ORTHOGONAL ENNUPLES IN A KAEHLER MANIFOLD

by Miss Gita Halder *and* Ram Behari, F.N.I., *University of Delhi*

(*Received October 3, 1955 ; after revision January 24 ; read March 2, 1956*)

1. Introduction

Consider a real $2n$-dimensional manifold V_{2n} of class C^r with a given covering by neighbourhoods each endowed with a co-ordinate system

$$z^a = x^a + ix^A$$

$$\bar{z}^a = x^a - ix^A$$

where $$a = 1, 2, \ldots, n ; A = n+1, \ldots, 2n.$$

Then we have a one to one correspondence

$$(z^a, \bar{z}^a) \rightleftarrows (x^\alpha),\ \alpha = 1, \ldots, n, (n+1), \ldots 2n$$

and $(z^a, \bar{z}^a)$ may be considered to be co-ordinates of a point in the real $2n$-dimensional manifold V_{2n} (Yano and Bochner, 1953). In what follows, the small Latin indices will take the values $1, 2, \ldots \ldots n$, and the capital Latin indices will take the values $n+1, \ldots \ldots 2n$, (here $A = n+a$, $B = n+b, \ldots$, for $1 \leq a, b, \ldots \leq n$) and the Greek indices will take the values $1, 2, \ldots \ldots \ldots \ldots 2n$.

Now if we can cover the manifold entirely by a system of co-ordinate neighbourhoods each endowed with complex co-ordinates $(z^a, \bar{z}^a)$ and if, U_1 and U_2 being two complex co-ordinate neighbourhoods of the manifold, a point P belongs to $U_1 \cap U_2$ then the complex co-ordinates z'^a of the point P in one of these complex co-ordinate neighbourhoods are complex analytic functions with non-vanishing Jacobian of the complex co-ordinates z^a of the same point in the other co-ordinate neighbourhood,

i.e. $$z'^a = \psi^a(z),\ \bar{z}'^a = \bar{\psi}^a(\bar{z}) \quad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

where $\bar{\psi}^a(z)$ denotes the complex conjugate of the function $\psi^a(z)$. Also if we put $\bar{z}^a = z^A$ and assume that capital Latin indices take the values $n+1, \ldots . 2n$, then for $(z^a, \bar{z}^a)$ we can write z^α, $\alpha = 1 \ldots ., n, (n+1) \ldots 2n$, and for the transformation (1.2) we can write

$$z'^\alpha = f^\alpha(z)$$

The Jacobian of (1.2) is easily seen to be $\left|\dfrac{\partial z'^\alpha}{\partial z^\beta}\right|$ which is real and greater than zero.

Thus the manifold is always orientable.

We shall denote a complex analytic manifold of complex dimensions n by C_n. In C_n, vectors, tensors, affine connections, etc., are defined with respect to co-ordinate transformation (1.2) in the same way as in the case of real manifolds.

We define a tensor to be self-adjoint if barring and unbarring all indices simultaneously (i.e. changing the small Latin indices into capital Latin indices and the capital Latin indices into small Latin indices) of any component the tensor changes the component into its complex conjugate.

Now assume that in our complex analytic manifold, there is given a positive definite quadratic differential form,

$$ds^2 = g_{\alpha\beta} dz^\alpha dz^\beta$$

where the symmetric tensor $g_{\alpha\beta}$ is self-adjoint.

$$g_{ab} = g_{AB} = 0$$

so that the metric form can be written as

$$ds^2 = 2g_{aB} dz^a dz^B,$$

where

$$g_{aB} = g_{Ba} = \overline{g_{Ab}} = \overline{g_{bA}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

This metric is called a Hermitian metric. Taking account of

$$g^{ab} = g^{AB} = 0,\ g^{aB} = g^{Ba} = \overline{g^{bA}} = \overline{g^{Ab}}$$

we obtain the Christoffel symbols

$$\{b^a c\} = \frac{1}{2} g^{aD} \left(\frac{\partial g_{Db}}{\partial z^c} + \frac{\partial g_{Dc}}{\partial z^b} \right),$$

$$\{b^a C\} = \frac{1}{2} g^{aD} \left(\frac{\partial g_{bD}}{\partial \bar{z}^c} - \frac{\partial g_b C}{\partial \bar{z}^d} \right), \{B^a C\} = 0,$$

and the values of other components are given by symmetry and self-adjointness. If in addition we assume

$$\frac{\partial g_{Ab}}{\partial z^c} = \frac{\partial g_{Ac}}{\partial z^b} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.4)$$

or further

$$g_{aB} = \frac{\partial^2 \phi}{\partial z^a \partial \bar{z}^b} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.5)$$

then the condition (1.4) or (1.5) is called 'Kaehler's condition' and the metric satisfying (1.3) and (1.4) is called a 'Kaehler's metric' and the manifold is called a 'Kaehler manifold' (Yano and Bochner, 1953). Thus in a Kaehler manifold we have

$$\{b^a c\} = g^{aD} \frac{\partial g_{Db}}{\partial z^c},$$

$$\{b^a C\} = 0,\ \{B^a C\} = 0,$$

the other components follow from symmetry and self-adjointness.

