TRIGONOMETRICAL SERIES METHOD IN SOLVING THE PROBLEM OF SOFTENING AND DEGREE OF POLARIZATION OF RADIATION IN AN ELECTRON ATMOSPHERE, SCATTERING ACCORDING TO RAYLEIGH'S LAW AND INVOLVING COMPTON CHANGE IN WAVELENGTH

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1. INTRODUCTION

Chandrasekhar (1948), in his paper on the problem of softening of radiation in a free electron atmosphere in the isotropic, conservative, plane-parallel case, suggested that a strict theory of this problem should make use of Rayleigh's phase function and should take account of the polarization of the radiation field. But as the calculations were only of the first order, consideration of such fineness of field property was thought unnecessary for obtaining the intensity distribution.

In the present paper, the transfer equation for axially symmetric plane-parallel, electron scattering atmosphere has been solved, not merely for the purpose of studying the modification of radiation of particular distribution, but also for determining the distribution of the degree of polarization of the radiation emerging from the outer boundary of the atmosphere. The phase function is supposed to be one of Rayleigh's type, and the coefficient of scattering is assumed to be independent of wavelength as in Thomson scattering and the Compton change in wavelength is taken into account. A Taylor's expansion of the scattered intensity is made in powers of γ (Compton wavelength), and only the term proportional to γ is retained for subsequent calculations. The boundary conditions are as follows: (*a*) there is no incident radiation from outside at the outer surface, and (*b*) the lower boundary is the photospheric surface in the form of infinite plane which is supposed to radiate with a known spectral distribution, and also the outward radiation at the same surface is taken to be entirely unpolarized. In the first approximation, the integro-differential equation of transfer is replaced by a system of two linear equations by Chandrasekhar's method of discrete ordinates. Two simultaneous partial differential equations with constant coefficients are obtained, and these are solved by the method of trigonometrical series. The intensity distribution and the distribution of the degree of polarization of the emergent radiation are calculated, and the former is compared with the corresponding distribution for the field where polarization is ignored in the same approximation. The calculations are carried through for two different types of intensity distribution at the lower boundary of the atmosphere, viz. that for monochromatic radiation, and a Gaussian distribution of intensity.

2. THE EQUATIONS AND BOUNDARY CONDITIONS

The equation of transfer appropriate to the problem of plane-parallel, electron scattering atmosphere, having axial symmetry and with no incident radiation from outside, may be written as

$$\mu\frac{\partial}{\partial\tau}\begin{pmatrix}I_l(\tau,\mu,\lambda)\\ I_r(\tau,\mu,\lambda)\end{pmatrix}=\begin{pmatrix}I_l(\tau,\mu,\lambda)\\ I_r(\tau,\mu,\lambda)\end{pmatrix}-\frac{1}{4\pi}\int_{-1}^{+1}\int_{0}^{2\pi}P(\mu,0;\mu',\phi')\times$$
$$\times\begin{pmatrix}I_l(\tau,\mu',\lambda-\gamma[1-\cos\theta])\\ I_r(\tau,\mu',\lambda-\gamma[1-\cos\theta])\end{pmatrix}d\mu'\,d\phi';\ldots \quad (1)$$

here $P(\mu, 0; \mu', \phi')$ is the phase matrix for an atmosphere scattering according to Rayleigh's law and is given by

$$P(\mu,0;\mu',\phi')=Q[P^{(0)}(\mu,\mu')+(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}P^{(1)}(\mu,0;\mu',\phi')+$$
$$+P^{(2)}(\mu,0;\mu',\phi')] \quad \ldots \quad (2)$$

where

$$P^{(0)}(\mu,\mu')=\frac{3}{4}\begin{pmatrix}2(1-\mu^2)(1-\mu'^2)+\mu^2\mu'^2 & \mu^2\\ \mu'^2 & 1\end{pmatrix} \quad \ldots \quad \ldots \quad (3)$$

$$P^{(1)}(\mu,0;\mu',\phi')=\frac{3}{4}\begin{pmatrix}4\mu\mu'\cos\phi' & 0\\ 0 & 0\end{pmatrix} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

$$P^{(2)}(\mu,0;\mu',\phi')=\frac{3}{4}\begin{pmatrix}\mu^2\mu'^2\cos 2\phi' & -\mu^2\cos 2\phi'\\ -\mu'^2\cos 2\phi' & \cos 2\phi'\end{pmatrix}\ldots \quad \ldots \quad (5)$$

and

$$Q=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

and γ, the Compton wavelength is given by

$$\gamma=\frac{h}{mc}=0{\cdot}024\,\mathring{A} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

$\mu = \cos\vartheta$, and τ, the optical thickness

$$\tau=\int_{z}^{\infty}\sigma\rho\,dz \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

where ρ is the density and σ, the scattering coefficient.

$I(\tau, \mu, \lambda)$ is the specific intensity of radiation of wavelength λ in the direction ϑ to the outward drawn normal at a depth τ. θ, the angle of scattering, is given by

$$\cos\theta=\mu\mu'+(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos\phi' \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

The form of specific intensity inside the integral representing the source function is due to the fact that a radiation of wavelength $\lambda-\gamma[1-\cos\theta]$ in the direction (μ', ϕ') after being scattered in the direction $(\mu, 0)$ through an angle θ will have the wavelength λ.

We shall suppose that $I(\tau, \mu', \lambda-\gamma[1-\cos\theta])$ is capable of expansion into Taylor's series, so that

$$I(\tau,\mu',\lambda-\gamma[1-\cos\theta])=I(\tau,\mu',\lambda)-\gamma(1-\cos\theta)\frac{\partial I(\tau,\mu',\lambda)}{\partial\lambda}+$$
$$+\frac{\gamma^2}{2}(1-\cos\theta)^2\frac{\partial^2 I(\tau,\mu',\lambda)}{\partial\lambda^2}-\cdots \quad \ldots \quad (10)$$

Retaining the term proportional to γ in the expansion (10) the equation of transfer may be written in the form

$$\mu \frac{\partial}{\partial \tau}\begin{pmatrix} I_l(\tau, \mu, \lambda) \\ I_r(\tau, \mu, \lambda) \end{pmatrix} = \begin{pmatrix} I_l(\tau, \mu, \lambda) \\ I_r(\tau, \mu, \lambda) \end{pmatrix} - \frac{1}{4\pi}\int_{-1}^{+1}\int_{0}^{2\pi} P(\mu, 0; \mu', \phi') \times$$

$$\times \begin{pmatrix} I_l(\tau, \mu', \lambda-\gamma[1-\cos\theta]) \\ I_r(\tau, \mu', \lambda-\gamma[1-\cos\theta]) \end{pmatrix} d\mu' \, d\phi' \quad .. \quad (11)$$

where

$$\left.\begin{aligned} I_l(\tau, \mu', \lambda-\gamma[1-\cos\theta]) &= I_l(\tau, \mu', \lambda)-\gamma(1-\cos\theta)\frac{\partial I_l(\tau, \mu', \lambda)}{\partial \lambda} \\ I_r(\tau, \mu', \lambda-\gamma[1-\cos\theta]) &= I_r(\tau, \mu', \lambda)-\gamma(1-\cos\theta)\frac{\partial I_r(\tau, \mu', \lambda)}{\partial \lambda} \end{aligned}\right\} \quad .. \quad (12)$$

and $P(\mu, 0; \mu', \phi')$ is the phase matrix given by equations (2), (3), (4), (5), and (6). Therefore we get the following two equations for $I_l(\tau, \mu, \lambda)$ and $I_r(\tau, \mu, \lambda)$. (Henceforward I_l and I_r will be used to mean $I_l(\tau, \mu, \lambda)$ and $I_r(\tau, \mu, \lambda)$.)

$$\mu \frac{\partial I_l}{\partial \tau} = I_l - \frac{3}{8}\int_{-1}^{+1}\Big[\{2(1-\mu^2)(1-\mu'^2)+\mu^2\mu'^2\}I_l-\gamma(1-\mu\mu')\{2(1-\mu^2)(1-\mu'^2)+$$

$$\mu^2\mu'^2\}\frac{\partial I_l}{\partial \lambda}+\mu^2 I_r-\gamma\mu^2(1-\mu\mu')\frac{\partial I_r}{\partial \lambda}+2\gamma(1-\mu^2)(1-\mu'^2)\mu\mu'\frac{\partial I_l}{\partial \lambda}\Big]\,d\mu' \quad .. \quad (13)$$

and

$$\mu \frac{\partial I_r}{\partial \tau} = I_r - \frac{3}{8}\int_{-1}^{+1}\Big[\mu'^2 I_l-\gamma(1-\mu\mu')\mu'^2\frac{\partial I_l}{\partial \lambda}+I_r-\gamma(1-\mu\mu')\frac{\partial I_r}{\partial \lambda}\Big]\,d\mu' \quad .. \quad (14)$$

Replacing the integrals by Gaussian sums according to Chandrasekhar's method each of the equations (13) and (14) may be replaced by $2n$ linear equations as shown below:—

$$\mu_i \frac{\partial I_{l,i}}{\partial \tau} = I_{l,i} - \frac{3}{8}\Big[\sum_j a_j\{2(1-\mu_i^2)(1-\mu_j^2)+\mu_i^2\mu_j^2\}I_{l,j}-$$

$$-\sum_j a_j\gamma(1-\mu_i\mu_j)\{2(1-\mu_i^2)(1-\mu_j^2)+\mu_i^2\mu_j^2\}\frac{\partial I_{l,j}}{\partial \lambda}+$$

$$+\sum_j a_j\mu_i^2 I_{r,j}-\sum_j a_j\gamma\mu_i^2(1-\mu_i\mu_j)\frac{\partial I_{r,j}}{\partial \lambda}+$$

$$+\sum_j 2a_j\gamma(1-\mu_i^2)(1-\mu_j^2)\mu_i\mu_j\frac{\partial I_{l,j}}{\partial \lambda}\Big] \quad .. \quad .. \quad .. \quad (15)$$

and

$$\mu_i \frac{\partial I_{r,i}}{\partial \tau} = I_{r,i} - \frac{3}{8}\Big[\sum_j a_j I_{l,j}\mu_j^2-\sum a_j\gamma(1-\mu_i\mu_j)\mu_j^2\frac{\partial I_{l,j}}{\partial \lambda}+$$

$$+\sum_j a_j I_{r,j}-\sum_j a_j\gamma(1-\mu_i\mu_j)\frac{\partial I_{r,j}}{\partial \lambda}\Big] \quad .. \quad .. \quad .. \quad (16)$$

$$[i = \pm 1, \pm 2 \ldots . \pm n \text{ and } j = \pm 1, \pm 2 \ldots . \pm n]$$

where μ_i's are the zeros of Legendre polynomial $P_{2n}(\mu)$, and a_j's are the appropriate Gaussian weights,

$$a_j = a_{-j},\ \mu_j = -\mu_j \quad \ldots \quad (17)$$

If we restrict ourselves to the first approximation only, we can write down

$$a_{+1} = a_{-1} = 1, \text{ and } \mu_{+1} = -\mu_{-1} = \frac{1}{\sqrt{3}} \quad \ldots \quad (18)$$

