INTERNAL BALLISTICS OF A LEAKING GUN

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1. Corner (1947, 1950) established the equations determining the ballistics of a leaking gun for the case when the rate of burning is proportional to p^{α} where p is the pressure. For the case in which the rate of burning is proportional to the pressure, he gave a simple but approximate solution based on semi-empirical methods. Recently Thiruvenkatachar and Venkatesan (1953) reconsidered the problem and suggested that a reasonably accurate solution may be obtained in the form of a series in powers of Ψ, the leakage parameter, and they worked out the solution for a tubular propellant ($\theta = 0$) correct up to the first power of Ψ. They applied their solution to an hypothetical recoilless gun and found that the solution up to the first power of Ψ is in fair agreement with that given by Corner. Subsequently Aggarwal (1955) extended this method to the propellants of general shape ($\theta \neq 0$), using Z the fraction of the mass of the charge burnt up to the time t, as independent variable. He obtains the coefficient of Ψ in the form of integrals to be numerically evaluated.

The aim of the present note is to form some idea about the approximate range of Ψ for which the solution up to the first power of Ψ gives sufficiently accurate result for practical purposes, and to see if in the general case ($\theta \neq 0$) the analysis can be made elegant by suitable choice of independent variable.

We have obtained for $\theta = 0$ the expression for the temperature correct up to Ψ^2 in an analytic form to estimate the comparative contribution of the Ψ^2-term. This solution has been applied to an hypothetical example which shows that it is not advisable in all cases to neglect the contribution of Ψ^2-term, even for smooth-bore leaking guns. For a recoilless gun, for which Ψ ranges from 0·4 to 0·6, if we neglect Ψ^2 and higher powers of Ψ, we shall be introducing appreciable error ; we may conclude, therefore, that the present expansion method will not give as rapidly convergent a result as we would like to have for recoilless guns and the numerical method of solution has to be used.

If we use f, the fraction of the ballistic size propellant remaining at time t, as the independent variable, the treatment is considerably simplified even for $\theta \neq 0$. However, in this particular case we have obtained the coefficient of Ψ explicitly in an analytical form and have indicated the method how using this expression the coefficient of Ψ^2 can be obtained.

We may mention in passing that in the particular case $\theta = 0$, it is possible to express the coefficient of Ψ in terms of elliptic integrals.

2. The equations of internal ballistics of a leaking or recoilless gun with the usual assumptions made by Corner (1950) are

$$\zeta\xi = NT'\left(1+\frac{kCN}{6W}\right) \qquad \text{(1a)}$$

$$\eta\frac{d\eta}{d\xi} = \left(M/1+\frac{kCN}{2W}\right)\zeta \qquad \text{(1b)}$$

$$\frac{df}{d\tau} = -\zeta \qquad \text{(1c)}$$

$$z = (1-f)(1+\theta f) \qquad \text{(1d)}$$

$$\frac{dN}{d\tau} = \frac{dz}{d\tau} - \Psi\zeta(T')^{-\frac{1}{2}} \qquad \text{(1e)}$$

$$\frac{d(NT')}{d\tau} = -(\bar{\gamma}-1)\zeta\frac{d\xi}{d\tau} + \frac{dz}{d\tau} - \gamma\Psi\zeta(T')^{\frac{1}{2}} \qquad \text{(1f)}$$

where

$$Al = U - \frac{C}{\delta} \qquad \text{(2a)}$$

$$\xi = 1 + \frac{x}{l} \qquad \text{(2b)}$$

$$\tau = \left(\frac{\beta CRT_0}{ADl}\right)t \qquad \text{(2c)}$$

$$\zeta = \left(\frac{Al}{CRT_0}\right)p \qquad \text{(2d)}$$

$$\eta = \frac{d\xi}{d\tau} = \frac{AD}{C\beta RT_0}\frac{dx}{dt} \qquad \text{(2e)}$$

$$T' = \frac{T}{T_0} \qquad \text{(2f)}$$

$$M = \frac{A^2D^2}{\beta^2 CRT_0 W} \qquad \text{(2g)}$$

$$\Psi = \frac{\psi SD}{\beta C(RT_0)^{\frac{1}{2}}} \qquad \text{(2h)}$$

Here ξ, η, ζ and T' are dimensionless variables corresponding to shot-travel, velocity, pressure and temperature respectively, whereas M and Ψ are the dimensionless central ballistic parameter and leakage parameter respectively.