Before we proceed to define orthogonality of two vectors or an orthogonal ennuple in a Kaehler manifold we shall discuss certain properties of vectors having real magnitude.

Let $u^\alpha = (u^a, u^A)$ be any arbitrary contravariant vector. The magnitude of (u^a, u^A) is defined to be U where

$$2g_{aB} u^a u^B = U^2.$$

Here U^2 is a scalar but not necessarily real. Now barring both sides we obtain

$$\overline{2g_{aB} u^a u^B} = \bar{U}^2.$$

Since $\overline{g_{aB}} = g_{Ab}$, we have

$$2g_{Ab}\bar{u}^a\bar{u}^B = \bar{U}^2.$$

In order that U^2 be real, it is necessary and sufficient that the vector u^α be self-adjoint, i.e. $\bar{u}^a = u^A$, where $\bar{u}^a$ is the complex conjugate of u^a. Hence we get the result that:

In order that the Kaehler manifold should admit a contravariant vector u^α, of real magnitude, it is necessary and sufficient that the contravariant vector be self-adjoint. In particular, every unit vector is self-adjoint.

We consider two contravariant vector fields u^α and v^α each of which is self-adjoint. Let

$$g_{aB}u^a v^B = re^{i\theta}$$

then because of self-adjoint property it follows that

$$g_{Ab}u^A v^b = re^{-i\theta}$$

$$\therefore \quad 2r\cos\theta = g_{aB}u^a v^B + g_{Ab}u^A v^b$$

where θ may be taken to represent the angle between the two vectors symbolically. A vector v^α can now be defined to be orthogonal to u^α if

$$g_{aB}u^a v^B + g_{Ab}u^A v^b = 0.$$

This definition can be extended as follows:—

Let there be a self-adjoint vector $u^\alpha = (u^a, u^A)$ and let $(u^a, 0)$ and $(0, u^A)$ be two vectors which can be formed from the components of u^α. Let

$$\xi^a = \lambda u^a,\ \xi^A = \mu u^A,\ [\xi^\alpha = \lambda u^\alpha_{(a,0)} + \mu u^\alpha_{(0,A)}]$$

where λ and μ are parameters. Hence these vectors $(u^a, 0)$ and $(0, u^A)$ define a real geodesic V_2 as in the case of Riemannian manifolds. For $\lambda = \mu = 1$ we get the original vector (u^a, u^A). Therefore this vector (u^a, u^A) lies on the geodesic V^u_2 formed by $(u^a, 0)$ and $(0, u^A)$. Also for $\lambda = e$, $\mu = \bar{e}$, where $e = \pm i$ and $\bar{e}$ is the conjugate of e, we find that $(eu^a, \bar{e}u^A)$ also lies on this geodesic surface. We shall call this vector the 'Automorphic Equivalent' of (u^a, u^A). The orthogonal property is identically satisfied for a vector and its automorphic equivalent.

Let (v^a, v^A) be another vector not lying on the geodesic V^u_2. Let (v^a, v^A) and $(ev^a, \bar{e}v^A)$ define another geodesic V^v_2. We define u^α to be orthogonal to v^α if every vector lying in V^u_2 is orthogonal to every vector lying in V^v_2. That is if

$$\left.\begin{aligned} g_{aB}u^a v^B &= 0 \\ g_{Ab}u^A v^b &= 0 \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.6)$$

It is assumed that the vectors are not automorphic equivalents, for in that case the vectors become null vectors, a case we exclude.

2. Automorphic equivalence

We shall define two tensors $T^{\alpha\beta\ldots}_{\tau\epsilon\ldots}$ and $L^{\alpha\beta\ldots}_{\tau\epsilon\ldots}$ of the same order $2l+1$ (l being a positive integer or zero) and the same kind, to be automorphically equivalent, if the components of each are holomorphic functions of the co-ordinates (i.e. locally expressible as a power series) and one is the transform of the other under the automorphism (i.e. a non-singular linear transformation of co-ordinates which maps a space on to itself)

$$z'^{a} = ez^{a},\ \bar{z}'^{a} = \bar{e}\bar{z}^{a}$$

where $e = \pm i$ and $\bar{e}$ is the conjugate of e, i.e. $\bar{e} = -e$ and $e\bar{e} = 1$. The above transformation is an allowable complex analytic transformation since the Jacobian matrix $\left.\frac{\partial z'^{\alpha}}{\partial z^{\beta}}\right|$ is