Then the equations (15) and (16) take the following forms :—

$$\frac{1}{\sqrt{3}}\frac{\partial I_{l(+1)}}{\partial \tau} = \frac{5}{8}I_{l(+1)} - \frac{3}{8}I_{l(-1)} + \frac{5}{36}\gamma\frac{\partial I_{l(+1)}}{\partial \lambda} + \frac{11}{18}\gamma\frac{\partial I_{l(-1)}}{\partial \lambda} - $$
$$-\frac{1}{8}\left(I_{r(+1)} + I_{r(-1)}\right) + \frac{\gamma}{12}\frac{\partial I_{r(+1)}}{\partial \lambda} + \frac{\gamma}{6}\frac{\partial I_{r(-1)}}{\partial \lambda} \quad \ldots \quad (19)$$

$$-\frac{1}{\sqrt{3}}\frac{\partial I_{l(-1)}}{\partial \tau} = -\frac{3}{8}I_{l(+1)} + \frac{5}{8}I_{l(-1)} + \frac{11}{18}\gamma\frac{\partial I_{l(+1)}}{\partial \lambda} + \frac{5}{36}\gamma\frac{\partial I_{l(-1)}}{\partial \lambda} - $$
$$-\frac{1}{8}\left(I_{r(+1)} + I_{r(-1)}\right) + \frac{\gamma}{6}\frac{\partial I_{r(+1)}}{\partial \lambda} + \frac{\gamma}{12}\frac{\partial I_{r(-1)}}{\partial \lambda} \quad \ldots \quad (20)$$

$$\frac{1}{\sqrt{3}}\frac{\partial I_{r(+1)}}{\partial \tau} = -\frac{1}{8}\left(I_{l(+1)} + I_{l(-1)}\right) + \frac{\gamma}{12}\frac{\partial I_{l(+1)}}{\partial \lambda} + \frac{\gamma}{6}\frac{\partial I_{l(-1)}}{\partial \lambda} + $$
$$+\frac{5}{8}I_{r(+1)} - \frac{3}{8}I_{r(-1)} + \frac{\gamma}{4}\frac{\partial I_{r(+1)}}{\partial \lambda} + \frac{\gamma}{2}\frac{\partial I_{r(-1)}}{\partial \lambda} \quad \ldots \quad (21)$$

$$-\frac{1}{\sqrt{3}}\frac{\partial I_{r(-1)}}{\partial \tau} = -\frac{1}{8}\left(I_{l(+1)} + I_{l(-1)}\right) + \frac{\gamma}{6}\frac{\partial I_{l(+1)}}{\partial \lambda} + \frac{\gamma}{12}\frac{\partial I_{l(-1)}}{\partial \lambda} - $$
$$-\frac{3}{8}I_{r(+1)} + \frac{5}{8}I_{r(-1)} + \frac{\gamma}{2}\frac{\partial I_{r(+1)}}{\partial \lambda} + \frac{\gamma}{4}\frac{\partial I_{r(-1)}}{\partial \lambda} \quad \ldots \quad (22)$$

It is to be remembered that $I_{l(+1)}$ and $I_{r(+1)}$ denote the intensities in the first approximation in the outward direction and $I_{l(-1)}$ and $I_{r(-1)}$, those in the inward direction.

Now introducing the variables

$$x = \frac{3}{2}\tau \text{ and } y = \frac{3}{2\gamma}(\lambda - \lambda_0) \quad \ldots \quad (23)$$

where λ_0 is some suitably chosen wavelength of constant value, equations (19), (20), (21), (22) can be written as

$$\frac{\sqrt{3}}{2}\frac{\partial I_{l(+1)}}{\partial x} = \frac{5}{8}I_{l(+1)} - \frac{3}{8}I_{l(-1)} + \frac{5}{24}\frac{\partial I_{l(+1)}}{\partial y} + \frac{11}{12}\frac{\partial I_{l(-1)}}{\partial y} - $$
$$-\frac{1}{8}\left(I_{r(+1)} + I_{r(-1)}\right) + \frac{1}{8}\frac{\partial I_{r(+1)}}{\partial y} + \frac{1}{4}\frac{\partial I_{r(-1)}}{\partial y} \quad \ldots \quad (24)$$

$$\frac{\sqrt{3}}{2}\frac{\partial I_{l(-1)}}{\partial x} = \frac{3}{8}I_{l(+1)} - \frac{5}{8}I_{l(-1)} - \frac{11}{12}\frac{\partial I_{l(+1)}}{\partial y} - \frac{5}{24}\frac{\partial I_{l(-1)}}{\partial y} + $$
$$+\frac{1}{8}\left(I_{r(+1)} + I_{r(-1)}\right) - \frac{1}{4}\frac{\partial I_{r(+1)}}{\partial y} - \frac{1}{8}\frac{\partial I_{r(-1)}}{\partial y} \quad \ldots \quad (25)$$

$$\frac{\sqrt{3}}{2}\frac{\partial I_{r(+1)}}{\partial x} = -\frac{1}{8}\left(I_{l(+1)}+I_{l(-1)}\right)+\frac{1}{8}\frac{\partial I_{l(+1)}}{\partial y}+\frac{1}{4}\frac{\partial I_{l(-1)}}{\partial y}+$$
$$+\frac{3}{8}\frac{\partial I_{r(+1)}}{\partial y}+\frac{3}{4}\frac{\partial I_{r(-1)}}{\partial y}+\frac{5}{8}I_{r(+1)}-\frac{3}{8}I_{r(-1)} \quad \ldots \quad \ldots \quad (26)$$

$$\frac{\sqrt{3}}{2}\frac{\partial I_{r(-1)}}{\partial x} = \frac{1}{8}\left(I_{l(+1)}+I_{l(-1)}\right)-\frac{1}{4}\frac{\partial I_{l(+1)}}{\partial y}-\frac{1}{8}\frac{\partial I_{l(-1)}}{\partial y}+$$
$$+\frac{3}{8}I_{r(+1)}-\frac{5}{8}I_{r(-1)}-\frac{3}{4}\frac{\partial I_{r(+1)}}{\partial y}-\frac{3}{8}\frac{\partial I_{r(-1)}}{\partial y}. \quad \ldots \quad (27)$$

Rearranging these equations and remembering that

$$\left.\begin{array}{ll} K_l = I_{l(+1)}+I_{l(-1)}; & K_r = I_{r(+1)}+I_{r(-1)} \\ H_l = I_{l(+1)}-I_{l(-1)}; & H_r = I_{r(+1)}-I_{r(-1)} \end{array}\right\} \quad \ldots \quad \ldots \quad (28)$$

we get the following four equations:—

$$\frac{\sqrt{3}}{2}\frac{\partial K_l}{\partial x} = H_l-\frac{17}{24}\frac{\partial H_l}{\partial y}-\frac{1}{8}\frac{\partial H_r}{\partial y} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

$$\frac{\sqrt{3}}{2}\frac{\partial H_l}{\partial x} = \frac{1}{4}K_l+\frac{9}{8}\frac{\partial K_l}{\partial y}-\frac{1}{4}K_r+\frac{3}{8}\frac{\partial K_r}{\partial y} \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

$$\frac{\sqrt{3}}{2}\frac{\partial K_r}{\partial x} = -\frac{1}{8}\frac{\partial H_l}{\partial y}+H_r-\frac{3}{8}\frac{\partial H_r}{\partial y} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

$$\frac{\sqrt{3}}{2}\frac{\partial H_r}{\partial x} = -\frac{1}{4}K_l+\frac{1}{4}K_r+\frac{3}{8}\frac{\partial K_l}{\partial y}+\frac{9}{8}\frac{\partial K_r}{\partial y} \quad \ldots \quad \ldots \quad (32)$$

Again putting

$$\left.\begin{array}{ll} K_l+K_r = K^+; & H_l+H_r = H^+ \\ K_l-K_r = K^-; & H_l-H_r = H^- \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

and substituting the values of K_l, K_r, H_l and H_r in equations (29), (30), (31), and (32) from (33) and rearranging the equations we get

$$\frac{\sqrt{3}}{2}\frac{\partial K^+}{\partial x}+\frac{2}{3}\frac{\partial H^+}{\partial y}+\frac{1}{6}\frac{\partial H^-}{\partial y} = H^+ \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

$$\frac{\sqrt{3}}{2}\frac{\partial K^-}{\partial x}+\frac{1}{6}\frac{\partial H^+}{\partial y}+\frac{5}{12}\frac{\partial H^-}{\partial y} = H^- \quad \ldots \quad \ldots \quad \ldots \quad (35)$$

$$\frac{\partial H^+}{\partial x} = \sqrt{3}\frac{\partial K^+}{\partial y} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

$$\sqrt{3}\frac{\partial H^-}{\partial x} = \frac{3}{2}\frac{\partial K^-}{\partial y}+K^- \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

To satisfy equation (36) we write

$$K^+ = \frac{\partial S(x, y)}{\partial x} \text{ and } H^+ = \sqrt{3}\frac{\partial S(x, y)}{\partial y} \quad \ldots \quad \ldots \quad (38)$$

and to satisfy equation (37) we write

$$K^- = \sqrt{3}\frac{\partial F(x, y)}{\partial x} \text{ and } H^- = \left[F(x, y)+\frac{3}{2}\frac{\partial F(x, y)}{\partial y}\right] \quad \ldots \quad \ldots \quad (39)$$

Substituting (38) and (39) in (34) and (35), we get

$$\frac{\partial F}{\partial y}+\frac{3}{2}\frac{\partial^2 F}{\partial y^2} = -3\sqrt{3}\frac{\partial^2 S}{\partial x^2}-4\sqrt{3}\frac{\partial^2 S}{\partial y^2}+6\sqrt{3}\frac{\partial S}{\partial y} \quad \ldots \quad \ldots \quad (40)$$

$$\frac{\partial^2 S}{\partial y^2} = 2\sqrt{3}F+\frac{13\sqrt{3}}{6}\frac{\partial F}{\partial y}-\frac{5\sqrt{3}}{4}\frac{\partial^2 F}{\partial y^2}-3\sqrt{3}\frac{\partial^2 F}{\partial x^2} \quad \ldots \quad \ldots \quad (41)$$

These two simultaneous partial differential equations are to be solved under the proper boundary conditions.

The boundary conditions are assumed as follows :—

At the lower boundary denoted by $\tau = \tau_1$, and $x = x_1$

(i) the outward intensity is supposed to have a given spectral distribution,

(ii) radiation is supposed to be unpolarized.

Again at the upper boundary denoted by $\tau = 0$, $x = 0$,

(iii) inward intensity is supposed to be absent, i.e. both $I_{l(-1)}(0, y)$ and $I_{r(-1)}(0, y)$ are separately equal to zero.

These boundary conditions are equivalent to the following :—

(i) (from equations (38), (33) and (28))

$$I_{l(+1)}(x_1, y)+I_{r(+1)}(x_1, y) = \frac{1}{2}\left(K^{+}+H^{+}\right) = \frac{1}{2}\left[\frac{\partial S}{\partial x}+\sqrt{3}\frac{\partial S}{\partial y}\right]_{x=x_1}$$

$$= \text{a known distribution in } y = \psi(y) \quad \ldots \quad (42)$$

(ii) (from equations (39), (33) and (28))

$$I_{l(+1)}(x_1, y)-I_{r(+1)}(x_1, y) = \frac{1}{2}\left(K^{-}+H^{-}\right) =$$

$$= \frac{1}{2}\left[\sqrt{3}\frac{\partial F}{\partial x}+F+\frac{3}{2}\frac{\partial F}{\partial y}\right]_{x=x_1} = 0 \quad \ldots \quad (43)$$

(iii) (*a*) (from equations (38), (33) and (28))

$$I_{l(-1)}(0, y)+I_{r(-1)}(0, y) = \frac{1}{2}\left(K^{+}-H^{+}\right) =$$

$$= \frac{1}{2}\left[\frac{\partial S}{\partial x}-\sqrt{3}\frac{\partial S}{\partial y}\right]_{x=0} = 0 \quad \ldots \quad \ldots \quad (44)$$

and (*b*) (from equations (39), (33) and (28))

$$I_{l(-1)}(0, y)-I_{r(-1)}(0, y) = \frac{1}{2}\left(K^{-}-H^{-}\right) =$$

$$= \frac{1}{2}\left[\sqrt{3}\frac{\partial F}{\partial x}-F-\frac{3}{2}\frac{\partial F}{\partial y}\right]_{x=0} = 0 \quad \ldots \quad (45)$$

3. Solution

The boundary value problem formulated in art. 2 can be solved by the method of trigonometrical series.