Denoting the values of f at the shot-start and nozzle-start by f_0 and f_N respectively, we have

$$\eta = 0 \text{ for } f = f_0 \qquad \text{(3a)}$$

$$N = Z \text{ for } f \geqslant f_N \qquad \text{(3b)}$$

combining (1*b*) and (1*c*) we get

$$\frac{d\eta}{d\tau} = \eta\frac{d\eta}{d\xi} = -\frac{M}{\left(1+\frac{kCN}{2W}\right)}\frac{df}{d\tau} \qquad \text{(4)}$$

Since the factor $1+\frac{kCN}{2W}$ has a value close to unity throughout the period of burning, we may replace it by average value σ ($\sigma \sim 1$).

Thus

$$\frac{d\eta}{d\tau} = -\frac{M}{\sigma}\frac{df}{d\tau} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Integrating the above differential equation and using the initial condition $f = f_0$ when $\eta = 0$, we have

$$\eta = \frac{M}{\sigma}(f_0 - f) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

From (1c), (1d) and (1e) we get

$$\frac{dN}{d\tau} = \frac{d}{d\tau}(1-f)(1+\theta f) + \Psi(T')^{-\frac{1}{2}}\frac{df}{d\tau} \qquad \ldots \quad \ldots \quad (7)$$

On integration,

$$N = (1-f)(1+\theta f) + \Psi\int_{f_N}^{f}(T')^{-\frac{1}{2}}\,df,\ f \le f_N \qquad \ldots \quad \ldots \quad (8)$$

Using equations (1b), (1c), (1d) and (1f) we obtain

$$\frac{d}{d\tau}(NT') = -\frac{(\bar{\gamma}-1)}{M}\sigma\eta\frac{d\eta}{d\tau} + \frac{d}{d\tau}(1-f)(1+\theta f) + \gamma\Psi(T')^{\frac{1}{2}}\frac{df}{d\tau} \qquad \ldots \quad (9)$$

Assuming that the nozzle opens before or at the instant when the shot starts to move, the integration of (9) gives

$$NT' = -\frac{(\bar{\gamma}-1)\sigma}{2M}\eta^2 + (1-f)(1+\theta f) + \gamma\Psi\int_{f_N}^{f}(T')^{\frac{1}{2}}\,df \qquad \ldots \quad (10)$$

Substituting for η and N from (6) and (8), we get

$$T'\left[(1-f)(1+\theta f) + \Psi\int_{f_N}^{f}(T')^{-\frac{1}{2}}\,df\right]$$

$$= -\frac{(\bar{\gamma}-1)\sigma}{2M}\cdot\frac{M^2}{\sigma^2}(f_0-f)^2 + (1-f)(1+\theta f) + \gamma\Psi\int_{f_N}^{f}(T')^{\frac{1}{2}}\,df \qquad \ldots \quad (11a)$$

or

$$(1-f)(1+f)T' + \Psi T'\int_{f_N}^{f}(T')^{-\frac{1}{2}}df - \gamma\Psi\int_{f_N}^{f}(T')^{\frac{1}{2}}df = A(f) \qquad \ldots \quad (11b)$$

where

$$A(f) = (1-f)(1+\theta f) - \alpha(f_0-f)^2 \qquad \ldots \quad \ldots \quad \ldots \quad (12a)$$

and

$$\alpha = \frac{1}{2}\frac{M(\bar{\gamma}-1)}{\sigma} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12b)$$

It may be noted that if the shot starts before the nozzle opens, the equation (9) with $\Psi = 0$ is valid up to the instant when the nozzle opens.