$$\begin{vmatrix} e & 0 & \dots & \dots & 0 \\ 0 & e & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \bar{e} & 0 \dots & 0 \\ \dots & \dots & \dots & \dots & \vdots \\ \dots & \dots & \dots & \dots & \bar{e} \end{vmatrix} = (e)^{n}(\bar{e})^{n} = (e\bar{e})^{n} = 1$$

which is real and greater than zero. We shall show that the components of T and L are related by

$$L = eT,$$

where $e = \pm i$ and that if a component of L is given by $L^{ab\dots}_{pq\dots} = e\,T^{ab\dots}_{pq\dots}$ then the corresponding conjugate component

$$L^{AB\dots}_{PQ\dots} = \bar{e}T^{AB\dots}_{PQ\dots}$$

Let the tensors $L^{\alpha_1\alpha_2\dots\alpha_p}_{\beta_1\beta_2\dots\beta_q}$ and $T^{\alpha_1\alpha_2\dots\alpha_p}_{\beta_1\dots\beta_q}$ be of order $p+q = 2l+1$. Taking only small Latin indices we have

$$L^{a_1a_2\dots a_p}_{b_1b_2\dots b_q} = T^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q} \dots \frac{\partial z^{\beta_q}}{\partial z'^{b_q}} \cdot \frac{\partial z'^{a_1}}{\partial z^{\alpha_1}} \dots \frac{\partial z'^{a_p}}{\partial z^{\alpha_p}}$$

$$= T^{a_1\dots a_p}_{b_1\dots b_q} \cdot \bar{e}\cdot\bar{e}\cdots q \text{ times} \cdot e\cdot e\cdots p \text{ times}$$

$$= (-1)^{q}T^{a_1\dots a_p}_{b_1\dots b_q}(e)^{p+q}. \qquad (2.1)$$

For its conjugate component we have

$$L^{A_1A_2\dots A_p}_{B_1\dots B_q} = T^{A_1\dots A_p}_{B_1\dots B_q}(e)^{q}(\bar{e})^{p}$$

$$= (-1)^{p}T^{A_1\dots A_p}_{B_1\dots B_q}(e)^{p+q}$$

Since $p+q = 2l+1$, therefore either p is odd, q is even, or vice versa. Now $e^{p+q} = e\,.\,(\pm i)^{2l} = (-1)^{l}e$. Therefore, if the coefficient of $T^{a_1\dots a_p}_{b_1\dots b_q}$ in (2.1) is e, i.e. $(-1)^{q+l}\,.\,e = e$, then the coefficient of $T^{A_1\dots A_p}_{B_1\dots B_q}$ in (2.2) is

$$(-1)^{p+l}e = -e = \bar{e}.$$

For the other components too we can show similarly that this relation holds. Thus we find that if (v^{a}, v^{A}) is the transform of (u^{a}, u^{A}) under the automorphism

$$z'^{a} = ez^{a},\ \bar{z}'^{a} = \bar{e}\bar{z}^{a}$$

then $v^a = eu^a$, $v^A = \bar{e}u^A$, and hence $(eu^a, \bar{e}u^A)$ is the automorphic equivalent of (u^a, u^A).

We now proceed to determine an orthogonal ennuple in a Kaehlerian C_n.

Let $\left(u^a_{r|}, u^A_{r|}\right)$, $r = 1, \ldots n$, denote n self-adjoint unit vectors which are mutually orthogonal but no two of them are automorphically equivalent.

Let $\left(v^a_{r|}, v^A_{r|}\right)$, $r = 1, \ldots n$, be the automorphic equivalents of $\left(u^a_{r|}, u^A_{r|}\right)$ so that $v^a_{r|} = e\, u^a_{r|}$, $v^A_{r|} = \bar{e}\, u^A_{r|}$. Now, $2\, g_{aB}\, v^a_{r|}\, v^B_{r|} = e\,.\,\bar{e}\,.\, 2g_{aB}\, u^a_{r|}\, u^B_{r|} = 1$.

Hence the v's are self-adjoint unit vectors.

From the conditions of orthogonality we have

$$g_{aB}u^a_{r|}u^B_{s|} = 0 \quad r \neq s \qquad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

Also

$$2g_{aB}u^a_{r|}u^B_{r|} = 1 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

The conditions (2.3), (2.4) are identically satisfied if $\left(u^a_{r|}, u^A_{r|}\right)$ is replaced by its automorphic equivalent. Hence the $2n$ vectors $\left(u^a_{r|}, u^A_{r|}\right)$ and $\left(v^a_{r|}, v^A_{r|}\right)$ form a mutually orthogonal system. We define this system as an orthogonal ennuple of a Kaehler manifold C_n.