Let us take as trial solution of (40) and (41)

$$S(x, y) = A_0(x^2+y)+B_0x+Ae^{mx+iny} \quad \ldots \quad (46)$$

and

$$F(x, y) = Ee^{mx+iny} \quad \ldots \quad (47)$$

Substituting these in equations (40) and (41), we get

$$\frac{E}{A}\left[in-\frac{3}{2}n^2\right] = 6\sqrt{3}in-3\sqrt{3}m^2+4\sqrt{3}n^2 \quad \ldots \quad (48)$$

and

$$-\frac{A}{E}n^2 = 2\sqrt{3}+\frac{13\sqrt{3}}{6}in+\frac{5\sqrt{3}}{4}n^2-3\sqrt{3}m^2 \quad \ldots \quad (49)$$

From the product of (48) and (49), we obtain a relation between m and n, which is given below

$$27m^4-3m^2\left(6+\frac{49}{2}in+\frac{63}{4}n^2\right)+\left(\frac{27}{2}n^4+\frac{99}{2}in^3-15n^2+36in\right) = 0 \quad \ldots \quad (50)$$

which leads to four values of m, for each value of n, viz.

$$\pm(q_n+ir_n) \text{ and } \pm(q_n'+ir_n') \text{ for positive values of } n, \quad \ldots \quad (51)$$

$$\text{and } \pm(q_n-ir_n) \text{ and } \pm(q_n'-ir_n') \text{ for negative values of } n. \quad \ldots \quad (52)$$

Here

$$q_n = \left[\frac{1}{36}\left\{\sqrt{\left(\frac{63}{4}n^2+6+Q_n\right)^2+\left(\frac{49}{2}n+R_n\right)^2}+\left(\frac{63}{4}n^2+6+Q_n\right)\right\}\right]^{\frac{1}{2}} \quad \ldots \quad (53)$$

$$r_n = \left[\frac{1}{36}\left\{\sqrt{\left(\frac{63}{4}n^2+6+Q_n\right)^2+\left(\frac{49}{2}n+R_n\right)^2}-\left(\frac{63}{4}n^2+6+Q_n\right)\right\}\right]^{\frac{1}{2}} \quad \ldots \quad (54)$$

$$q_n' = \left[\frac{1}{36}\left\{\sqrt{\left(\frac{63}{4}n^2+6-Q_n\right)^2+\left(\frac{49}{2}n-R_n\right)^2}+\left(\frac{63}{4}n^2+6-Q_n\right)\right\}\right]^{\frac{1}{2}} \quad \ldots \quad (55)$$

$$r_n' = \left[\frac{1}{36}\left\{\sqrt{\left(\frac{63}{4}n^2+6-Q_n\right)^2+\left(\frac{49}{2}n-R_n\right)^2}-\left(\frac{63}{4}n^2+6-Q_n\right)\right\}\right]^{\frac{1}{2}} \quad \ldots \quad (56)$$

where

$$Q_n = \left[\frac{1}{2}\left\{\sqrt{\left(\frac{1377}{16}n^4-\frac{925}{4}n^2+36\right)^2+\left(\frac{711}{4}n^3-138n\right)^2}+\left(\frac{1377}{16}n^4-\frac{925}{4}n^2+36\right)\right\}\right]^{\frac{1}{2}} \quad \ldots \quad (57)$$

$$R_n = \left[\frac{1}{2}\left\{\sqrt{\left(\frac{1377}{16}n^4-\frac{925}{4}n^2+36\right)^2+\left(\frac{711}{4}n^3-138n\right)^2}-\left(\frac{1377}{16}n^4-\frac{925}{4}n^2+36\right)\right\}\right]^{\frac{1}{2}} \quad \ldots \quad (58)$$

The general solutions of (40) and (41) may be obtained by considering all possible values of n. So they can be written as (from equations (46), (47), (51), (52))

$$S(x, y) = A_0(x^2+y)+B_0x+\sum_{n=0}^{\infty}\Big\{A_{n1}e^{(q_n+ir_n)x+iny}+A_{n2}e^{-(q_n+ir_n)x+iny}+$$
$$+A_{n3}e^{(q_n-ir_n)x-iny}+A_{n4}e^{-(q_n-ir_n)x-iny}+$$
$$+A'_{n1}e^{(q'_n+ir'_n)x+iny}+A'_{n2}e^{-(q'_n+ir'_n)x+iny}+$$
$$+A'_{n3}e^{(q'_n-ir'_n)x-iny}+A'_{n4}e^{-(q'_n-ir'_n)x-iny}\Big\} \quad (59)$$

and

$$F(x, y) = \sum_{n=0}^{\infty}\Big\{E_{n1}e^{(q_n+ir_n)x+iny}+E_{n2}e^{-(q_n+ir_n)x+iny}+$$
$$+E_{n3}e^{(q_n-ir_n)x-iny}+E_{n4}e^{-(q_n-ir_n)x-iny}+$$
$$+E'_{n1}e^{(q'_n+ir'_n)x+iny}+E'_{n2}e^{-(q'_n+ir'_n)x+iny}+$$
$$+E'_{n3}e^{(q'_n-ir'_n)x-iny}+E'_{n4}e^{-(q'_n-ir'_n)x-iny}\Big\}. \quad .. \quad (60)$$

We write the solutions in the real form as

$$S(x, y) = A_0(x^2+y)+B_0x+\sum_{n=0}^{\infty}\Big[e^{q_nx}\Big\{a_n\cos(r_nx+ny)+b_n\sin(r_nx+ny)\Big\}+$$
$$+e^{-q_nx}\Big\{c_n\cos(r_nx-ny)+d_n\sin(r_nx-ny)\Big\}+$$
$$+e^{q'_nx}\Big\{a'_n\cos(r'_nx+ny)+b'_n\sin(r'_nx+ny)\Big\}+$$
$$+e^{-q'_nx}\Big\{c'_n\cos(r'_nx-ny)+d'_n\sin(r'_nx-ny)\Big\}\Big] \quad .. \quad (61)$$

and

$$F(x, y) = \sum_{n=0}^{\infty}\Big[e^{q_nx}\Big\{f_n\cos(r_nx+ny)+g_n\sin(r_nx+ny)\Big\}+$$
$$+e^{-q_nx}\Big\{h_n\cos(r_nx-ny)+k_n\sin(r_nx-ny)\Big\}+$$
$$+e^{q'_nx}\Big\{f'_n\cos(r'_nx+ny)+g'_n\sin(r'_nx+ny)\Big\}+$$
$$+e^{-q'_nx}\Big\{h'_n\cos(r'_nx-ny)+k'_n\sin(r'_nx-ny)\Big\}\Big] \quad .. \quad (62)$$

Substitution of equations (61) and (62) in the differential equations (40) and (41) shows that the terms corresponding to $n = 0$ will not occur in the solutions.

Therefore the general solutions are given by

$$S(x, y) = A_0(x^2+y)+B_0x+\sum_{n=1}^{\infty}\Big[\qquad\Big] \quad \ldots \quad \ldots \quad (63)$$

and

$$F(x, y) = \sum_{n=1}^{\infty}\Big[\qquad\Big] \quad \ldots \quad \ldots \quad \ldots \quad (64)$$

The quantities within the brackets [ ] in (63) and (64) are the same as those in (61) and (62) except for the fact that the terms corresponding to $n = 0$ have been omitted. n is supposed to have only positive integral values.

4. Evaluation of Coefficients

Now substituting the values of x and y derivatives of $S(x, y)$ and $F(x, y)$ obtained from (63) and (64) in equation (40), we can obtain the relation between the sets of coefficients representing $F(x, y)$ and those representing $S(x, y)$. The following relations are thus obtained:

$$f_n = a_n\alpha_{n2}-b_n\alpha_{n1} \quad \ldots \quad (65)$$

$$g_n = a_n\alpha_{n1}+b_n\alpha_{n2} \quad \ldots \quad (66)$$

$$h_n = c_n\alpha_{n2}+d_n\alpha_{n1} \quad \ldots \quad (67)$$

$$k_n = -c_n\alpha_{n1}+d_n\alpha_{n2} \quad \ldots \quad (68)$$

$$f_n' = a_n'\alpha_{n2}'-b_n'\alpha_{n1}' \quad \ldots \quad (69)$$

$$g_n' = a_n'\alpha_{n1}'+b_n'\alpha_{n2}' \quad \ldots \quad (70)$$

$$h_n' = c_n'\alpha_{n2}'+d_n'\alpha_{n1}' \quad \ldots \quad (71)$$

$$k_n' = -c_n'\alpha_{n1}'+d_n'\alpha_{n2}' \quad \ldots \quad (72)$$

where

$$\alpha_{n1} = \frac{1}{n\left(1+\frac{9}{4}n^2\right)}\left[13\sqrt{3}n^2-3\sqrt{3}\left\{(q_n^2-r_n^2)+3nq_nr_n\right\}\right] \quad \ldots \quad (73)$$

$$\alpha_{n2} = \frac{1}{n\left(1+\frac{9}{4}n^2\right)}\left[6\sqrt{3}n(1-n^2)-3\sqrt{3}\left\{2r_nq_n-\frac{3}{2}n\,(q_n^2-r_n^2)\right\}\right] \quad \ldots \quad (74)$$

$$\alpha_{n1}' = \frac{1}{n\left(1+\frac{9}{4}n^2\right)}\left[13\sqrt{3}n^2-3\sqrt{3}\left\{(q_n'^2-r_n'^2)+3nq_n'r_n'\right\}\right] \quad \ldots \quad (75)$$

$$\alpha_{n2}' = \frac{1}{n\left(1+\frac{9}{4}n^2\right)}\left[6\sqrt{3}n(1-n^2)-3\sqrt{3}\left\{2r_n'q_n'-\frac{3}{2}n\,(q_n'^2-r_n'^2)\right\}\right] \quad \ldots \quad (76)$$

Thus if the constants a_n, b_n, c_n, d_n . . . . d_n' are known f_n, g_n, h_n, k_n . . . . h_n' and k_n' can be evaluated.

Now substituting the values of $\frac{\partial S}{\partial x}$ and $\frac{\partial S}{\partial y}$ in the boundary condition (44), we obtain

$$(B_0-\sqrt{3}A_0)+\sum_{n=1}^{\infty}\Big[\Big\{a_nq_n+b_n(r_n-\sqrt{3}n)-c_nq_n+d_n(r_n+\sqrt{3}n)+ +a_n'q_n'+b_n'(r_n'-\sqrt{3}n)-c_n'q_n'+d_n'(r_n'+\sqrt{3}n)\Big\}\cos ny+ \Big\{b_nq_n-a_n(r_n-\sqrt{3}n)+c_n(r_n+\sqrt{3}n)+d_nq_n+b_n'q_n'-a_n'(r_n'-\sqrt{3}n)+ +c_n'(r_n'+\sqrt{3}n)+d_n'q_n'\Big\}\sin ny\Big]=0 \quad \ldots \quad \ldots \quad \ldots \quad (77)$$

From this it is clear that the individual coefficients of cos ny and sin ny are zeros. Therefore,

$$B_0=\sqrt{3}A_0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (78)$$

$$\Big\{a_nq_n+b_n(r_n-\sqrt{3}n)-c_nq_n+d_n(r_n+\sqrt{3}n)+a_n'q_n'+b_n'(r_n'-\sqrt{3}n)-c_n'q_n' +d_n'(r_n'+\sqrt{3}n)\Big\}=0 \quad \ldots \quad \ldots \quad \ldots \quad (79)$$

$$\Big\{b_nq_n-a_n(r_n-\sqrt{3}n)+c_n(r_n+\sqrt{3}n)+d_nq_n+b_n'q_n'-a_n'(r_n'-\sqrt{3}n) +c_n'(r_n'+\sqrt{3}n)+d_n'q_n'\Big\}=0 \quad \ldots \quad \ldots \quad (80)$$

