ON THE EQUILIBRIUM OF A SMALL CONDUCTING LIQUID DROP IN A UNIFORM EXTERNAL ELECTRIC FIELD

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1. Introduction

The problem of the equilibrium configuration of an incompressible and infinitely conducting gravitating fluid sphere in a magnetic field has been investigated by several authors (Chandrasekhar and Fermi, 1953; Ferraro, 1954; Auluck and Kothari, 1955). It is found that in the presence of the magnetic field the sphere gets deformed into a spheroid of revolution with its axis along the field.

In the present note we investigate the equilibrium configuration of a conducting liquid drop in a uniform external electric field by two different methods, namely, the energy method and the equilibrium method. The drop is assumed to be small enough so that the gravitational effect can be neglected. In the energy method we calculate the change in the electric energy and the surface tension energy when the drop is subjected to a small deformation. The equilibrium configuration satisfying the minimal energy condition is found to be a prolate spheroid of ellipticity,

$$\frac{\epsilon}{R} = \frac{3E_0^2R}{16\pi T}, \left(\text{for } \frac{\epsilon}{R} << 1\right)$$

where E_0 is the strength of the electric field, R the radius of the undeformed liquid drop and T the surface tension.

In the equilibrium method we obtain the expressions for electric pressure and the surface tension pressure at the surface of the drop. Equating the net pressure to zero we obtain an expression for the ellipticity of the resulting configuration which, indeed, is the same as that obtained by the energy method. Experimentally it is also observed that the liquid drop, subject to a uniform external electric field, becomes a prolate spheroid in accordance with the above relation.

2. Formulation of the problem

In this section the energy method (Auluck and Kothari, 1955) is followed. Consider a small conducting liquid drop of radius R placed in a uniform external electric field E_0 applied along the Z-axis. We assume that this field is due to a charge of strength $+e$ ($= E_0 d^2$) situated on the axis of symmetry of the liquid drop and at a large distance d ($d >> R$) from it. This makes the electric field almost uniform at the liquid drop and at the same time the electric field vanishes at infinity. The liquid drop in the presence of the electric field is deformed and its new boundary is defined by

$$r = R\left[1+\frac{\epsilon}{R}P_l(\mu)\right], \quad \ldots \quad (1)$$

where $\frac{\epsilon}{R}(<<1)$ is the ellipticity, $P_l(\mu)$ is the Legendre polynomial of order l and $\mu = \cos\theta$. Since the liquid drop is conducting, the electric field does not exist inside it. The potential Φ satisfies the Laplacian equation $\nabla^2\Phi = 0$ and the tangential component of the electric field at the surface of the conducting drop is zero. In Fig. 1, let the charge $+e$ be placed at O at a distance d from the centre C of the liquid drop.

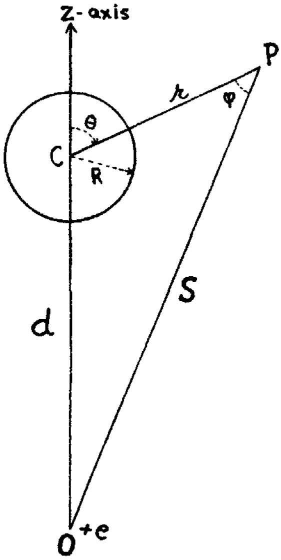

FIGURE 1

The radial and the transverse components of the electric field at any point P outside the liquid drop are given by

$$E_r^0 = \frac{E_0 d^2}{S^2}\cos\phi + \frac{2E_0R^3}{r^3}\cos\theta$$

and

$$E_\theta^0 = -\frac{E_0 d^2}{S^2}\sin\phi + \frac{E_0R^3}{r^3}\sin\theta\,, \qquad \ldots \quad \ldots \quad \ldots \quad (2)$$

where S is the distance of the point P from the charge, ϕ is the angle that OP makes with CP. We have

$$\sin\phi = \frac{d\sin\theta}{S} \text{ and } \cos\phi = \frac{r+d\cos\theta}{S}, \qquad \ldots \quad \ldots \quad \ldots \quad (3)$$

where θ is the polar angle.