Hence we find that, although the space is of 2n real dimensions, it suffices to obtain n mutually orthogonal vectors $\left(u^a_{r|}, u^A_{r|}\right)$, $r = 1, \ldots, n$, *no two of which are automorphically equivalent, and then the other n vectors completing the ennuple are automatically obtained.*

3. Ricci coefficients in a Kaehler manifold

Differentiating covariantly $g^{aB}\, u_{r\,|\,a}\, u_{s\,|\,B} = 0$ with respect to z^c and multiplying by $u_{t\,|}{}^c$ we get

$$u_{r\,|\,a,c}u^a_{s\,|}\, u^c_{t\,|} + u^B_{r\,|}\, u_{s\,|\,B,c}u^c_{t\,|} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

Denoting Ricci coefficients (Eisenhart, 1949) by

$$\gamma_{rst} = u_{r\,|\,a,b}u^a_{s\,|}\, u^b_{t\,|}$$

$$\gamma_{RSt} = u_{r\,|\,A,b}u^A_{s\,|}\, u^b_{t\,|}$$

$$\gamma_{rsT} = u_{r\,|\,a,B}u^a_{s\,|}\, u^B_{t\,|}$$

$$\gamma_{RST} = u_{r\,|\,A,B}u^A_{s\,|}\, u^B_{t\,|}$$

We find from (3.1) that

$$\gamma_{rst} + \gamma_{SRt} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

Similarly differentiating covariantly $g^{aB}\, u_{r\,|\,a}\, u_{s\,|\,B} = 0$ with respect to $\bar{z}^d$ and multiplying by $u^D_{t\,|}$ we get

$$\gamma_{SRT} + \gamma_{rsT} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

Comparing (3.2) with (3.3) we see that barring and unbarring the indices do not change the relation. Hence extending the definition of self-adjointness to these scalars we get the result that *Ricci coefficients are self-adjoint.*

Note—(i) In the above discussion if we denote the vectors $\left(u^a_{r|}, u^A_{r|}\right)$ by $u^\alpha_{r|}$ and differentiate covariantly $g^{aB} u_{r|a} u_{s|B} = 0$ with respect to z^α and multiply by $u^\alpha_{t|}$ we get

$$u_{r|a,c} u^a_{s|} u^c_{t|} + u_{s|B,c} u^B_{r|} u^c_{t|} + u_{r|a,C} u^a_{s|} u^C_{t|} + u_{s|B,C} u^B_{r|} u^C_{t|} = 0 \quad .. \quad (3.4)$$

Again differentiating covariantly $g^{aB} u_{r|a} u_{s|B} = 0$ with respect to z^α and multiplying by $v^\alpha_{t|}$ where $v^\alpha_{t|}$ is the automorphic equivalent of $u^\alpha_{t|}$ and substituting $v^a_{t|} = eu^a_{t|}$, $v^A_{t|} = \bar{e}\, u^A_{t|}$ we get

$$eu_{r|a,c} u^a_{s|} u^c_{t|} + eu_{s|B,c} u^B_{r|} u^c_{t|} + \bar{e} u_{r|a,C} u^a_{s|} u^C_{t|} + \bar{e} u_{s|B,C} u^B_{r|} u^C_{t|} = 0 \quad .. \quad (3.5)$$

Putting $\bar{e} = -e$ and comparing (3.4) and (3.5) we get

$$u_{r|a,c} u^a_{s|} u^c_{t|} + u_{s|B,c} u^B_{r|} u^c_{t|} = 0$$

$$u_{r|a,C} u^a_{s|} u^C_{t|} + u_{s|B,C} u^B_{r|} u^C_{t|} = 0$$

i.e. $$\gamma_{rst} + \gamma_{SRt} = 0,\ \gamma_{SRT} + \gamma_{rsT} = 0$$

which are exactly the same as the equations (3.2) and (3.3) showing thereby the equivalence of the two methods followed.

Note—(ii) We also observe that in (3.1) if we change any u by its automorphic equivalent v and note that

$$v^a = eu^a,\ v_a = \bar{e} u_a,\ v^A = \bar{e} u^A,\ v_A = eu_A$$

we find the same relations as (3.2) and (3.3). Hence in order to find relations amongst Ricci coefficients, it is sufficient to deal with only the n non-automorphic equivalent vectors $\left(u^a_{r|}, u^A_{r|}\right)$ of the ennuple.