From (79) and (80), we get

$$a_n'=\frac{1}{2\sqrt{3}nq_n'}\Big[a_n\Big\{q_n'(r_n-\sqrt{3}n)-q_n(r_n'+\sqrt{3}n)\Big\}-b_n\Big\{q_nq_n'+(r_n-\sqrt{3}n)(r_n'+\sqrt{3}n)\Big\} +c_n\Big\{q_n(r_n'+\sqrt{3}n)-q_n'(r_n+\sqrt{3}n)\Big\}-d_n\Big\{q_nq_n'+(r_n+\sqrt{3}n)(r_n'+\sqrt{3}n)\Big\} -b_n'\Big\{r_n'^2+q_n'^2-3n^2\Big\}-d_n'\Big\{q_n'^2+(r_n'+\sqrt{3}n)^2\Big\}\Big] \quad \ldots \quad \ldots \quad (81)$$

and

$$c_n'=\frac{1}{2\sqrt{3}nq_n'}\Big[a_n\Big\{q_n'(r_n-\sqrt{3}n)-q_n(r_n'-\sqrt{3}n)\Big\}-b_n\Big\{q_nq_n'+(r_n-\sqrt{3}n)(r_n'-\sqrt{3}n)\Big\} +c_n\Big\{q_n(r_n'-\sqrt{3}n)-q_n'(r_n+\sqrt{3}n)\Big\}-d_n\Big\{q_nq_n'+(r_n+\sqrt{3}n)(r_n'-\sqrt{3}n)\Big\} -b_n'\Big\{q_n'^2+(r_n'-\sqrt{3}n)^2\Big\}-d_n'\Big\{r_n'^2+q_n'^2-3n^2\Big\}\Big] \quad \ldots \quad \ldots \quad (82)$$

Now we substitute in the boundary condition (45) the values of $\frac{\partial F}{\partial x}$, F, $\frac{\partial F}{\partial y}$ and replace the constants f_n, g_n . . . etc. by a_n, b_n etc. by virtue of relations (65) to (72). The form of the equation obtained suggests that the individual coefficients of cos ny and sin ny are zeros.

Therefore, we get

$$a_nP_{n1}+b_nP_{n2}-c_nP_{n3}+d_nP_{n4}+a_n'P_{n1}'+b_n'P_{n2}'-c_n'P_{n3}'+d_n'P_{n4}'=0 \quad \ldots \quad (83)$$

and

$$-a_nP_{n2}+b_nP_{n1}+c_nP_{n4}+d_nP_{n3}-a_n'P_{n2}'+b_n'P_{n1}'+c_n'P_{n4}'+d_n'P_{n3}'=0 \quad \ldots \quad (84)$$

where

$$P_{n1} = \alpha_{n2}(\sqrt{3}q_n - 1) + \alpha_{n1}\left(\sqrt{3}r_n - \frac{3}{2}n\right) \quad \ldots \quad (85)$$

$$P_{n2} = \alpha_{n2}\left(\sqrt{3}r_n - \frac{3}{2}n\right) - \alpha_{n1}(\sqrt{3}q_n - 1) \quad \ldots \quad (86)$$

$$P_{n3} = \alpha_{n2}(\sqrt{3}q_n + 1) + \alpha_{n1}\left(\sqrt{3}r_n + \frac{3}{2}n\right) \quad \ldots \quad (87)$$

$$P_{n4} = \alpha_{n2}\left(\sqrt{3}r_n + \frac{3}{2}n\right) - \alpha_{n1}(\sqrt{3}q_n + 1) \quad \ldots \quad (88)$$

$$P'_{n1} = \alpha'_{n2}(\sqrt{3}q'_n - 1) + \alpha'_{n1}\left(\sqrt{3}r'_n - \frac{3}{2}n\right) \quad \ldots \quad (89)$$

$$P'_{n2} = \alpha'_{n2}\left(\sqrt{3}r'_n - \frac{3}{2}n\right) - \alpha'_{n1}(\sqrt{3}q'_n - 1) \quad \ldots \quad (90)$$

$$P'_{n3} = \alpha'_{n2}(\sqrt{3}q'_n + 1) + \alpha'_{n1}\left(\sqrt{3}r'_n + \frac{3}{2}n\right) \quad \ldots \quad (91)$$

$$P'_{n4} = \alpha'_{n2}\left(\sqrt{3}r'_n + \frac{3}{2}n\right) - \alpha'_{n1}(\sqrt{3}q'_n + 1) \quad \ldots \quad (92)$$

Substituting the values of a'_n and c'_n from (81) and (82) in equations (83) and (84) and rearranging, we get

$$a_n M_{n1} + b_n M_{n2} - c_n M_{n3} + d_n M_{n4} + b'_n M_{n5} + d'_n M_{n6} = 0 \quad \ldots \quad (93)$$

and

$$-a_n M_{n7} + b_n M_{n8} + c_n M_{n9} + d_n M_{n10} + b'_n M_{n11} + d'_n M_{n12} = 0 \quad \ldots \quad (94)$$

where

$$M_{n1} = P_{n1} + \frac{P'_{n1}}{2\sqrt{3}nq'_n}\left\{q'_n(r_n - \sqrt{3}n) - q_n(r'_n + \sqrt{3}n)\right\} - \\ - \frac{P'_{n3}}{2\sqrt{3}nq'_n}\left\{q'_n(r_n - \sqrt{3}n) - q_n(r'_n - \sqrt{3}n)\right\} \quad \ldots \quad (95)$$

$$M_{n2} = P_{n2} - \frac{P'_{n1}}{2\sqrt{3}nq'_n}\left\{q_n q'_n + (r_n - \sqrt{3}n)(r'_n + \sqrt{3}n)\right\} + \\ + \frac{P'_{n3}}{2\sqrt{3}nq'_n}\left\{q_n q'_n + (r_n - \sqrt{3}n)(r'_n - \sqrt{3}n)\right\} \quad \ldots \quad (96)$$

$$M_{n3} = P_{n3} - \frac{P'_{n1}}{2\sqrt{3}nq'_n}\left\{q_n(r'_n + \sqrt{3}n) - q'_n(r_n + \sqrt{3}n)\right\} + \\ + \frac{P'_{n3}}{2\sqrt{3}nq'_n}\left\{q_n(r'_n - \sqrt{3}n) - q'_n(r_n + \sqrt{3}n)\right\} \quad \ldots \quad (97)$$

$$M_{n4} = P_{n4} - \frac{P'_{n1}}{2\sqrt{3}nq'_n}\left\{q_n q'_n + (r_n + \sqrt{3}n)(r'_n + \sqrt{3}n)\right\} + \\ + \frac{P'_{n3}}{2\sqrt{3}nq'_n}\left\{q_n q'_n + (r_n + \sqrt{3}n)(r'_n - \sqrt{3}n)\right\} \quad \ldots \quad (98)$$

$$M_{n5} = P'_{n2} - \frac{P'_{n1}}{2\sqrt{3}nq'_n}\left\{r'^2_n + q'^2_n - 3n^2\right\} + \frac{P'_{n3}}{2\sqrt{3}nq'_n}\left\{q'^2_n + (r'_n - \sqrt{3}n)^2\right\} \quad .. \quad (99)$$

$$M_{n6} = P'_{n4} - \frac{P'_{n1}}{2\sqrt{3}nq'_n}\left\{q'^2_n + (r'_n + \sqrt{3}n)^2\right\} + \frac{P'_{n3}}{2\sqrt{3}nq'_n}\left\{r'^2_n + q'^2_n - 3n^2\right\} \quad (100)$$

$$M_{n7} = P_{n2} + \frac{P'_{n2}}{2\sqrt{3}nq'_n}\left\{q'_n(r_n - \sqrt{3}n) - q_n(r'_n + \sqrt{3}n)\right\} - $$
$$- \frac{P'_{n4}}{2\sqrt{3}nq'_n}\left\{q'_n(r_n - \sqrt{3}n) - q_n(r'_n - \sqrt{3}n)\right\} \quad .. \quad (101)$$

$$M_{n8} = P_{n1} + \frac{P'_{n2}}{2\sqrt{3}nq'_n}\left\{q_n q'_n + (r_n - \sqrt{3}n)(r'_n + \sqrt{3}n)\right\} - $$
$$- \frac{P'_{n4}}{2\sqrt{3}nq'_n}\left\{q_n q'_n + (r_n - \sqrt{3}n)(r'_n - \sqrt{3}n)\right\} \quad .. \quad (102)$$

$$M_{n9} = P_{n4} - \frac{P'_{n2}}{2\sqrt{3}nq'_n}\left\{q_n(r'_n + \sqrt{3}n) - q'_n(r_n + \sqrt{3}n)\right\} + $$
$$+ \frac{P'_{n4}}{2\sqrt{3}nq'_n}\left\{q_n(r'_n - \sqrt{3}n) - q'_n(r_n + \sqrt{3}n)\right\} \quad .. \quad (103)$$

$$M_{n10} = P_{n3} + \frac{P'_{n2}}{2\sqrt{3}nq'_n}\left\{q_n q'_n + (r_n + \sqrt{3}n)(r'_n + \sqrt{3}n)\right\} - $$
$$- \frac{P'_{n4}}{2\sqrt{3}nq'_n}\left\{q_n q'_n + (r_n + \sqrt{3}n)(r'_n - \sqrt{3}n)\right\} \quad .. \quad (104)$$

$$M_{n11} = P'_{n1} + \frac{P'_{n2}}{2\sqrt{3}nq'_n}\left\{r'^2_n + q'^2_n - 3n^2\right\} - \frac{P'_{n4}}{2\sqrt{3}nq'_n}\left\{q'^2_n + (r'_n - \sqrt{3}n)^2\right\} \quad (105)$$

$$M_{n12} = P'_{n3} + \frac{P'_{n2}}{2\sqrt{3}nq'_n}\left\{q'^2_n + (r'_n + \sqrt{3}n)^2\right\} - \frac{P'_{n4}}{2\sqrt{3}nq'_n}\left\{q'^2_n + r'^2_n - 3n^2\right\} \quad (106)$$

From equations (93) and (94), we can obtain the values of b'_n and d'_n in terms of other four coefficients; as for example

$$b'_n = -\frac{M_{n1}M_{n12} + M_{n7}M_{n6}}{M_{n5}M_{n12} - M_{n11}M_{n6}}a_n + \frac{M_{n8}M_{n6} - M_{n2}M_{n12}}{M_{n5}M_{n12} - M_{n11}M_{n6}}b_n$$
$$+ \frac{M_{n3}M_{n12} + M_{n6}M_{n9}}{M_{n5}M_{n12} - M_{n11}M_{n6}}c_n + \frac{M_{n6}M_{n10} - M_{n4}M_{n12}}{M_{n5}M_{n12} - M_{n11}M_{n6}}d_n \quad .. \quad .. \quad (107)$$

and

$$d'_n = -\frac{M_{n1}M_{n11} + M_{n5}M_{n7}}{M_{n6}M_{n11} - M_{n5}M_{n12}}a_n + \frac{M_{n5}M_{n8} - M_{n2}M_{n11}}{M_{n6}M_{n11} - M_{n5}M_{n12}}b_n$$
$$+ \frac{M_{n8}M_{n11} + M_{n5}M_{n7}}{M_{n6}M_{n11} - M_{n5}M_{n12}}c_n + \frac{M_{n5}M_{n10} - M_{n4}M_{n11}}{M_{n6}M_{n11} - M_{n5}M_{n12}}d_n. \quad .. \quad (108)$$

