t) = \int_0^t \phi(u)du = o\left(t^{\Delta+1-\frac{1}{\Delta}}\right), \tag{1.8}$$

as $t \to 0$, *and*

$$\int_0^t \left| d\left\{u^{\Delta+\frac{1}{\Delta}-1}\phi(u)\right\}\right| = O(t), \tag{1.9}$$

$0 \leqslant t \leqslant \pi$, *the Fourier series of* $f(t)$ *is summable* $\left(C, -\frac{1}{\Delta}\right)$ *to* S, *at the point* $t = x$.

2. For proving the theorem we shall require the following Lemma.

LEMMA. *If*

$$\psi(u) = \left(2\sin\frac{u}{2}\right)^{\Delta+\frac{1}{\Delta}-1}\phi(u), \tag{2.1}$$

then

$$\Psi(t) = \int_0^t |d\psi(u)| = O(t), \tag{2.2}$$

and

$$\psi(t) = O(t) \tag{2.3}$$

provided (1.9) *holds.*

Here (2.2) follows from (1.9) and from (2.2)

$$|\psi(t)-\psi(0)| = O(t).$$

Since $\psi(0) = 0$, we have $\psi(t) = O(t)$.

3. Proof of the theorem.

Let $s^{-\frac{1}{\Delta}}(n, t)$ be the nth Cesàro mean of order $-\dfrac{1}{\Delta}$ of

$$\frac{1}{\pi}+\frac{2}{\pi}\sum_{\nu=1}^{n}\cos \nu t.$$

Gergen (1930, p. 264) has proved that, for $0 < t < \pi$,

$$\left|s^{-\frac{1}{\Delta}}(n, t)\right| \leqslant An, \tag{3.1}$$

$$\left|\frac{d}{dt}s^{-\frac{1}{\Delta}}(n, t)\right| \leqslant An^2, \tag{3.2}$$

$$|\sigma(n, t)| \leqslant An^{-1}t^{-2}, \tag{3.3}$$

$$\left|\frac{d}{dt}\sigma(n, t)\right| \leqslant An^{-1}t^{-3}, \tag{3.4}$$

where

$$\sigma(n, t) = s^{-\frac{1}{\Delta}}(n, t)-S(n, t),$$

$$S(n, t) = \frac{2}{\pi}\frac{1}{A_n^{-\frac{1}{\Delta}}}\frac{\sin (n\,;\,t)}{(2\sin\frac{1}{2}t)^{1-\frac{1}{\Delta}}},$$

$$(n\,;\,t) = (n+\tfrac{1}{2}-\tfrac{1}{2}\alpha)t+\tfrac{1}{2}\alpha\pi,\ \alpha = \frac{1}{\Delta},$$

$$A_n^{\delta} = \frac{\Gamma(n+1+\delta)}{\Gamma(n+1)\,\Gamma(1+\delta)} \sim \frac{n^{\delta}}{\Gamma(\delta+1)},$$

and A denotes a number independent of n and t.

It is well known that a necessary and sufficient condition for the summability $\left(C, -\frac{1}{\Delta}\right)$ to S of (1.1), without the analysis of the Kernel, is that

$$I = \int_0^\pi \phi(t) s^{-\frac{1}{\Delta}}(n, t)dt = o(1), \tag{3.5}$$

as $n \to \infty$.

Let K be a constant, which is sufficiently large and $\mu = (K/n)^{\frac{1}{\Delta}}$.

Now, write

$$I = \int_0^{K/n} \phi(t) s^{-\frac{1}{\Delta}}(n, t)dt + \int_{K/n}^{\pi} \phi(t)\sigma(n, t)dt \tag{3.6}$$

$$+ \int_{K/n}^{\mu} \phi(t)S(n, t)dt + \int_{\mu}^{\pi} \phi(t)S(n, t)dt$$

$$= I_1 + I_2 + I_3 + I_4, \text{ say.}$$

By integration by parts

$$I_1 = \left[\Phi(t) s^{-\frac{1}{\Delta}}(n, t)\right]_0^{K/n} - \int_0^{K/n} \Phi(t) \frac{d}{dt}\left\{s^{-\frac{1}{\Delta}}(n, t)\right\} dt$$

$$= I_{1.1} - I_{1.2}, \text{ say.}$$

Then, by (1.8) and (3.1),

$$I_{1.1} = o(1);$$

and, by (1.8) and (3.2),

$$I_{1.2} \leqslant A \int_0^{K/n} o\left(t^{\Delta+1-\frac{1}{\Delta}}\right) O(n^2)\, dt$$

$$= o(1).$$

Again, by integration by parts,

$$I_2 = \left[\Phi(t)\sigma(n, t)\right]_{K/n}^{\pi} - \int_{K/n}^{\pi} \Phi(t) \frac{d}{dt}\{\sigma(n, t)\}\, dt$$

$$= I_{2.1} - I_{2.2}, \text{ say.}$$

Then, by (1.8) and (3.3),

$$I_{2.1} = o(1);$$

and, by (1.8) and (3.4),

$$I_{2.2} = \int_{K/n}^{\pi} o\left(t^{\Delta+1-\frac{1}{\Delta}}\right) O(n^{-1}t^{-3})dt$$