The variation in temperature with f is determined by (11b) and we shall solve it by assuming the following expansion in powers of Ψ as has been suggested by Thiruvenkatachar and Venkatesan

$$T' = T_{(0)} + \Psi T_{(1)} + \Psi^2 T_{(2)} + \ldots \qquad \ldots \quad \ldots \quad \ldots \quad (13)$$

Once the value of T' has been obtained as function of f, we can proceed to solve the other ballistic equations as follows: We find N from (8) and NT' from (10). To obtain ξ, we eliminate ζ between (1a) and (1b). Thus

$$\eta\frac{d\eta}{d\xi}=\frac{M}{\sigma}\frac{NT'}{\xi}\left(1+\frac{kCN}{6W}\right)=\frac{(2+\sigma)M}{3\sigma}\frac{NT'}{\xi} \quad \ldots \quad (14a)$$

Integrating we get

$$\log\xi=\frac{3\sigma}{(2+\sigma)M}\int_0^\eta\frac{\eta d\eta}{NT'} \quad \ldots \quad (14b)$$

Using (6) we obtain

$$\log\xi=\frac{3M}{\sigma(2+\sigma)}\int_{f_0}^{f}\frac{(f-f_0)}{NT'}df \quad \ldots \quad (14c)$$

The pressure ζ is given by (1a) as

$$\zeta=\frac{2+\sigma}{3}\frac{NT'}{\xi} \quad \ldots \quad (15)$$

3. 'The Approximate Solution of the Equation'

Substituting the expression (13) for T' in (11b) and equating the coefficients of various powers of Ψ to zero we have

$$T_{(0)}=\frac{A(f)}{(1-f)(1+\theta f)} \quad \ldots \quad (16)$$

$$T_{(1)}=\frac{\gamma}{(1-f)(1+\theta f)}\int_{f_N}^{f}T_{(0)}{}^{\frac{1}{2}}df-\frac{T_{(0)}}{(1-f)(1+\theta f)}\int_{f_N}^{f}T_{(0)}{}^{-\frac{1}{2}}df \quad \ldots \quad (17)$$

$$T_{(2)}=\tfrac{1}{2}\frac{\gamma}{(1-f)(1+\theta f)}\int_{f_N}^{f}T_{(1)}T_{(0)}^{-\frac{1}{2}}df+\tfrac{1}{2}\frac{T_{(0)}}{(1-f)(1+\theta f)}\int_{f_N}^{f}T_{(1)}T_{(0)}^{-\frac{3}{2}}df$$
$$-\frac{T_{(1)}}{(1-f)(1+\theta f)}\int_{f_N}^{f}T_{(0)}^{-\frac{1}{2}}df \quad \ldots \quad (18)$$

Retaining terms up to Ψ^2, we have from (8) and (10)

$$N=(1-f)(1+\theta f)+\Psi\int_{f_N}^{f}T_{(0)}^{-\frac{1}{2}}df-\tfrac{1}{2}\Psi^2\int_{f_N}^{f}T_{(1)}T_{(0)}^{-\frac{3}{2}}df \quad \ldots \quad (19)$$

and

$$NT'=-\frac{(\bar{\gamma}-1)\sigma}{2M}\eta^2+(1-f)(1+\theta f)+\gamma\Psi\int_{f_N}^{f}T_{(0)}^{\frac{1}{2}}df+\tfrac{1}{2}\gamma\Psi^2\int_{f_N}^{f}T_{(1)}T_{(0)}^{-\frac{1}{2}}df \quad \ldots \quad (20a)$$

$$= A(f)+\gamma\Psi\int_{f_N}^{f} T_{(0)}^{\frac{1}{2}}\,df+\tfrac{1}{2}\gamma\Psi^2\int_{f_N}^{f} T_{(1)}T_{(0)}^{-\frac{1}{2}}\,df \qquad (20b)$$

Equations (19) and (20b) give N and NT' as explicit functions of f. Substituting this value of NT' in equations (14c) and (15), we get ξ and ζ as functions of f.