The components of the electric field outside the deformed liquid drop at any point are given by

$$E_r^{(e)} = \frac{E_0d^2}{S^3}(r+d\cos\theta) + \frac{2E_0R^3}{r^3}\cos\theta + \frac{\epsilon}{R}E_0\sum_n a_n\,.\,n(n+1)\left(\frac{R}{r}\right)^{n+2}P_n(\mu)$$

and

$$E_\theta^{(e)} = \frac{-E_0 d^3}{S^3}\sin\theta + \frac{E_0 R^3}{r^3}\sin\theta + \frac{\epsilon}{R} E_0 \sum_n a_n \,.\, n \left(\frac{R}{r}\right)^{n+2} P_n'(\mu)\sin\theta. \quad .. \quad (4)$$

In order to evaluate the constant a_n we use the boundary condition that the tangential component of the electric field at the surface of the conducting drop is zero, i.e.

$$\left[E_\theta^{(e)}\right]_{R+\epsilon P_l} + \left[E_r^{(e)}\right]_R \left[-\frac{\epsilon}{R}\sin\theta P_l'(\mu)\right] = \left[E_\theta^0\right]_R = 0 \quad .. \quad (5)$$

From (4) and (5) we have

$$\sum_n a_n \,.\, nP_n'(\mu) = \frac{3}{(2l+1)}\left[(l+1)P_{l+1}'(\mu) + lP_{l-1}'(\mu)\right] \quad .. \quad .. \quad (6)$$

which shows that a_n is zero except for $n = l-1$ *or* $l+1$.
Hence we have

$$\left.\begin{aligned} a_{l+1} &= \frac{3}{2l+1}, \\ a_{l-1} &= \frac{3l}{(2l+1)\,(l-1)}. \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad (7)$$

From (4) and (7) we get

$$E_r^{(e)} = \frac{E_0 d^2}{S^3}(r + d\cos\theta) + \frac{2E_0 R^3}{r^3}\cos\theta + \frac{\epsilon}{R} E_0 \frac{3}{(2l+1)}$$

$$\times\left[(l+1)(l+2)\left(\frac{R}{r}\right)^{l+3} P_{l+1}(\mu) + l^2\left(\frac{R}{r}\right)^{l+1} P_{l-1}(\mu)\right], \quad .. \quad (8)$$

$$E_\theta^{(e)} = \frac{-E_0 d^3}{S^3}\sin\theta + \frac{E_0 R^3}{r^3}\sin\theta + \frac{\epsilon}{R} E_0 \frac{3}{(2l+1)}$$

$$\times\left[(l+1)\left(\frac{R}{r}\right)^{l+3} P_{l+1}'(\mu) + \left(\frac{R}{r}\right)^{l+1} l \,.\, P_{l+1}'(\mu)\right]\sin\theta.$$

The change in the electric energy outside the liquid drop up to first order in $\frac{\epsilon}{R}$ due to P_l-deformation is given by

$$\delta\mathcal{E} = -\frac{1}{8\pi}\int_R^{R+\epsilon P_l}\int_{-1}^{+1}\int_0^{2\pi}\left[(E_r^0)^2 + (E_\theta^0)^2\right] r^2 dr\, d\mu\, d\phi$$

$$+\frac{1}{8\pi}\int_R^{\infty}\int_{-1}^{+1}\int_0^{2\pi} 2\left[E_r^{(0)}\,\delta E_r^{(e)} + E_\theta^0\,\delta E_\theta^{(e)}\right] r^2 dr\, d\mu\, d\phi. \quad .. \quad (9)$$

From (2), (8) and (9) we get

$$\delta\mathcal{E} = \frac{-E_0^2R^6}{2}\int_{-1}^{+1}\int_{R}^{R+\epsilon P_l}\frac{1+P_2}{r^4}\,dr\,d\mu - E_0^2R^3\int_{-1}^{+1}\int_{R}^{R+\epsilon P_l}\frac{rP_1+P_2d}{rS}\,dr\,d\mu$$

$$+\frac{3}{2}E_0^2\frac{\epsilon}{R}\frac{l^2}{2l+1}d^2\int_{-1}^{+1}\int_{R}^{\infty}\frac{rP_{l-1}+P_l\,d}{S^3}\left(\frac{R}{r}\right)^{l+1}r^2\,dr\,d\mu$$

$$+\frac{3}{2}E_0^2\frac{\epsilon}{R}\frac{(l+1)(l+2)}{(2l+1)}d^2\int_{-1}^{+1}\int_{R}^{\infty}\frac{rP_{l+1}'+P_{l+2}\,d}{S^3}\left(\frac{R}{r}\right)^{l+3}r^2\,dr\,d\mu$$