Next we substitute the values of F, $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ in the boundary condition (43). The expression takes up the form of a trigonometrical series with periodicity in y, which by the boundary condition stated above is equal to zero. As a consequence the individual coefficients of cos ny and sin ny will also be zero. This consideration yields the following two equations :—

$$f_n\beta_{n1}+g_n\beta_{n2}-h_n\beta_{n3}+k_n\beta_{n4}+f_n'\beta_{n1}'+g_n'\beta_{n2}'-h_n'\beta_{n3}'+k_n'\beta_{n4}' = 0 \quad .. \quad (109)$$

$$-f_n\beta_{n2}+g_n\beta_{n1}+h_n\beta_{n4}+k_n\beta_{n3}-f_n'\beta_{n2}'+g_n'\beta_{n1}'+h_n'\beta_{n4}'+k_n'\beta_{n3}' = 0 \quad .. \quad (110)$$

where

$$\beta_{n1} = e^{q_nx_1}\left\{(\sqrt{3}q_n+1)\cos r_nx_1-\left(\sqrt{3}r_n+\frac{3}{2}n\right)\sin r_nx_1\right\} \quad .. \quad (111)$$

$$\beta_{n2} = e^{q_nx_1}\left\{\left(\sqrt{3}r_n-\frac{3}{2}n\right)\cos r_nx_1+(\sqrt{3}q_n+1)\sin r_nx_1\right\} \quad .. \quad (112)$$

$$\beta_{n3} = e^{-q_nx_1}\left\{\left(\sqrt{3}r_n-\frac{3}{2}n\right)\sin r_nx_1+(\sqrt{3}q_n-1)\cos r_nx_1\right\} \quad .. \quad (113)$$

$$\beta_{n4} = e^{-q_nx_1}\left\{\left(\sqrt{3}r_n-\frac{3}{2}n\right)\cos r_nx_1-(\sqrt{3}q_n-1)\sin r_nx_1\right\} \quad .. \quad (114)$$

$$\beta_{n1}' = e^{q_n'x_1}\left\{(\sqrt{3}q_n'+1)\cos r_n'x_1-\left(\sqrt{3}r_n'+\frac{3}{2}n\right)\sin r_n'x_1\right\} \quad .. \quad (115)$$

$$\beta_{n2}' = e^{q_n'x_1}\left\{\left(\sqrt{3}r_n'-\frac{3}{2}n\right)\cos r_n'x_1+(\sqrt{3}q_n'+1)\sin r_n'x_1\right\} \quad .. \quad (116)$$

$$\beta_{n3}' = e^{-q_n'x_1}\left\{\left(\sqrt{3}r_n'-\frac{3}{2}n\right)\sin r_n'x_1+(\sqrt{3}q_n'-1)\cos r_n'x_1\right\} \quad .. \quad (117)$$

$$\beta_{n4}' = e^{-q_n'x_1}\left\{\left(\sqrt{3}r_n'-\frac{3}{2}n\right)\cos r_n'x_1-(\sqrt{3}q_n'-1)\sin r_n'x_1\right\} \quad .. \quad (118)$$

If we now replace the values of constants f_n, g_n, h_n, k_n, etc., by a_n, b_n, c_n, etc., with the help of relations (65) to (72), and substitute for a_n', b_n', c_n' and d_n' values given by equations (81), (82), (107) and (108), we get

$$a_nN_{n1}+b_nN_{n2}-c_nN_{n3}+d_nN_{n4} = 0 \quad .. \quad .. \quad .. \quad (119)$$

and

$$a_nN_{n5}+b_nN_{n6}+c_nN_{n7}+d_nN_{n8} = 0 \quad .. \quad .. \quad .. \quad (120)$$

where

$$N_{n1} = \Bigg[(\alpha_{n2}\beta_{n1}+\alpha_{n1}\beta_{n2})+ + \frac{\alpha_{n2}'\beta_{n1}'+\alpha_{n1}'\beta_{n2}'}{2\sqrt{3}nq_n'}\left\{q_n'(r_n-\sqrt{3}n)-q_n(r_n'+\sqrt{3}n)\right\} - - \frac{\alpha_{n2}'\beta_{n3}'+\alpha_{n1}'\beta_{n4}'}{2\sqrt{3}nq_n'}\left\{q_n'(r_n-\sqrt{3}n)-q_n(r_n'-\sqrt{3}n)\right\} - - \frac{M_{n1}M_{n12}+M_{n6}M_{n7}}{M_{n5}M_{n12}-M_{n11}M_{n6}}B_{n1}-\frac{M_{n1}M_{n11}+M_{n5}M_{n7}}{M_{n6}M_{n11}-M_{n5}M_{n12}}B_{n2}\Bigg] \quad .. \quad (121)$$

$$N_{n2} = \Bigg[(\alpha_{n2}\beta_{n2}-\alpha_{n1}\beta_{n1})-$$

$$-\frac{\alpha'_{n2}\beta'_{n1}+\alpha'_{n1}\beta'_{n2}}{2\sqrt{3}nq'_n}\left\{q_nq'_n+(r_n-\sqrt{3}n)(r'_n+\sqrt{3}n)\right\}+$$

$$+\frac{\alpha'_{n2}\beta'_{n3}+\alpha'_{n1}\beta'_{n4}}{2\sqrt{3}nq'_n}\left\{q_nq'_n+(r_n-\sqrt{3}n)(r'_n-\sqrt{3}n)\right\}+$$

$$+\frac{M_{n8}M_{n6}-M_{n2}M_{n12}}{M_{n5}M_{n12}-M_{n11}M_{n6}}B_{n1}+\frac{M_{n5}M_{n8}-M_{n2}M_{n12}}{M_{n6}M_{n11}-M_{n5}M_{n12}}B_{n2}\Bigg] \quad .. \quad (122)$$

$$N_{n3} = \Bigg[(\alpha_{n2}\beta_{n3}+\alpha_{n1}\beta_{n4})-$$

$$-\frac{\alpha'_{n2}\beta'_{n1}+\alpha'_{n1}\beta'_{n2}}{2\sqrt{3}nq'_n}\left\{q_n(r'_n+\sqrt{3}n)-q'_n(r_n+\sqrt{3}n)+\right.$$

$$\left.+\frac{\alpha'_{n2}\beta'_{n3}+\alpha'_{n1}\beta'_{n4}}{2\sqrt{3}nq'_n}\left\{q_n(r'_n-\sqrt{3}n)-q'_n(r_n+\sqrt{3}n)\right\}-\right.$$

$$-\frac{M_{n3}M_{n12}-M_{n6}M_{n9}}{M_{n5}M_{n12}-M_{n11}M_{n6}}B_{n1}-\frac{M_{n3}M_{n11}+M_{n5}M_{n9}}{M_{n6}M_{n11}-M_{n12}M_{n5}}B_{n2}\Bigg] \quad .. \quad (123)$$

$$N_{n4} = \Bigg[(\alpha_{n2}\beta_{n3}+\alpha_{n1}\beta_{n4})-$$

$$-\frac{\alpha'_{n2}\beta'_{n1}+\alpha'_{n1}\beta'_{n2}}{2\sqrt{3}nq'_n}\left\{q_nq'_n+(r_n+\sqrt{3}n)(r'_n+\sqrt{3}n)\right\}-$$

$$-\frac{\alpha'_{n2}\beta'_{n1}+\alpha'_{n1}\beta'_{n4}}{2\sqrt{3}nq'_n}\left\{q_nq'_n+(r_n+\sqrt{3}n)(r'_n-\sqrt{3}n)\right\}+$$

$$+\frac{M_{n6}M_{n10}-M_{n4}M_{n12}}{M_{n5}M_{n12}-M_{n11}M_{n6}}B_{n1}+\frac{M_{n5}M_{n10}-M_{n4}M_{n11}}{M_{n6}M_{n11}-M_{n5}M_{n12}}B_{n2}\Bigg] \quad .. \quad (124)$$

$$N_{n5} = \Bigg[(\alpha_{n1}\beta_{n1}-\alpha_{n2}\beta_{n2})-$$

$$-\frac{\alpha'_{n1}\beta'_{n1}-\alpha'_{n2}\beta'_{n2}}{2\sqrt{3}nq'_n}\left\{q'_n(r_n+\sqrt{3}n)-q_n(r'_n+\sqrt{3}n)\right\}+$$

$$+\frac{\alpha'_{n2}\beta'_{n4}-\alpha'_{n1}\beta'_{n3}}{2\sqrt{3}nq'_n}\left\{q'_n(r_n-\sqrt{3}n)-q_n(r'_n-\sqrt{3}n)\right\}$$

$$-\frac{M_{n1}M_{n12}+M_{n6}M_{n7}}{M_{n5}M_{n12}-M_{n11}M_{n6}}B_{n3}-\frac{M_{n1}M_{n11}+M_{n5}M_{n7}}{M_{n6}M_{n11}-M_{n5}M_{n12}}B_{n4}\Bigg] \quad .. \quad (125)$$

$$N_{n6} = \Bigg[(\alpha_{n1}\beta_{n2}+\alpha_{n2}\beta_{n1}) -$$

$$- \frac{\alpha'_{n1}\beta'_{n1}-\alpha'_{n2}\beta'_{n2}}{2\sqrt{3}nq'_n}\left\{q_nq'_n+(r_n-\sqrt{3}n)(r'_n+\sqrt{3}n)\right\} -$$

$$- \frac{\alpha'_{n2}\beta'_{n4}-\alpha'_{n1}\beta'_{n3}}{2\sqrt{3}nq'_n}\left\{q_nq'_n+(r_n+\sqrt{3}n)(r'_n-\sqrt{3}n)\right\} +$$

$$+ \frac{M_{n8}M_{n6}-M_{n2}M_{n12}}{M_{n5}M_{n12}-M_{n11}M_{n6}}B_{n3}+\frac{M_{n5}M_{n8}-M_{n2}M_{n11}}{M_{n6}M_{n11}-M_{n12}M_{n5}}B_{n4}\Bigg] \quad .. \quad (126)$$

$$N_{n7} = \Bigg[(\alpha_{n2}\beta_{n4}-\alpha_{n1}\beta_{n3}) +$$

$$+ \frac{\alpha'_{n1}\beta'_{n1}-\alpha'_{n2}\beta'_{n2}}{2\sqrt{3}nq'_n}\left\{q_n(r'_n+\sqrt{3}n)-q'_n(r_n+\sqrt{3}n)\right\} +$$

$$+ \frac{\alpha'_{n2}\beta'_{n4}-\alpha'_{n1}\beta'_{n3}}{2\sqrt{3}nq'_n}\left\{q(r'_n-\sqrt{3}n)-q'_n(r_n+\sqrt{3}n)\right\} +$$

$$+ \frac{M_{n8}M_{n12}+M_{n6}M_{n9}}{M_{n5}M_{n12}-M_{n11}M_{n6}}B_{n3}+\frac{M_{n8}M_{n11}+M_{n5}M_{n9}}{M_{n6}M_{n11}-M_{n12}M_{n5}}B_{n4}\Bigg] \quad .. \quad (127)$$

$$N_{n8} = \Bigg[(\alpha_{n2}\beta_{n3}+\alpha_{n1}\beta_{n4}) -$$

$$- \frac{\alpha'_{n1}\beta'_{n1}-\alpha'_{n2}\beta'_{n2}}{2\sqrt{3}nq'_n}\left\{q_nq'_n+(r_n+\sqrt{3}n)(r'_n+\sqrt{3}n)\right\} -$$

$$- \frac{\alpha'_{n2}\beta'_{n4}-\alpha'_{n1}\beta'_{n3}}{2\sqrt{3}nq'_n}\left\{q_nq'_n+(r_n+\sqrt{3}n)(r'_n-\sqrt{3}n)\right\} +$$

$$+ \frac{M_{n6}M_{n10}-M_{n4}M_{n12}}{M_{n5}M_{n12}-M_{n11}M_{n6}}B_{n3}+\frac{M_{n5}M_{n10}-M_{n4}M_{n11}}{M_{n6}M_{n11}-M_{n5}M_{n12}}B_{n4}\Bigg] \quad .. \quad (128)$$