4. Tubular Propellants

When $\theta = 0$, we have from (12a) and (16)

$$T_{(0)} = 1-\alpha\frac{(f_0-f)^2}{(1-f)} \qquad (21)$$

so that

$$\int_{f_N}^{f} T_{(0)}^{\frac{1}{2}}\,df = \sqrt{\alpha}\int_{f_N}^{f}\sqrt{\frac{(\lambda-f)\,(\mu+f)}{(1-f)}}\,df \qquad (22)$$

and

$$\int_{f_N}^{f} T_{(0)}^{-\frac{1}{2}}\,df = \frac{1}{\sqrt{\alpha}}\int_{f_N}^{f}\sqrt{\frac{(1-f)}{(\lambda-f)(\mu+f)}}\,df \qquad (23)$$

where

$$\lambda = \frac{1}{2\alpha}\left[-(1-2\alpha f_0)+\sqrt{1+4\alpha(1-f_0)}\right] \qquad (24)$$

and

$$\mu = \frac{1}{2\alpha}\left[(1-2\alpha f_0)+\sqrt{1+4\alpha(1-f_0)}\right] \qquad (25)$$

Putting

$$f = \lambda\sin^2\phi-\mu\cos^2\phi \qquad (26)$$

in (22) and (23), we have

$$\int_{f_N}^{f} T_{(0)}^{\frac{1}{2}}\,df = 2\sqrt{\alpha}\,\frac{(\lambda+\mu)^2}{\sqrt{1+\mu}}\int_{\phi_0}^{\phi}\frac{\sin^2\phi\cos^2\phi}{\sqrt{1-\left(\frac{\lambda+\mu}{1+\mu}\right)\sin^2\phi}}\,d\phi \qquad (27)$$

and

$$\int_{f_N}^{f} T_{(0)}^{-\frac{1}{2}}\,df = \frac{2}{\sqrt{\alpha}}\sqrt{1+\mu}\int_{\phi_0}^{\phi}\sqrt{1-\left(\frac{\lambda+\mu}{1+\mu}\right)\sin^2\phi}\,d\phi \qquad (28)$$

where

$$\phi = \sin^{-1}\sqrt{\frac{\mu+f}{\lambda+\mu}} \qquad (29)$$

and

$$\phi_0 = \sin^{-1}\sqrt{\frac{\mu+f_N}{\lambda+\mu}} \qquad (30)$$

If we take

$$\epsilon = \alpha(1-f_0) \quad \ldots \quad (31)$$

we find that in all practical cases ϵ^2 is small and hence we can express the values of λ and μ in powers of ϵ and retain terms containing up to ϵ^2.

In the following table, we have collected the values of ϵ, ϵ^2, and ϵ^3 for $f_0 = 0{\cdot}95$, $\bar{\gamma} = 1{\cdot}3$, $\sigma = 1{\cdot}1$ for some values of M ranging from 0·5 to 4.

TABLE I

M	0·5	1	2	3	4
ϵ ..	0·003,409	0·006,818	0·013,636	0·020,455	0·027,273
ϵ^2 ..	0·000,012	0·000,047	0·000,186	0·000,418	0·000,744
ϵ^3 ..	0·000,000	0·000,000	0·000,003	0·000,009	0·000,020

Neglecting powers of ϵ higher than the second we have

$$\lambda = 1 - \frac{1}{\alpha}\epsilon^2 \quad \ldots \quad (32)$$

$$\mu = \frac{1}{\alpha}\left[(1-\alpha)+2\epsilon-\epsilon^2\right] \quad \ldots \quad (33)$$

$$T_{(0)} = \left[1+2\epsilon-\left(\frac{1+2\cos^2\phi}{\cos^2\phi}\right)\epsilon^2\right]\sin^2\phi \quad \ldots \quad (34)$$

$$\int_{f_N}^{f} T_{(0)}^{\frac{1}{2}}\,df = \frac{2}{3\alpha}(1+3\epsilon)(\sin^3\phi-\sin^3\phi_0) + \frac{\epsilon^2}{\alpha}\left[(\sin\phi-\sin\phi_0)-(\sin^3\phi-\sin^3\phi_0)-\log\frac{\tan\left(\frac{\phi}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\phi_0}{2}+\frac{\pi}{4}\right)}\right] \quad \ldots \quad (35)$$