$$= o(1).$$

Writing

$$I_3 = \frac{2}{\pi} \frac{1}{A_n^{-\frac{1}{\Delta}}} \int_{K/n}^{\mu} \phi(t) \frac{\sin(n;t)}{(2\sin t/2)^{1-\frac{1}{\Delta}}} dt$$

$$= \frac{2}{\pi} \frac{1}{A_n^{-\frac{1}{\Delta}}} \left[\Phi(t) \frac{\sin(n;t)}{(2\sin t/2)^{1-\frac{1}{\Delta}}} \right]_{K/n}^{\mu}$$

$$- \frac{2}{\pi} \frac{1}{A_n^{-\frac{1}{\Delta}}} (n+\tfrac{1}{2}-\tfrac{1}{2}\alpha) \int_{K/n}^{\mu} \Phi(t) \frac{\cos(n;t)}{(2\sin t/2)^{1-\frac{1}{\Delta}}} dt$$

$$+ \frac{(1-\frac{1}{\Delta})}{\pi A_n^{-\frac{1}{\Delta}}} \int_{K/n}^{\mu} \Phi(t) \frac{\sin(n;t)}{(2\sin t/2)^{2-\frac{1}{\Delta}}} dt$$

$$= I_{3.1} - I_{3.2} + I_{3.3}, \text{ say.}$$

Then, by (1.8), as $n \to \infty$, we have

$$I_{3.1} = n^{\frac{1}{\Delta}} \left[o(t^{\Delta}) \right]_{K/n}^{\mu}$$

$$= o(1);$$

$$I_{3.2} = O(1) n^{1+\frac{1}{\Delta}} \int_{K/n}^{\mu} o(t^{\Delta}) dt$$

$$= o\left(K^{1+\frac{1}{\Delta}}\right) + o\left(K^{\Delta+1} / n^{\Delta-\frac{1}{\Delta}}\right)$$

$$= o(1);$$

and

$$I_{3.3} = A n^{\frac{1}{\Delta}} \int_{K/n}^{\mu} o(t^{\Delta-1}) dt$$

$$= o(1).$$

Thus we have

(3.7) $$|I_1| + |I_2| + |I_3| = o(1),$$

as $n \to \infty$.

Hence it follows from (3.6) and (3.7) that whenever (1.8) is satisfied, (3.5) can be replaced by

(3.8) $$I_4 = o(1), \quad \text{as } n \to \infty.$$

To prove (3.8), we write

$$\wedge = \wedge(n, t) = \int_t^{\pi} \frac{\sin(n;t)}{(2\sin t/2)^{\Delta}} dt,$$

hence, by the second mean value theorem, we have

$$(3.9) \qquad \wedge = O(n^{-1}t^{-\Delta}).$$

Now by the Lemma we have

$$|I_4| \leqslant An^{\frac{1}{\Delta}} \left| \int_\mu^\pi \psi(t) d\wedge \right|$$

$$= An^{\frac{1}{\Delta}} \left| \Big[\psi(t) \wedge\Big]_\mu^\pi - \int_\mu^\pi \wedge\, d\{\psi(t)\} \right|$$

$$\leqslant An^{\frac{1}{\Delta}} \left| \Big[\psi(t) \wedge\Big]_\mu^\pi \right| + An^{\frac{1}{\Delta}} \left| \int_\mu^\pi \wedge\, d\{\psi(t)\} \right|$$

$$= I_{4.1} + I_{4.2}, \text{ say.}$$

Then, by (2.3) and (3.9),

$$I_{4.1} = O\left[n^{\frac{1}{\Delta}-1} t^{1-\Delta}\right]_\mu^\pi$$

$$= O\left(n^{\frac{1}{\Delta}-1}\right) + O\left(K^{\frac{1}{\Delta}-1}\right)$$

$$= o(1),$$

as $n \to \infty$, and by (2.2) and (3.9)

$$I_{4.2} \leqslant An^{\frac{1}{\Delta}} \int_\mu^\pi |\wedge||d\psi(t)|$$

$$= \frac{A}{n^{1-\frac{1}{\Delta}}} \Big[\Psi(t)/t^\Delta\Big]_\mu^\pi - \frac{A}{n^{1-\frac{1}{\Delta}}} \int_\mu^\pi \Psi(t) t^{-\Delta-1}\, dt$$

$$= O\left(n^{\frac{1}{\Delta}-1}\right) + O\left(K^{\frac{1}{\Delta}-1}\right) + O\left(n^{1-\frac{1}{\Delta}}\right) \int_\mu^\pi t^{-\Delta}\, dt$$

$$= O\left(n^{\frac{1}{\Delta}-1}\right) + O\left(K^{\frac{1}{\Delta}-1}\right)$$

$$= o(1),$$

as $n \to \infty$, and K is sufficiently large constant.

Thus, finally, we get the result of (3.8). This completes the proof of the theorem.

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