where

$$B_{n1} = (\alpha'_{n2}\beta'_{n2}-\alpha'_{n1}\beta'_{n1}) -$$

$$- \frac{\alpha'_{n2}\beta'_{n2}+\alpha'_{n1}\beta'_{n2}}{2\sqrt{3}nq'_n}\left\{r'^2_n+q'^2_n-3n^2\right\}+\frac{\alpha'_{n2}\beta'_{n3}+\alpha'_{n1}\beta'_{n4}}{2\sqrt{3}nq'_n}\left\{q'^2_n+(r'_n-\sqrt{3}n)^2\right\} \quad (129)$$

$$B_{n2} = (\alpha'_{n1}\beta'_{n4}-\alpha'_{n1}\beta'_{n3}) -$$

$$- \frac{\alpha'_{n2}\beta'_{n1}+\beta'_{n2}\alpha'_{n1}}{2\sqrt{3}nq'_n}\left\{q'^2_n+(r'_n+\sqrt{3}n)^2\right\}+\frac{\alpha'_{n3}\beta'_{n3}+\alpha'_{n1}\beta'_{n4}}{2\sqrt{3}nq'_n}\left\{q'^2_n+r'_n-\sqrt{3}n^2\right\} \quad (130)$$

$$B_{n3} = (\alpha'_{n1}\beta'_{n2}+\alpha'_{n2}\beta'_{n1}) -$$

$$-\frac{\alpha'_{n1}\beta'_{n1}-\alpha'_{n2}\beta'_{n2}}{2\sqrt{3}nq'_n}\left\{r'^2_n+q'^2_n-3n^2\right\}-\frac{\alpha'_{n2}\beta'_{n4}-\alpha'_{n1}\beta'_{n3}}{2\sqrt{3}nq'_n}\left\{q'^2_n+(r'_n-3n)^2\right\} \quad (131)$$

$$B_{n4} = (\alpha'_{n2}\beta'_{n3}+\alpha'_{n1}\beta'_{n4}) -$$

$$-\frac{\alpha'_{n1}\beta'_{n1}-\alpha'_{n2}\beta'_{n2}}{2\sqrt{3}nq'_n}\left\{q'^2_n+(r'_n+\sqrt{3}n)^2\right\}-\frac{\alpha'_{n2}\beta'_{n4}-\alpha'_{n1}\beta'_{n3}}{2\sqrt{3}nq'_n}\left\{r'^2_n+q'^2_n-3n^2\right\} \quad (132)$$

From equations (119) and (120), we have, therefore,

$$c_n = \frac{N_{n1}N_{n8}-N_{n5}N_{n4}}{N_{n3}N_{n8}+N_{n7}N_{n4}}a_n+\frac{N_{n2}N_{n8}-N_{n6}N_{n4}}{N_{n3}N_{n8}+N_{n7}N_{n4}}b_n \quad .. \quad .. \quad (133)$$

and

$$d_n = -\frac{N_{n7}N_{n1}+N_{n5}N_{n3}}{N_{n4}N_{n7}+N_{n8}N_{n3}}a_n-\frac{N_{n2}N_{n7}-N_{n6}N_{n4}}{N_{n3}N_{n8}+N_{n7}N_{n4}}b_n \quad .. \quad (134)$$

Lastly, we use the following boundary condition (42), in order to complete the evaluation of constants.

$$\frac{1}{2}\left[\frac{\partial S}{\partial x}+\sqrt{3}\frac{\partial S}{\partial y}\right]_{x=x_1} = \psi(y) = \text{a known distribution in } y.$$

$\psi(y)$, the distribution at the base of the atmosphere is now supposed to admit a Fourier expansion, and hence, it may be written in the form,

$$\psi(y) = A'_0+\sum_1^\infty A'_n \cos ny+\sum_1^\infty B'_n \sin ny \quad .. \quad .. \quad (135)$$

Now substituting the values of $\frac{\partial s}{\partial x}$ and $\frac{\partial s}{\partial y}$ in the boundary condition (42), the values derived from (63), and comparing the coefficients of cos ny and sin ny on both sides of the sign of equality, we get the following equations.

$$A_0(x_1+\sqrt{3}) = A'_0 \quad .. \quad .. \quad .. \quad .. \quad (136)$$

$$a_n\gamma_{n1}+b_n\gamma_{n2}-c_n\gamma_{n3}+d_n\gamma_{n4}+a'_n\gamma'_{n1}+b'_n\gamma'_{n2}-c'_n\gamma'_{n3}+d'_n\gamma'_{n4} = A'_n \quad .. \quad (137)$$

$$-a_n\gamma_{n2}+b_n\gamma_{n1}+c_n\gamma_{n4}+d_n\gamma_{n3}-a'_n\gamma'_{n2}+b'_n\gamma'_{n1}+c'_n\gamma'_{n4}+d'_n\gamma'_{n3} = B'_n \quad .. \quad (138)$$

where

$$\gamma_{n1} = \frac{e^{q_nx_1}}{2}\left\{q_n \cos r_nx_1-(r_n+\sqrt{3}n)\sin r_nx_1\right\} \quad .. \quad .. \quad (139)$$

$$\gamma_{n2} = \frac{e^{q_nx_1}}{2}\left\{(r_n+\sqrt{3}n)\cos r_nx_1+q_n \sin r_nx_1\right\} \quad .. \quad .. \quad (140)$$

$$\gamma_{n3} = \frac{e^{-q_nx_1}}{2}\left\{q_n \cos r_nx_1+(r_n-\sqrt{3}n)\sin r_nx_1\right\} \quad .. \quad .. \quad (141)$$

$$\gamma_{n4} = \frac{e^{-q_nx_1}}{2}\left\{(r_n-\sqrt{3}n)\cos r_nx_1-q_n \sin r_nx_1\right\} \quad .. \quad .. \quad (142)$$

$$\gamma'_{n1} = \frac{e^{q'_n x_1}}{2} \left\{ q'_n \cos r'_n x_1 - (r'_n + \sqrt{3}n) \sin r'_n x_1 \right\} \quad \ldots \quad \ldots \quad (143)$$

$$\gamma'_{n2} = \frac{e^{q'_n x_1}}{2} \left\{ (r'_n + \sqrt{3}n) \cos r'_n x_1 + q'_n \sin r'_n x_1 \right\} \quad \ldots \quad \ldots \quad (144)$$

$$\gamma'_{n3} = \frac{e^{-q'_n x_1}}{2} \left\{ q'_n \cos r'_n x_1 + (r'_n - \sqrt{3}n) \sin r'_n x_1 \right\} \quad \ldots \quad \ldots \quad (145)$$

$$\gamma'_{n4} = \frac{e^{-q'_n x_1}}{2} \left\{ (r'_n - \sqrt{3}n) \cos r'_n x_1 - q'_n \sin r'_n x_1 \right\} \quad \ldots \quad \ldots \quad (146)$$

In equations (137) and (138) replacement of constants $a'_n,$ $b'_n,$ $c'_n,$ $d'_n,$ c_n and d_n is done in steps. At the first instance, if we substitute for a'_n and c'_n values obtained from (81) and (82), we get

$$a_n l_{n1} + b_n l_{n2} - c_n l_{n3} + d_n l_{n4} + b'_n l_{n5} + d'_n l_{n6} = A'_n \quad \ldots \quad \ldots \quad (147)$$

$$-a_n l_{n7} + b_n l_{n8} + c_n l_{n9} + d_n l_{n10} + b'_n l_{n11} + d'_n l_{n12} = B'_n \quad \ldots \quad \ldots \quad (148)$$

where

$$l_{n1} = \gamma_{n1} + \frac{\gamma'_{n1}}{2\sqrt{3}nq'_n} \left\{ q'_n (r_n - \sqrt{3}n) - q_n (r'_n + \sqrt{3}n) \right\} - \\ - \frac{\gamma'_{n3}}{2\sqrt{3}nq'_n} \left\{ q'_n (r_n - \sqrt{3}n) - q_n (r'_n - \sqrt{3}n) \right\} \quad \ldots \quad (149)$$

$$l_{n2} = \gamma_{n2} - \frac{\gamma'_{n1}}{2\sqrt{3}nq'_n} \left\{ q_n q'_n + (r_n - \sqrt{3}n)(r'_n + \sqrt{3}n) \right\} + \\ + \frac{\gamma'_{n3}}{2\sqrt{3}nq'_n} \left\{ q_n q'_n + (r_n - \sqrt{3}n)(r'_n - \sqrt{3}n) \right\} \quad \ldots \quad (150)$$

$$l_{n3} = \gamma_{n3} - \frac{\gamma'_{n1}}{2\sqrt{3}nq'_n} \left\{ q_n (r'_n + \sqrt{3}n) - q'_n (r_n + \sqrt{3}n) \right\} + \\ + \frac{\gamma'_{n3}}{2\sqrt{3}nq'_n} \left\{ q_n (r'_n - \sqrt{3}n) - q'_n (r_n + \sqrt{3}n) \right\} \quad \ldots \quad (151)$$

$$l_{n4} = \gamma_{n4} - \frac{\gamma'_{n1}}{2\sqrt{3}nq'_n} \left\{ q_n q'_n + (r_n + \sqrt{3}n)(r'_n + \sqrt{3}n) \right\} + \\ + \frac{\gamma'_{n3}}{2\sqrt{3}nq'_n} \left\{ q_n q'_n + (r_n + \sqrt{3}n)(r'_n - \sqrt{3}n) \right\} \quad \ldots \quad (152)$$

$$l_{n5} = \gamma'_{n2} - \frac{\gamma'_{n1}}{2\sqrt{3}nq'_n} \left\{ r'^2_n + q'^2_n - 3n^2 \right\} + \frac{\gamma'_{n3}}{2\sqrt{3}nq'_n} \left\{ q'^2_n + (r'_n - \sqrt{3}n)^2 \right\} \quad \ldots \quad (153)$$

$$l_{n6} = \gamma'_{n4} - \frac{\gamma'_{n1}}{2\sqrt{3}nq'_n} \left\{ q'^2_n + (r'_n + \sqrt{3}n)^2 \right\} + \frac{\gamma'_{n3}}{2\sqrt{3}nq'_n} \left\{ q'^2_n + r'^2_n - 3n^2 \right\} \quad \ldots \quad (154)$$

$$l_{n7} = \gamma_{n2} + \frac{\gamma'_{n2}}{2\sqrt{3}nq'_n}\left\{q'_n(r_n-\sqrt{3}n)-q_n(r'_n+\sqrt{3}n)\right\} - $$
$$- \frac{\gamma'_{n4}}{2\sqrt{3}nq'_n}\left\{q'_n(r_n-\sqrt{3}n)-q_n(r'_n-\sqrt{3}n)\right\} \quad .. \quad (155)$$

$$l_{n8} = \gamma_{n1} + \frac{\gamma'_{n2}}{2\sqrt{3}nq'_n}\left\{q_nq'_n+(r_n-\sqrt{3}n)(r'_n+\sqrt{3}n)\right\} - $$
$$- \frac{\gamma'_{n4}}{2\sqrt{3}nq'_n}\left\{q'_n(r_n-\sqrt{3}n)-q_n(r'_n-\sqrt{3}n)\right\} \quad .. \quad (156)$$

$$l_{n9} = \gamma_{n4} - \frac{\gamma'_{n2}}{2\sqrt{3}nq'_n}\left\{q_n(r'_n+\sqrt{3}n)-q'_n(r_n+\sqrt{3}n)\right\} + $$
$$+ \frac{\gamma'_{n4}}{2\sqrt{3}nq'_n}\left\{q_n(r'_n-\sqrt{3}n)-q'_n(r_n+\sqrt{3}n)\right\} \quad .. \quad (157)$$

$$l_{n10} = \gamma_{n3} + \frac{\gamma'_{n2}}{2\sqrt{3}nq'_n}\left\{q_nq'_n+(r_n+\sqrt{3}n)(r'_n+\sqrt{3}n)\right\} - $$
$$- \frac{\gamma'_{n4}}{2\sqrt{3}nq'_n}\left\{q_nq'_n+(r_n+\sqrt{3}n)(r'_n-\sqrt{3}n)\right\} \quad .. \quad (158)$$

$$l_{n11} = \gamma'_{n1} + \frac{\gamma'_{n2}}{2\sqrt{3}nq'_n}\left\{r'^2_n+q'^2_n-3n^2\right\} - \frac{\gamma'_{n4}}{2\sqrt{3}nq'_n}\left\{q'^2_n+(r'_n-\sqrt{3}n)^2\right\} \quad .. \quad (159)$$

$$l_{n12} = \gamma'_{n3} + \frac{\gamma'_{n3}}{2\sqrt{3}nq'_n}\left\{q'^2_n+(r'_n+\sqrt{3}n)^2\right\} - \frac{\gamma'_{n4}}{2\sqrt{3}nq'_n}\left\{r'^2_n+q'^2_n-3n^2\right\} \quad .. \quad (160)$$