If all the vectors forming the ennuple have their covariant components complex analytic, then $u_{r|A,c} = 0$, $u_{r|a,C} = 0$, hence

$$\gamma_{RSt} = \gamma_{rsT} = 0 \quad r \neq s$$

$$\therefore \quad \gamma_{srt} = 0,\ \gamma_{SRT} = 0, \quad \text{from (3.2) and (3.3).}$$

Again differentiating covariantly $2g^{aB} u_{r|a} u_{r|B} = 1$ with respect to z^c and multiplying by $u^c_{r|}$ we find $\gamma_{rrr} = 0$, $\gamma_{RRr} = 0$ and similarly $\gamma_{RRR} = 0$, $\gamma_{rrR} = 0$.

Hence

If the self-adjoint unit contravariant vectors $\left(u^a_{r|}, u^A_{r|}\right)$ and $\left(v^a_{r|}, v^A_{r|}\right)$ (where $\left(v^a_{r|}, v^A_{r|}\right)$ is the automorphic equivalent of $\left(u^a_{r|}, u^A_{r|}\right)$ for any particular r) forming an orthogonal ennuple in C_n have their corresponding covariant components complex analytic then all the Ricci coefficients vanish.

In any equation involving the u's of the ennuple if we replace any of the u's by its automorphic equivalent, the equation is identically satisfied; we do not have to consider the Ricci coefficients of the v's separately. When the Ricci coefficients are formed from the u's we find that replacing any of the u's by its corresponding v does not change the Ricci coefficients except in case of a factor e, where $e = \pm i$.

Again, let only one of the vectors, say $(u_{s|a}, u_{s|A})$, have its components complex analytic. Then its automorphic equivalent $(v_{s|a}, v_{s|A})$ also has its components complex analytic. Differentiating covariantly $g^{aB}\, u_{s\,|\,a}\, u_{t\,|\,B} = 0$, $s \neq t$ with respect to $\bar{z}^c$ and multiplying by $u^{C}_{r\,|}$ we get

$$u_{t\,|\,B,\,C}\, u^{B}_{s\,|}\, u^{C}_{r\,|} = 0$$

i.e. $\gamma_{TSR} = 0$, and $\gamma_{stR} = 0$, follows from (3.3).

From the self-adjointness of Ricci coefficients we have

$$\gamma_{tsr} = 0\,,\; \gamma_{STr} = 0,\quad s \neq t.$$

Also differentiating $2g^{aB}\, u_{s\,|\,a}\, u_{s\,|\,B} = 1$ covariantly and taking into account the complex analytic character of $(u_{s\,|\,a},\, u_{s\,|\,A})$ we find

$$\gamma_{ssr} = 0,\; \gamma_{SSR} = 0.$$

Hence

If only one and therefore two of the covariant vectors of an orthogonal ennuple in a Kaehlerian C_n, *viz.* $(u_{s\,|\,a},\, u_{s\,|\,A})$ *and its automorphic equivalent* $(v_{s\,|\,a},\, v_{s\,|\,A})$ *have their components complex analytic, then the Ricci coefficients*

$$\gamma_{tsr}\,,\quad \gamma_{TSR}\,,\quad \gamma_{stR}\,,\quad \gamma_{STr}\,,$$

vanish.

When the vectors $(u_{r\,|\,a},\, u_{r\,|\,A})$ and their automorphic equivalents $(v_{r\,|\,a},\, v_{r\,|\,A})$ $r = 1, \ldots, n$, are not complex analytic, let

$$\Gamma^{u}_{rst} = \begin{pmatrix} \gamma_{rst} & \gamma_{RSt} \\ \gamma_{rsT} & \gamma_{RST} \end{pmatrix} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.6)$$

We shall denote the Ricci coefficients formed by replacing all the u's by their corresponding automorphic equivalents by $\overset{v}{\gamma}_{rst}$ and correspondingly Γ^{v}_{rst}. Then

$$\overset{v}{\gamma}_{rst} = v_{r\,|\,a,\,b}\, v^{a}_{s\,|}\, v^{b}_{t\,|} = \bar{e}\,.\,e\,.\,e u_{r\,|\,a,\,b}\, u^{a}_{s\,|}\, u^{b}_{t\,|} = e\gamma_{rst}.$$

Similarly,

$$\overset{v}{\gamma}_{RSt} = e\gamma_{RSt}$$

$$\overset{v}{\gamma}_{rsT} = \bar{e}\gamma_{rsT}$$

$$\overset{v}{\gamma}_{RST} = \bar{e}\gamma_{RST}$$

$$\therefore\quad \Gamma^{v}_{rst} = \begin{pmatrix} e\gamma_{rst} & e\gamma_{RSt} \\ \bar{e}\gamma_{rsT} & \bar{e}\gamma_{RST} \end{pmatrix} \quad \ldots \quad \ldots \quad \ldots \quad (3.7)$$

Hence

Extending the definition of automorphic equivalence to the Ricci coefficients we find from (3.6) *and* (3.7) *that the components of* Γ^{u}_{rst} *are the automorphic equivalents of the components of* Γ^{v}_{rst}.