and

$$\int_{f_N}^{f} T_{(0)}^{-\frac{1}{2}}\,df = \frac{2}{\alpha}(1+\epsilon)(\sin\phi-\sin\phi_0) + \frac{\epsilon^2}{\alpha}\left[\log\frac{\tan\left(\frac{\phi}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\phi_0}{2}+\frac{\pi}{4}\right)} - 3(\sin\phi-\sin\phi_0)\right] \quad \ldots \quad (36)$$

so that

$$T_{(1)}=\frac{2}{3\cos^2\phi}(1+\epsilon)[3\sin^2\phi\sin\phi_0-\gamma\sin^3\phi_0-(3-\gamma)\sin^3\phi]$$

$$+\frac{\epsilon^2}{3\cos^2\phi}\Bigg[-3(\gamma+\sin^2\phi)\log\frac{\tan\left(\frac{\phi}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\phi_0}{2}+\frac{\pi}{4}\right)}$$

$$+\frac{\sin\phi}{\cos^2\phi}\{2(6-\gamma)-(3-2\gamma)\cos^2\phi-3(3-\gamma)\cos^4\phi\}$$

$$-\frac{\sin\phi_0}{\cos^2\phi}\{3(\gamma\cos^2\phi_0-1)\cos^2\phi-9\cos^4\phi+12-2\gamma\sin^2\phi_0\}\Bigg] \quad .. \quad (37)$$

With the values of $T_{(0)}$ and $T_{(1)}$ given by (34) and (37) respectively, the integrals occurring in (18) with $\theta=0$ can now be evaluated:

$$\int_{f_N}^{f}T_{(1)}T_{(0)}^{-\frac{1}{2}}\,df=\frac{4}{3\alpha}(1+2\epsilon)\Bigg[(3-\gamma)\log\frac{\cos\phi}{\cos\phi_0}+(3\sin\phi_0-\gamma\sin^3\phi_0)\log\frac{\tan\left(\frac{\phi}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\phi_0}{2}+\frac{\pi}{4}\right)}$$

$$-\tfrac{1}{2}(\sin\phi-\sin\phi_0)\{(3+\gamma)\sin\phi_0-(3-\gamma)\sin\phi\}\Bigg]$$

$$+\frac{\epsilon^2}{3\alpha}\Bigg[3\Bigg\{2\sin\phi+3\gamma\sin^3\phi_0-(2\gamma+5)\sin\phi_0$$

$$-(\gamma+1)\log\frac{\tan\left(\frac{\phi}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\phi_0}{2}+\frac{\pi}{4}\right)}\Bigg\}\log\frac{\tan\left(\frac{\phi}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\phi_0}{2}+\frac{\pi}{4}\right)}$$

$$+4(\sin\phi-\sin\phi_0)\{(3+\gamma)\sin\phi_0-(3-\gamma)\sin\phi\}$$

$$+(9-\gamma)\left(\frac{1}{\cos^2\phi}-\frac{1}{\cos^2\phi_0}\right)+(9\sin\phi_0-\gamma\sin^3\phi_0)\left(\frac{\sin\phi_0}{\cos^2\phi_0}-\frac{\sin\phi}{\cos^2\phi}\right)\Bigg] \quad .. \quad (38)$$

and

$$\int_{f_N}^{f}T_{(1)}T_{(0)}^{-\frac{3}{2}}\,df=\frac{4}{3\alpha}\Bigg[(3-\gamma)\log\frac{\cos\phi}{\cos\phi_0}+(3\sin\phi_0-\gamma\sin^3\phi_0)\log\frac{\tan\left(\frac{\phi}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\phi_0}{2}+\frac{\pi}{4}\right)}$$

$$+\gamma\sin^3\phi_0\left(\frac{1}{\sin\phi}-\frac{1}{\sin\phi_0}\right)\Bigg]$$

$$+\frac{\epsilon^2}{3\alpha}\Bigg[3\left\{\frac{2\gamma}{\sin\phi}-(2\gamma+1)\sin\phi_0-\gamma\sin^3\phi_0-(\gamma+1)\log\frac{\tan\left(\frac{\phi}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\phi_0}{2}+\frac{\pi}{4}\right)}\right\}\log\frac{\tan\left(\frac{\phi}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\phi_0}{2}+\frac{\pi}{4}\right)}$$

$$+(3+\gamma)\left(\frac{1}{\cos^2\phi}-\frac{1}{\cos^2\phi_0}\right)+2\gamma(3\sin\phi_0+\sin^3\phi_0)\left(\frac{1}{\sin\phi}-\frac{1}{\sin\phi_0}\right)$$

$$+(3\sin\phi_0+\gamma\sin^3\phi_0)\left(\frac{\sin\phi_0}{\cos^2\phi_0}-\frac{\sin\phi}{\cos^2\phi}\right)\Bigg] \quad \ldots \quad (39)$$