Again substituting in (147) and (148), the values of b'_n and d' from (107) and (108), we obtain

$$a_np_{n1}+b_np_{n2}-c_np_{n3}+d_np_{n4} = A'_n \quad .. \quad .. \quad .. \quad (161)$$

$$-a_np_{n5}+b_np_{n6}+c_np_{n7}+d_np_{n8} = B'_n \quad .. \quad .. \quad .. \quad (162)$$

where

$$p_{n1} = l_{n1} - \frac{M_{n1}M_{n12}+M_{n7}M_{n6}}{M_{n5}M_{n12}-M_{n11}M_{n6}}l_{n5} - \frac{M_{n1}M_{n11}+M_{n5}M_{n7}}{M_{n6}M_{n11}-M_{n5}M_{n12}}l_{n6} \quad .. \quad (163)$$

$$p_{n2} = l_{n2} + \frac{M_{n8}M_{n6}-M_{n2}M_{n12}}{M_{n5}M_{n12}-M_{n11}M_{n6}}l_{n5} + \frac{M_{n5}M_{n8}-M_{n2}M_{n11}}{M_{n6}M_{n11}-M_{n5}M_{n12}}l_{n6} \quad .. \quad (164)$$

$$p_{n3} = l_{n3} - \frac{M_{n3}M_{n12}+M_{n6}M_{n9}}{M_{n5}M_{n12}-M_{n11}M_{n6}}l_{n5} + \frac{M_{n3}M_{n11}+M_{n5}M_{n9}}{M_{n6}M_{n11}-M_{n5}M_{n12}}l_{n6} \quad .. \quad (165)$$

$$p_{n4} = l_{n4} + \frac{M_{n6}M_{n10}-M_{n4}M_{n12}}{M_{n5}M_{n12}-M_{n11}M_{n6}}l_{n5} + \frac{M_{n5}M_{n10}-M_{n4}M_{n11}}{M_{n6}M_{n11}-M_{n5}M_{n12}}l_{n6} \quad .. \quad .. \quad (166)$$

$$p_{n5} = l_{n7} + \frac{M_{n1}M_{n12}+M_{n7}M_{n6}}{M_{n5}M_{n12}-M_{n11}M_{n6}}l_{n11} + \frac{M_{n1}M_{n11}+M_{n5}M_{n7}}{M_{n6}M_{n11}-M_{n5}M_{n12}}l_{n12} \quad .. \quad (167)$$

$$p_{n6} = l_{n8}+\frac{M_{n8}M_{n6}-M_{n2}M_{n12}}{M_{n5}M_{n12}-M_{n11}M_{n6}}\cdot l_{n11}+\frac{M_{n8}M_{n5}-M_{n2}M_{n11}}{M_{n6}M_{n11}-M_{n5}M_{n12}}l_{n12} \quad .. \quad (168)$$

$$p_{n7} = l_{n9}+\frac{M_{n3}M_{n12}+M_{n6}M_{n9}}{M_{n5}M_{n12}-M_{n11}M_{n6}}l_{n11}+\frac{M_{n3}M_{n11}+M_{n5}M_{n9}}{M_{n6}M_{n11}-M_{n5}M_{n12}}l_{n12} \quad .. \quad (169)$$

$$p_{n8} = l_{n10}+\frac{M_{n6}M_{n10}-M_{n4}M_{n12}}{M_{n5}M_{n12}-M_{n11}M_{n6}}l_{n11}+\frac{M_{n5}M_{n10}-M_{n4}M_{n11}}{M_{n6}M_{n11}-M_{n5}M_{n12}}l_{n12} \quad .. \quad (170)$$

Next by the substitution of the values of c_n and d_n obtained from (133) and (134), in equations (161) and (162), we obtain the following two relations

$$a_nS_{n1}+b_nS_{n2} = A_n' \quad .. \quad .. \quad .. \quad .. \quad (171)$$

and

$$-a_nS_{n3}+b_nS_{n4} = B_n' \quad .. \quad .. \quad .. \quad .. \quad (172)$$

where

$$S_{n1} = p_{n1}-\frac{N_{n1}N_{n8}-N_{n5}N_{n3}}{N_{n3}N_{n8}+N_{n4}N_{n7}}p_{n3}-\frac{N_{n7}N_{n1}+N_{n5}N_{n3}}{N_{n4}N_{n7}+N_{n8}N_{n3}}p_{n4} \quad .. \quad (173)$$

$$S_{n2} = p_{n2}-\frac{N_{n2}N_{n8}-N_{n6}N_{n4}}{N_{n3}N_{n8}+N_{n4}N_{n7}}p_{n3}-\frac{N_{n2}N_{n7}+N_{n6}N_{n3}}{N_{n4}N_{n7}+N_{n8}N_{n3}}p_{n4} \quad .. \quad (174)$$

$$S_{n3} = p_{n5}-\frac{N_{n1}N_{n8}-N_{n5}N_{n4}}{N_{n3}N_{n8}+N_{n4}N_{n7}}p_{n7}+\frac{N_{n1}N_{n7}+N_{n5}N_{n3}}{N_{n4}N_{n7}+N_{n8}N_{n3}}p_{n8} \quad .. \quad (175)$$

$$S_{n4} = p_{n6}+\frac{N_{n2}N_{n8}-N_{n6}N_{n4}}{N_{n3}N_{n8}+N_{n4}N_{n7}}p_{n7}-\frac{N_{n2}N_{n7}+N_{n6}N_{n3}}{N_{n4}N_{n7}+N_{n8}N_{n3}}p_{n8} \quad .. \quad (176)$$

Therefore the equations (171) and (172) lead to the following values of a_n and b_n namely

$$a_n = \frac{A_n'S_{n4}-B_n'S_{n2}}{S_{n1}S_{n4}+S_{n3}S_{n2}} \quad .. \quad .. \quad .. \quad .. \quad (177)$$

and

$$b_n = \frac{A_n'S_{n3}+B_n'S_{n4}}{S_{n1}S_{n4}+S_{n3}S_{n2}} \quad .. \quad .. \quad .. \quad .. \quad (178)$$

Thus we see that we can evaluate all the constants occurring in the expression for $S(x, y)$ (equation (63) by the use of equations (177), (178), (136), (134), (133), (107), (108), (82) and (81), provided the values of the Fourier coefficients of the representation of the spectral distribution at the lower boundary are known as also the optical thickness of the atmosphere.

It is also clear that by virtue of relations (65)–(72), we can find the values of the sets of constants f_n, g_n, h_n, ... k_n' required for the representation of $F(x, y)$ from the calculated values of a_n, b_n, c_n, d_n . . . . d_n'.

5. Calculation of intensity and degree of polarization

Thus completing the determination of constants as given in art. 4, it is easy to find out the value of the intensities both outward and inward at any level of the atmosphere. But our main interest is the determination of the distribution of the emergent radiation at the outer boundary of the atmosphere; the total emergent intensity is given by (cf. equation (42))

$$I_{l(+1)}(0, y)+I_{r(+1)}(0, y) = \frac{1}{2}\left[K^{+}+H^{+}\right] = \frac{1}{2}\left[\frac{\partial S}{\partial x}+\sqrt{3}\,\frac{\partial S}{\partial y}\right]_{x=0} = \left[\frac{\partial S}{\partial x}\right]_{x=0}$$

[cf. equation (44)]

$$\begin{aligned} = \sqrt{3}A_0+\sum_{1}^{\infty}\Big[&\Big\{(a_n-c_n)q_n+(b_n+d_n)r_n\Big\}\cos ny \\ &+\Big\{(b_n+d_n)q_n-(a_n-c_n)r_n\Big\}\sin ny \\ &+\Big\{(a'_n-c'_n)q'_n+(b'_n+d'_n)r'_n\Big\}\cos ny \\ &+\Big\{(b'_n+d'_n)q'_n-(a'_n-c'_n)r'_n\Big\}\sin ny\Big] \end{aligned} \tag{179}$$

($B_0 = \sqrt{3}A_0$ by equation (78))

We can also calculate the value of $I_{l(+1)}-I_{r(+1)}$, the difference of the l and r components of the emergent intensity, at the outer boundary of the atmosphere and thence obtain the degree of polarization of the emergent radiation as a function of y, the shift.

Now,

$$I_{l(+1)}(0, y)-I_{r(+1)}(0, y) = \frac{1}{2}\left[K^{-}+H^{-}\right] = \frac{1}{2}\left[\sqrt{3}\,\frac{\partial F}{\partial x}+F+\frac{3}{2}\frac{\partial F}{\partial y}\right]_{x=0} = \sqrt{3}\left[\frac{\partial F}{\partial x}\right]_{x=0}$$

[cf. equation (45)]

$$\begin{aligned} = \sqrt{3}\sum_{1}^{\infty}\Big[&\Big\{[(a_n-c_n)\alpha_{n2}-(b_n+d_n)\alpha_{n1}]q_n+[(a_n-c_n)\alpha_{n1}+ \\ &\qquad +(b_n+d_n)\alpha_{n2}]r_n\Big\}\cos ny \\ &+\Big\{[(a_n-c_n)\alpha_{n1}+(b_n+d_n)\alpha_{n2}]q_n-[(a_n-c_n)\alpha_{n2}- \\ &\qquad -(b_n+d_n)\alpha_{n1}]r_n\Big\}\sin ny \\ &+\Big\{[(a'_n-c'_n)\alpha'_{n2}-(b'_n+d'_n)\alpha'_{n1}]q'_n+[(a'_n-c'_n)\alpha'_{n1}+ \\ &\qquad +(b'_n+d'_n)\alpha'_{n2}]\Big\}\cos ny \\ &+\Big\{[(a'_n-c'_n)\alpha'_{n1}+(b'_n+d'_n)\alpha'_{n2}]q'_n-[(a'_n-c'_n)\alpha'_{n2}- \\ &\qquad -(b'_n+d'_n)\alpha'_{n1}]r'_n\Big\}\sin ny\Big] \end{aligned} \tag{180}$$

It is to be noted that in the above equation the constants $f_n, g_n, \ldots g'_n, h'_n, k'_n$ which should have occurred in the expression for $\frac{\partial F}{\partial x}$ have been replaced by $a_n, b_n, \ldots c'_n, d'_n$ by the relations (65)–(72).

Hence from (179) and (180), we can calculate the value of degree of polarization of the emergent radiation which is given by

$$\delta = \frac{I_r(0, y)-I_l(0, y)}{I_r(0, y)+I_l(0, y)} \tag{181}$$

for different values of y.