4. Normal congruences in a Kaehler manifold

Definition: A congruence of curves determined by the self-adjoint complex analytic vector field $(u_a,\, u_A)$ is said to define a normal congruence if there exist functions

$f(z)$ and $F(\bar{z})$ in the manifold where $F(\bar{z})$ is the complex conjugate of $f(z)$, such that at every point of a co-ordinate neighbourhood the following relations are satisfied :—

$$\mu u_a = f,_a \; ; \; \bar{\mu} u_A = F,_A$$

where μ is an invariant.

Putting $\mu' = e\mu$, we find that the automorphic equivalent of (u_a, u_A) also defines a normal congruence, being normal to the system of hypersurfaces C_{n-1} defined by

$$f(z) = \text{const.,} \quad F(\bar{z}) = \text{const.}$$

From $f(z) = \text{const.}$, we find that grad $f(z)$ gives us a vector $(N_{a,0})$ such that $f,_a = \mu N_a$ and $f,_A = 0$, $f(z)$ being a function of z^a only. Similarly grad $F(\bar{z})$ gives us another vector $(0, N_A)$ where $F,_A = \bar{\mu} N_A$ and $F,_a = 0$, since $F(\bar{z})$ is a function of $\bar{z}^a$ only. These two vectors $(N_a, 0)$ and $(0, N_A)$ define a real geodesic V_2 which contains all the vectors including (N_a, N_A) normal to the hypersurface C_{n-1}, which justifies our definition of a normal congruence.

Let $\left(u^a_{r|}, u^A_{r|}\right)$ together with their automorphic equivalents $\left(v^a_{r|}, v^A_{r|}\right)$, $r = 1, .., n$, define an orthogonal ennuple in a Kaehlerian C_n, where the given vectors are self-adjoint unit vectors. Let $(u_{n|a}, u_{n|A})$ be a complex analytic self-adjoint unit vector field which defines a normal congruence, being normal to a system of hypersurfaces

$$f(z) = \text{const.,} \qquad F(\bar{z}) = \text{const.}$$

$$\left.\begin{array}{ll} f,_a = \mu u_{n|a} , & F,_A = \bar{\mu} u_{n|A} \\ f,_a = \mu' v_{n|a} , & F,_A = \bar{\mu}' v_{n|A} \end{array}\right\} \quad .. \quad .. \quad .. \quad (4.1)$$

where $\mu' = e\mu$ and $(v_{n|a}, v_{n|A})$ is the automorphic equivalent of $(u_{n|a}, u_{n|A})$. Now $(u_{n|a}, u_{n|A})$ will be a normal congruence if

$$X_r(f) = u^a_{r|} f,_a = 0,$$

$$X_R(F) = u^A_{r|} F,_A = 0$$

are integrable. As in Eisenhart, 1949, this is so if

$(X_s, X_r)f = X_s X_r(f) - X_r X_s(f) =$ a linear function of $X_t(f)$

$(X_S, X_r)f = X_S X_r(f) - X_r X_S(f) =$ a linear function of $X_t(f)$ $r, s, t = 1, .., (n-1)$.

$X_s X_r(f) = u^b_{s|} \left(u^a_{r|} f,_a\right),_b = f,_{ab} u^a_{r|} u^b_{s|} + f,_a u^a_{r|},_b u^b_{s|}$

$X_S X_r(f) = u^B_{s|} \left(u^a_{r|} f,_a\right),_B = f,_a u^a_{r|},_B u^B_{s|}$, $\because f_a,_B = 0.$

$X_S(f) = u^A_{s|} f,_A = 0$, $\therefore X_r X_S(f) = 0.$

Hence the conditions of integrability reduce to

$$(X_s, X_r)f = f,_a u^a_{r|},_b u^b_{s|} - f,_a u^a_{s|},_b u^b_{r|} = 0 \quad .. \quad .. \quad (4.2)$$

$$(X_S, X_r)f = f,_a u^a_{r|},_B u^B_{s|} = 0 \quad .. \quad .. \quad .. \quad (4.3)$$

Putting $f,_a = \mu u_{n|a}$ in the conditions (4.2) and (4.3) we get

$$\gamma_{RNS} = 0 \qquad (N = 2n) \quad .. \quad .. \quad .. \quad (4.4)$$

$$\gamma_{RNs} = \gamma_{SNr} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (4.5)$$

Also $$\gamma_{rns} = 0, \quad \gamma_{rnS} = \gamma_{snR}$$

which follows from the self-adjointness of Ricci coefficients. From (3.2) we find $\gamma_{srt}+\gamma_{RSt}=0$. Hence (4.5) gives us $\gamma_{nrs}=\gamma_{nsr}$. Then $\gamma_{NRS}=\gamma_{NSR}$ follows from self-adjointness.