To examine the validity of the method of expansion we have obtained the estimate of $T_{(0)}$, $T_{(1)}$, and $T_{(2)}$ in the following hypothetical case:

$$M = 2,\ \sigma = 1{\cdot}1,\ \gamma = 1{\cdot}25,\ \bar{\gamma} = 1{\cdot}3,\ f_N = 0{\cdot}97, \text{ and } f_0 = 0{\cdot}95$$

so that

$$\alpha = 0{\cdot}272{,}727 \text{ and } \epsilon = 0{\cdot}013{,}636.$$

TABLE II

f	$T_{(0)}$	$T_{(1)}$	$T_{(2)}$	N	NT'
0·9 ..	0·993,182	−0·183,112	0·281,046	0·092,834	0·090,779
0·7 ..	0·943,182	−0·255,652	0·337,213	0·271,938	0·250,526
0·5 ..	0·889,545	−0·289,998	0·337,948	0·450,379	0·389,220
0·3	0·835,390	−0·319,247	0·329,052	0·628,093	0·506,884
0·0 ..	0·753,864	−0·360,914	0·308,536	0·893,119	0·644,015

We may note from the table that the contribution of $T_{(1)}$ is negative while that of $T_{(2)}$ is positive. In the initial stage of burning say when $f = 0{\cdot}9$ for $\Psi = 0{\cdot}1$, the Ψ^2-term contributes about 15% of the Ψ-term. However, if $\Psi = 0{\cdot}5$, Ψ^2-term is about 77% of the Ψ-term. Hence, we may conclude that it will be desirable to include the Ψ^2-term in the value of T' to have a good approximation for the leaking guns, while for the recoilless guns the solution up to the Ψ^2-term is not at all reliable. We have seen that the analytical expression for the coefficient of Ψ and Ψ^2-terms become quite cumbersome and hence to have better approximation it might be easier to obtain the values of the coefficients of Ψ^3, Ψ^4, etc., which can be easily obtained from (11*b*), by actual numerical integration for a given data. We have included in Table II the values of N and NT' also as obtained from (19) and (20*b*) respectively, for this hypothetical data.

5. Coefficient of Ψ in terms of elliptical integrals for $\theta = 0$

As usual taking

$$E(k, \phi) = \int_0^{\phi} \sqrt{1-k^2 \sin^2 \phi}\, d\phi$$

and

$$F(k, \phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{1-k^2 \sin^2\phi}}$$

we have from (27) and (28)

$$\int_{f_N}^{f} T_{(0)}^{\frac{1}{2}}\, df = \tfrac{2}{3} \frac{\sqrt{\alpha}}{k^3} (\lambda+\mu)^{\frac{3}{2}}\big[(2-k^2)\{E(k, \phi)-E(k, \phi_0)\}$$
$$-2k'^2\{F(k, \phi)-F(k, \phi_0)\} -k^2(\Delta \sin \phi \cos \phi - \Delta_0 \sin \phi_0 \cos \phi_0)\big] \quad \ldots \quad (40)$$

and

$$\int_{f_N}^{f} T_{(0)}^{-\frac{1}{2}}\, df = \frac{2}{\sqrt{\alpha}} \sqrt{1+\mu}\, [E(k, \phi)-E(k, \phi_0)] \quad \ldots \quad \ldots \quad (41)$$

where

$$\Delta = \sqrt{1-k^2 \sin^2 \phi}$$
$$\Delta_0 = \sqrt{1-k^2 \sin^2 \phi}$$
$$k^2 = (\lambda+\mu)/(1+\mu)$$
$$k'^2 = 1-k^2$$