6. Application to a Gaussian distribution

Let us suppose that the spectral distribution of the unpolarized radiation across the photospheric surface is of the following simple type

$$\psi(y) = \frac{2}{\sqrt{\pi}} e^{-y^2} \quad \ldots \quad (182)$$

we can expand this as a Fourier series in cosines between the limits $(-\pi, \pi)$, so that

$$\psi(y) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{1}^{\infty} e^{-n^2/4} \cos ny. \quad \ldots \quad (183)$$

This range $(-\pi, \pi)$ practically covers the significant part of the function $\psi(y)$. Then in equation (135),

$$A_0' = \frac{1}{\pi} \quad \ldots \quad (184)$$

$$A_n' = \frac{2}{\pi} e^{-n^2/4} \quad \ldots \quad (185)$$

$$B_n' = 0. \quad \ldots \quad (186)$$

For particular values of τ_1 (the optical thickness of the atmosphere) or x_1, we can calculate the values of the constants a_n, b_n, c_n, d_n . . . . d_n' for different values of n (equations (178), (177), (134), (133), (108), (107), (82), (81) and also the value of A_0 (equation (136), as the values of Fourier coefficients for the distribution at the base of the atmosphere are known from equations (184), (185), and (186).

Substituting these in equation (179), we can find out the value of emergent intensity $I_{l(+1)}(0, y) + I_{r(+1)}(0, y)$, from the outer surface of the atmosphere, when the distribution at the base is given by (182). The results calculated numerically for $x_1 = 1$ or $\tau_1 = \frac{2}{3}$ have been given in the second column of Table I. The intensities obtained in this case of polarized field for every value of y are compared with those for the radiation field in which polarization is not taken account of. These latter values of intensity have been calculated in a previous paper of the author, by the method of trigonometrical series and the results obtained were found to be as accurate as those worked out by Chandrasekhar by the method of Green's function. These values of intensities are noted in the third column of Table I. The comparative results are exhibited graphically in Fig. 1. The curve drawn with dots and dashes represents the intensities for the polarized field and the curve in dashes only, those when polarization is ignored.

Table I

y	$I_{+1}(0, y, \psi)$ (for the polarized field)	$I_{+1}(0, y, \psi)$ (where polarization is ignored)
0·0	0·35	0·34
0·5	0·40	0·41
1·0	0·38	0·40
1·5	0·31	0·33
2·0	0·23	0·24
2·5	0·16	0·16
3·0	0·11	0·11

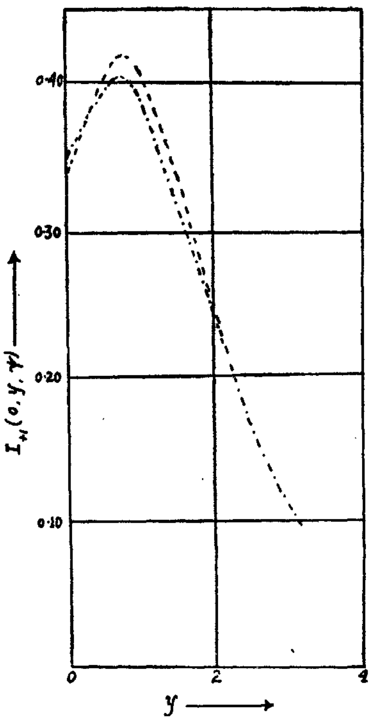

FIG. 1

Compton shift and distribution of intensity of the emergent radiation passing through an electron scattering atmosphere; the photospheric distribution is supposed to follow Gaussian law. y is the shift in units of 2/3 Compton wavelength.

—·—·—·— for the polarization field.
— — — — — — — when polarization is ignored.

The table and the adjoining figure show that the intensity distribution of the emergent radiation is slightly modified, when we consider the electron atmosphere to scatter according to Rayleigh's law and allow for the polarization of the radiation field. In any accurate calculation they should be the same. However, the position of the peak of the intensity curve remains the same.

Table II contains the values of degree of polarization of the emergent radiation (obtained from equations (179), (180) and (181) calculated numerically for an atmosphere of above description of optical thickness $\tau_1 = \frac{2}{3}$ or $x_1 = 1$. The variation of the degree of polarization with y has been shown graphically in Fig. 2.

TABLE II

y	$I_{r(+1)}(0, y, \psi) - I_{l(+1)}(0, y, \psi)$	$I_{r(+1)}(0, y, \psi) + I_{l(+1)}(0, y, \psi)$	$\delta = \frac{I_{r(+1)} - I_{l(+1)}}{I_{r(+1)} + I_{l(+1)}}$
0·0	0·09	0·35	0·26
0·5	0·04	0·40	0·10
1·0	0·00	0·38	0·00
1·5	0·00	0·31	0·00
2·0	0·02	0·23	0·09
2·5	0·02	0·16	0·12
3·0	−0·01	0·11	−0·09

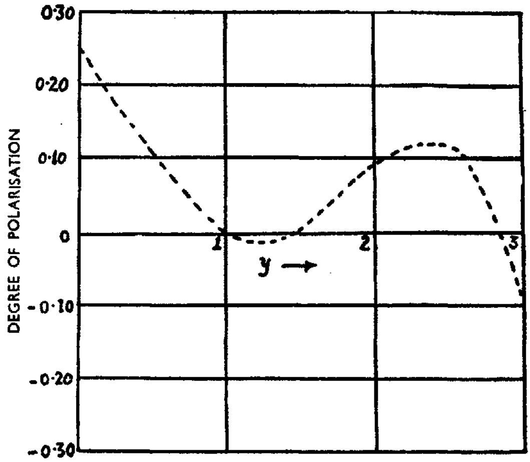

FIG. 2

Distribution of polarization of the emergent radiation passing through an electron scattering atmosphere; the photospheric distribution is supposed to follow Gaussian law. y is the shift in units of 2/3 Compton wavelength.

7. Application to monochromatic radiation

We shall now work out the case of a monochromatic radiation at the photospheric surface. One needs for this purpose a Fourier series representation of the δ-function. The recent theory of distribution by Schwartz classifies δ-function not as a point function but as a distribution and the expansibility of the distribution function δ has been rigorously established by Schwartz (Halperin, 1952). Below we formally obtain the expansion in the usual way.

Suppose we want to expand $\psi(y) = \delta(y)$ in the interval $-\pi$ and π. We have

$$\left.\begin{aligned} \psi(y) &= 0 \quad \text{for} \quad -\pi \leqslant y \leqslant -\epsilon \\ \psi(y) &= 0 \quad \text{for} \quad \epsilon \leqslant y \leqslant \pi \end{aligned}\right\} \qquad (187)$$

and

$$\underset{\epsilon \to 0}{\text{Lt}} \int_{-\epsilon}^{\epsilon} \psi(y)\, dy = 1 \qquad (188)$$

Considering this as a symmetric function let us assume the expansion

$$\psi(y) = c_0 + \sum_{n=1}^{\infty} c_n \cos ny \qquad (189)$$

Then multiplying by dy and integrating between $-\pi$ and $+\pi$, we have

$$\underset{\epsilon \to 0}{\text{Lt}} \int_{-\epsilon}^{\epsilon} \psi(y)\, dy = 2\pi c_0 \qquad (190)$$

Hence from (188), we have

$$c_0 = \frac{1}{2\pi} \qquad (191)$$

Similarly multiplying (189) by cos ny and integrating between the same limits, we have

$$\underset{\epsilon \to 0}{\mathrm{Lt}} \int_{-\epsilon}^{\epsilon} \psi(y) \cos ny \, dy = \pi c_n. \quad \ldots \quad (192)$$

As the left-hand side is unity, we have

$$c_n = \frac{1}{\pi} \quad \ldots \quad (193)$$

Hence we represent the above $\psi(y)$ function by the Fourier series

$$\psi(y) = \frac{1}{2\pi}\left[1+2\sum_{n=1}^{\infty} \cos ny\right] \quad \ldots \quad (194)$$

Thus

$$\left.\begin{aligned} A_0' &= \frac{1}{2\pi} \\ A_n' &= 2 \\ B_n' &= 0 \end{aligned}\right\} \quad \ldots \quad (195)$$

Hence we can evaluate the constants occurring in (179) and (180) for monochromatic radiation distribution at the inner surface of atmosphere and obtain the values of $I_{l(+1)}(0, y)+I_{r(+1)}(0, y)$ and the degree of polarization for different values of y, by the method described in art. 6.

In Table III is shown the comparative values of intensity in the above case for the polarized radiation field and for that where polarization is ignored. The table and the adjoining figure (Fig. 3) corroborate the conclusions arrived at in art. 6.

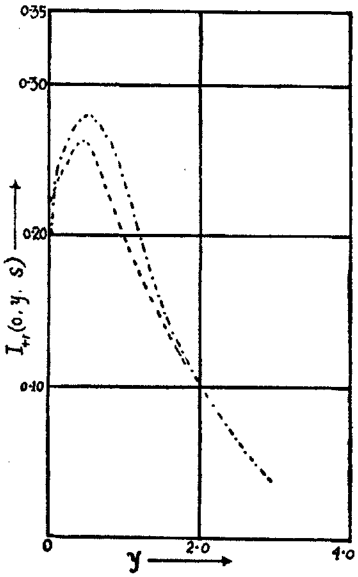

FIG. 3

Compton shift and distribution of intensity of the emergent radiation passing through an electron scattering atmosphere; the photospheric distribution is supposed to be monochromatic. y is the shift in units of 2/3 Compton wavelength.

—·—·—·— when polarization is ignored.

— — — — — for the polarized field.

TABLE III

y	$I_{+1}(0, y, \delta)$ (without polarization)	$I_{+1}(0, y, \delta)$ (with polarization)
0·0	0·20	0·22
0·5	0·28	0·26
1·0	0·24	0·20
1·5	0·16	0·15
2·0	0·11	0·11
2·5	0·07	0·07
3·0	0·04	0·04

Table IV contains the distribution of degree of polarization for different values of y, when the distribution of intensity at the base of the boundary is supposed to be monochromatic. The results are shown in the adjoining figure.

TABLE IV

y	$I_{r(+1)}(0, y) - I_{l(+1)}(0, y)$	$I_{r(+1)}(0, y) + I_{l(+1)}(0, y)$	$\frac{I_{r(+1)} - I_{l(+1)}}{I_{r(+1)} + I_{l(+1)}}$ (Degree of polarization)
0·0	0·11	0·22	0·50
0·5	0·00	0·26	0·00
1·0	−0·06	0·20	−0·30
1·5	−0·01	0·15	−0·07
2·0	0·03	0·11	0·27
2·5	0·03	0·07	0·43
3·0	0·01	0·04	0·25

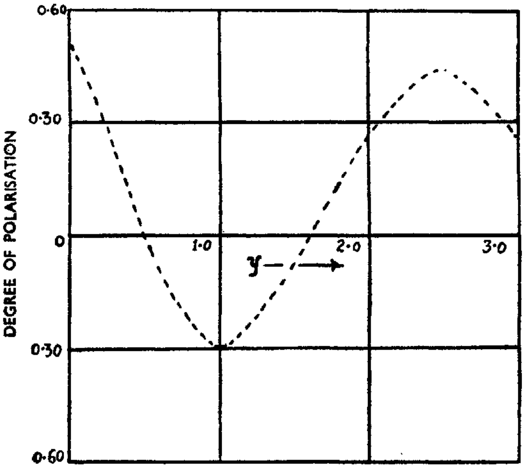

FIG. 4

Distribution of polarization of the emergent radiation passing through an electron scattering atmosphere; the photospheric distribution is supposed to be monochromatic. y is the shift in units of 2/3 Compton wavelength.

In conclusion it gives the author much pleasure to record his sincerest gratitude to Prof. N. R. Sen, F.N.I., for his kind guidance and advice, during the preparation of this work.

Abstract

The problem of the softening of radiation due to multiple Compton scattering in an axially symmetric, plane parallel, electron atmosphere, scattering according to Rayleigh's law and taking into account the polarization of the radiation field has been solved in the first approximation by the method of trigonometrical series. The intensity distribution and the distribution of the degree of polarization of the outcoming radiation for a monochromatic and a Gaussian distribution of the base have been calculated and the intensity distribution has been compared with the corresponding distribution when polarization is not taken account of.

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