Hence a vector $\left(u^{a}_{n|}, u^{A}_{n|}\right)$ *defines a normal congruence if the Ricci coefficients formed by* $\left(u^{a}_{r|}, u^{A}_{r|}\right)$, $r=1, \ldots, (n-1)$ *or their automorphic equivalents are related as*

$$\gamma_{nrs}=\gamma_{nsr}, \quad \gamma_{NRS}=\gamma_{NSR}, \quad \gamma_{rns}=\gamma_{RNS}=0$$

If the covariant vector fields $(u_{r|a}, u_{r|A})$ and $(v_{r|a}, v_{r|A})$ are all complex analytic, then from a previous result it follows that all the Ricci coefficients are zero. Hence the conditions of normal congruence are identically satisfied.
Hence we get the theorem that:

The necessary and sufficient condition that a system of mutually orthogonal n vectors and their automorphic equivalents forming an orthogonal ennuple in a Kaehler manifold form normal congruences of curves is that the vectors have their covariant components complex analytic.

5. Geodesics and geodesic congruences in a Kaehler manifold

A contravariant vector field u^{α} is said to define a geodesic if it undergoes a parallel displacement in Levi Civita's sense (Eisenhart, 1949), i.e. if

$$u^{\alpha}_{,\beta}u^{\beta}=0$$

or

$$u^{a}_{,b}u^{b}+u^{a}_{,B}u^{B}=0 \quad \ldots \quad (5.1)$$

$$u^{A}_{,b}u^{b}+u^{A}_{,B}u^{B}=0 \quad \ldots \quad (5.2)$$

Let $u^{a}=\dfrac{dz^{a}}{ds}$, $u^{A}=\dfrac{d\bar{z}^{a}}{ds}$, where ds is the element of arc of the curve with real parameter s. From (5.1) we have

$$\frac{\partial}{\partial z^{b}}\left(\frac{dz^{a}}{ds}\right)\frac{dz^{b}}{ds}+\{b^{a}c\}\frac{dz^{b}}{ds}\cdot\frac{dz^{c}}{ds}+\frac{\partial}{\partial \bar{z}^{b}}\left(\frac{dz^{a}}{ds}\right)\frac{dz^{b}}{ds}=0.$$

Now, unless (u^{a}, u^{A}) is complex analytic u will be a function of z^{a} and $\bar{z}^{a}$ both.

$$\therefore \quad \left[\frac{\partial}{\partial z^{b}}\left(\frac{dz^{a}}{ds}\right)\frac{dz^{b}}{ds}+\frac{\partial}{\partial \bar{z}^{b}}\left(\frac{dz^{a}}{ds}\right)\frac{d\bar{z}^{b}}{ds}\right]+\{b^{a}c\}\frac{dz^{b}}{ds}\cdot\frac{dz^{c}}{ds}=0,$$

or

$$\frac{d^{2}z^{a}}{ds^{2}}+\{b^{a}c\}\frac{dz^{b}}{ds}\cdot\frac{dz^{c}}{ds}=0 \quad \ldots \quad (5.3)$$

Similarly (5.2) can be reduced to

$$\frac{d^{2}\bar{z}^{a}}{ds^{2}}+\{B^{A}C\}\frac{d\bar{z}^{b}}{ds}\cdot\frac{d\bar{z}^{c}}{ds}=0 \quad \ldots \quad (5.4)$$

(5.3) and (5.4) together give us differential equations of geodesics (Weatherburn, 1950) of the same form as in Riemann spaces.

Now multiplying (5.1) by g_{aC} and (5.2) by g_{Ac} and summing we get

$$u_{C,b}u^{b}+u_{C,B}u^{B}=0 \quad \ldots \quad (5.5)$$

$$u_{c,b}u^{b}+u_{c,B}u^{B}=0 \quad \ldots \quad (5.6)$$

From (5.5) and (5.6) it follows that if a contravariant vector field defines a geodesic so does its covariant component. If the vector be complex analytic, i.e. if both the contravariant and covariant components of the vector are complex analytic, then

$$u^a_{\ ,B} = 0, \quad u_{a,B} = 0.$$

Multiplying $u^a{}_{,B} = 0$ by g_{aC} then $u_{C,B} = 0$ which is a much stronger condition than $u_{C,B}\, u^B = 0$. Hence if a complex analytic vector field undergoes a parallel displacement then

$$u^a_{\ ,b} = 0, \quad u^a_{\ ,B} = 0, \quad u^A_{\ ,b} = 0, \quad u^A_{\ ,B} = 0$$