so that

$$T_{(1)} = \tfrac{2}{3} \frac{\gamma}{1-f} \frac{\sqrt{\alpha}}{k^3} (\lambda+\mu)^{3/2}\big[(2-k^2)\{E(k, \phi)-E(k, \phi_0)\} -2k'^2\{F(k, \phi)-F(k, \phi_0)\}$$
$$-k^2(\Delta \sin \phi \cos \phi - \Delta_0 \sin \phi_0 \cos \phi_0)\big]-2\frac{T_{(0)}}{1-f} \frac{\sqrt{1+\mu}}{\sqrt{\alpha}}[E(k, \phi)-E(k, \phi_0)] \quad \ldots \quad (42)$$

6. General case $\theta \neq 0$

Restricting the discussion to the leaking guns only, we have in this section considered the general case when θ is not necessarily equal to zero. However, in the present case the analytical expression for the coefficient of Ψ becomes very lengthy. Consequently we have given the expressions for the integrals in (17) in analytical form retaining up to ϵ^2. From these we obtain the value of $T_{(1)}$ in the analytical form which when substituted in (18) will give the value of $T_{(2)}$.

Retaining terms up to ϵ^2

$$\int T^{\frac{1}{2}}_{(0)}\,df = \int \sqrt{\frac{(1+\theta f)-\alpha(1-f)}{1+\theta f}}\,df + \epsilon \int \frac{df}{\sqrt{1+\theta f}\,\sqrt{(1+\theta f)-\alpha(1-f)}}$$

$$- \frac{1}{2\alpha}\epsilon^2 \int \frac{\sqrt{1+\theta f}\,df}{(1-f)\left[(1+\theta f)-\alpha(1-f)\right]^{3/2}} \quad \ldots \quad (43a)$$

$$= \frac{1}{\theta}\sqrt{1+\theta f}\,\sqrt{(1+\theta f)-\alpha(1-f)}$$

$$- \frac{\alpha(1+\theta)-2\theta\epsilon}{2\theta^{3/2}\sqrt{\alpha+\theta}} \log\left[2\theta(\alpha+\theta)f+\theta(1-\alpha)+(\alpha+\theta) + 2\sqrt{\theta(\alpha+\theta)}\,\sqrt{1+\theta f}\,\sqrt{(1+\theta f)-\alpha(1-f)}\right]$$

$$- \frac{\epsilon^2}{2\alpha(1+\theta)}\left[\log\frac{1}{(1-f)}\left\{2(1+\theta f)-\alpha(1-f)+2\sqrt{1+\theta f}\,\sqrt{(1+\theta f)-\alpha(1-f)}\right\} - 2\frac{\sqrt{1+\theta f}}{\sqrt{(1+\theta f)-\alpha(1-f)}}\right] \quad \ldots \quad (43b)$$

$$\int T^{-\frac{1}{2}}_{(0)}\,df = \int \sqrt{\frac{1+\theta f}{(1+\theta f)-\alpha(1-f)}}\,df - \epsilon \int \frac{\sqrt{1+\theta f}}{\left[(1+\theta f)-\alpha(1-f)\right]^{3/2}}\,df$$

$$+ \frac{\epsilon^2}{2\alpha}\int \frac{\sqrt{1+\theta f}}{(1-f)\left[(1+\theta f)-\alpha(1-f)\right]^{3/2}}\,df + \tfrac{3}{2}\epsilon^2 \int \frac{\sqrt{1+\theta f}}{\left[(1+\theta f)-\alpha(1-f)\right]^{5/2}}\,df \quad \ldots \quad (44a)$$

$$= \frac{1}{\alpha+\theta}\,\frac{\sqrt{1+\theta f}\left[2\epsilon+(1+\theta f)-\alpha(1-f)\right]}{\sqrt{(1+\theta f)-\alpha(1-f)}}$$

$$+ \frac{\alpha(1+\theta)-2\theta\epsilon}{2\sqrt{\theta}\,(\alpha+\theta)^{3/2}} \log\left[2\theta(\alpha+\theta)f+\theta(1-\alpha)+(\alpha+\theta) + 2\sqrt{\theta(\alpha+\theta)}\,\sqrt{1+\theta f}\,\sqrt{(1+\theta f)-\alpha(1-f)}\right]$$