We now investigate the Ricci coefficients for geodesic congruences. Let $\left(u^a_{r|}\ ,\ u^A_{r|}\right)$ and their automorphic equivalents $\left(v^a_{r|}\ ,\ v^A_{r|}\right)$, $r = 1, .., n$, be $2n$ unit vectors mutually orthogonal, which define an orthogonal ennuple in a Kaehler C_n. Let $\left(u^a_{n|}\ ,\ u^A_{n|}\right)$ define a geodesic congruence. In (3.2) putting $s = t = n$ we get

$$\gamma_{NRn} + \gamma_{rnn} = 0 \qquad \ldots \qquad (5.7)$$

We know that for a geodesic congruence $\left(u^a_{n|}\ ,\ u^A_{n|}\right)$,

$$u^a_{n|,b}\, u^b_{n|} + u^a_{n|,B}\, u^B_{n|} = 0 \qquad \ldots \qquad (5.8)$$

Multiplying (5.8) by $g_{aB}\, u^B_{r|}$ we get

$$\gamma_{NRN} + \gamma_{NRn} = 0 \qquad \ldots \qquad (5.9)$$

From (5.7) and (5.9) we get

$$\gamma_{NRN} = \gamma_{rnn} \qquad \ldots \qquad (5.10)$$

Barring and unbarring the indices in (5.10) we get

$$\gamma_{nrn} = \gamma_{RNN} \qquad \ldots \qquad (5.11)$$

Equations (5.10) and (5.11) give us the conditions for a vector field $\left(u^a_{n|}\ ,\ u^A_{n|}\right)$ to be a geodesic congruence. If in (5.8) we replace $\left(u^a_{n|}\ ,\ u^A_{n|}\right)$ by its automorphic equivalent $\left(v^a_{n|}\ ,\ v^A_{n|}\right)$ then it is not identically satisfied. Hence the automorphic equivalent does not define a geodesic. But if the vector field $u^\alpha_{n|}$ and $u_{n|\alpha}$ be complex analytic then the conditions for geodesic congruence are $u^\alpha_{n|,\beta} = 0$ which are identically satisfied by its automorphic equivalent. Hence the automorphic equivalent also defines a geodesic. The conditions in terms of Ricci coefficients are

$$\gamma_{rnn} = 0\,, \quad \gamma_{nrn} = 0\,, \quad \gamma_{NRN} = 0\,, \quad \gamma_{RNN} = 0.$$

Also in this case

$$\gamma_{nrs} = g_{Ab}\, u^A_{n|,c}\, u^b_{r|}\, u^c_{s|} = 0 \because u^A_{n|,b} = 0\,, r, s \neq n.$$

But from the relations $\gamma_{RNS} + \gamma_{nrS} = 0$, we get

$$\gamma_{RNS} = 0.$$

Barring and unbarring the indices we get

$$\gamma_{rns} = 0\,, \quad \gamma_{RNS} = 0\,, \quad \gamma_{NRs} = 0\,, \quad \gamma_{nrS} = 0,$$

i.e. all the Ricci coefficients having n as the first or the second index vanish. We have shown earlier that $(e\underset{n|}{u^a}, \bar{e}\underset{n|}{u^A})$ is the transform of $(\underset{n|}{u^a}, \underset{n|}{u^A})$ under the non-singular linear transformation

$$z'^a = ez^a, \quad \bar{z}'^a = \bar{e}\bar{z}^a, \quad e = \pm i, \quad \bar{e} = -e.$$

Hence, under an automorphism $z'^a = ez^a$, $\bar{z}'^a = \bar{e}\bar{z}^a$, *a geodesic is preserved (i.e. a geodesic is transformed into a geodesic) if and only if, the vector field defining the geodesic is complex analytic.*

Abstract

Some properties of vectors having real magnitude have been discussed and then an orthogonal ennuple in a Kaehler space has been defined and studied. It has been proved that Ricci's coefficients in a Kaehler manifold are self-adjoint. Some other properties of Ricci coefficients have also been established. Normal and geodesic congruences have been studied in detail.

References

Yano, K., and Bochner, S. (1953). Curvature and Betti Numbers, 117.
———(1953). *Loc. cit.*, 123.
Eisenhart, L. P. (1949). Riemannian Geometry, 97.
———(1949). *Loc. cit.*, 115.
———(1949). *Loc. cit.*, 65.
Weatherburn, C. E. (1950). Tensor Calculus and Riemannian Geometry, 75.

Issued March 20, 1957.