$$+ \frac{\epsilon^2}{2\alpha(1+\theta)}\left[\log\frac{1}{(1-f)}\left\{2(1+\theta f)-\alpha(1-f)+2\sqrt{(1+\theta f)-\alpha(1-f)}\sqrt{1+\theta f}\right\} - 2\sqrt{\frac{1+\theta f}{(1+\theta f)-\alpha(1-f)}}\right]$$

$$- \frac{\epsilon^2}{2\alpha(1+\theta)(\alpha+\theta)^3\left[(1+\theta f)\left\{(1+\theta f)-\alpha(1-f)\right\}\right]^{3/2}} \times$$

$$\Big[6\theta(\alpha+\theta)\left\{3(\alpha+\theta)(1+\theta f)-\alpha(1+\theta)\right\}\left\{(1+\theta f)-\alpha(1-f)\right\}(1+\theta f)$$

$$+\alpha^3(1+\theta)^3\left\{2\theta(\alpha+\theta)f+\theta(1-\alpha)+(\alpha+\theta)\right\}$$

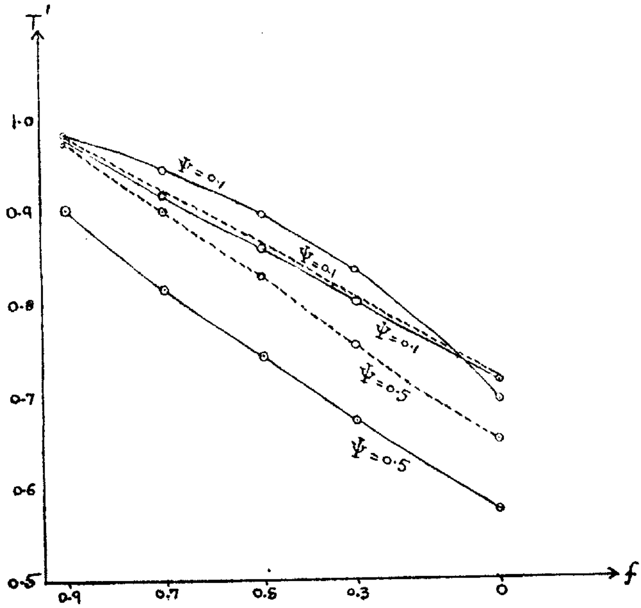
$$\left\{3\alpha^2(1+\theta)^2-2(2\theta(\alpha+\theta)f+\theta(1-\alpha)+(\alpha+\theta))^2\right\}\Big] \quad \ldots \quad (44b)$$

In order to have some idea of the effect of not taking $\theta = 0$, we collect in Table III the values of $T_{(0)}$ and $T_{(1)}$ for $\theta = 1$ when $f = 0{\cdot}9$, $0{\cdot}7$, $0{\cdot}5$, $0{\cdot}3$ and $0{\cdot}0$.

TABLE III

f	$T_{(0)}$	$T_{(1)}$
0·9	0·996,412	—0·151,330
0·7 ..	0·966,578	—0·229,438
0·5 ..	0·926,364	—0·290,598
0·3 ..	0·873,377	—0·371,619
0·0 ..	0·753,864	—0·583,360

The curves in the following figure exhibit the relation between T' and f. The top curve refers to $\theta = 1$ while the remaining four curves refer to $\theta = 0$. The complete curves represent T' correct to first power of Ψ while the dotted curves represent T' correct to Ψ^2-term. We may note the considerable deviation of the dotted curve from the corresponding continuous curve for $\Psi = 0{\cdot}5$.



SUMMARY

In the present paper the method of Thiruvenkatachar and Venkatesan for internal ballistics of a leaking gun has been extended and the coefficient of Ψ^2 has been obtained explicitly in an analytical form. By applying it to a hypothetical example, it has been shown that the contribution of Ψ^2-term can be substantial and it is not safe to neglect it especially in the case of recoilless guns.

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