$$+\frac{3}{2}E_0^2\frac{\epsilon}{R}\frac{l^2}{(2l+1)(2l-1)}\int_{-1}^{+1}\int_{R}^{\infty}\Big[(l+1)P_l+3(l-1)P_{l-2}\Big]\left(\frac{R}{r}\right)^{l+4}r^2\,dr\,d\mu$$

$$+\frac{3}{2}E_0^2\frac{\epsilon}{R}\frac{(l+1)(l+2)}{(2l+1)(2l+3)}\int_{-1}^{+1}\int_{R}^{\infty}\Big[3(l+1)P_l+(l+3)P_{l+2}\Big]\left(\frac{R}{r}\right)^{l+6}r^2\,dr\,d\mu.$$

Evaluating the integrals we get

$$\delta\mathcal{E} = -\frac{3}{5}E_0^2R^3\frac{\epsilon}{R} \quad \text{for} \quad l=2$$

$$= 0 \ \text{for} \ l\neq 2, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (10)$$

which is the required expression for the change in the electric energy due to P_2-deformation.

We shall now calculate the change in the surface tension energy due to P_2-deformation. As a result of deformation the surface area of the liquid drop changes from $4\pi R^2$ to $4\pi R^2\left[1+\frac{2}{5}\left(\frac{\epsilon}{R}\right)^2\right]$. Therefore the surface tension energy for a deformed liquid drop comes out to be

$$S = 4\pi R^2 . T\left[1+\frac{2}{5}\left(\frac{\epsilon}{R}\right)^2\right], \qquad \ldots \quad \ldots \quad \ldots \qquad (11)$$

where T is the surface tension energy per unit area.
Therefore the change in the surface tension energy is

$$\delta S = \frac{8}{5}\pi R^2T\left(\frac{\epsilon}{R}\right)^2. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (12)$$

Thus the total change in the energy of the system is given by

$$\delta W = \delta\mathcal{E}+\delta S,$$

$$= -\frac{3}{5}E_0^2R^3\left(\frac{\epsilon}{R}\right)+\frac{8}{5}\pi R^2T\left(\frac{\epsilon}{R}\right)^2. \qquad \ldots \quad \ldots \quad \ldots \qquad (13)$$

For stability this should be minimum. Therefore we have

$$\frac{16}{5}\pi R^2T\frac{\epsilon}{R} = \frac{3}{5}E_0^2R^3,$$

which gives

$$\frac{\epsilon}{R} = \frac{3E_0^2 R}{16\pi T}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

This is the required expression for the ellipticity of a prolate spheroid for $\frac{\epsilon}{R} << 1$.

3. In this section the equilibrium method (Ferraro, 1954) is followed. The electric pressure at the surface of the liquid drop is given by

$$[E_p]_s = \frac{E_0^2}{8\pi} = \frac{1}{8\pi}\left[(E_r^0)_s^2 + (E_\theta^0)_s^2\right]. \qquad \ldots \quad \ldots \quad \ldots \quad (15)$$

Making use of (2) and (3), and since at the surface of the liquid drop $S \sim d$, we have

$$[E_r^0]_s = 3E_0\mu,$$

$$[E_\theta^0]_s = 0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

which implies that the contribution to the electric pressure comes only from the radial part of the electric field. Thus the electric pressure at the surface of the liquid drop from (15) and (16) is

$$[E_p]_s = \frac{9E_0^2\mu^2}{8\pi}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

which elongates the liquid drop and is balanced by the surface tension pressure.

We shall now proceed to calculate the surface tension pressure.

The principal radii of curvature at any point on the surface of a spheroid (prolate) in spherical polar co-ordinates are given by

$$\rho^2 - \frac{\rho}{p} r^2 \left[\frac{b^2+c^2}{b^2}(1-\mu^2) + \frac{2b^2}{c^2}\mu^2\right] + \frac{c^2 b^4}{p^4} = 0, \qquad \ldots \quad \ldots \quad (18)$$

where

$$p^2 = \frac{1}{r^2\left[\frac{1-\mu^2}{b^4} + \frac{\mu^2}{c^4}\right]}.$$

If ρ_1, ρ_2 be the two principal radii of curvature, then $\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$ at any point on the surface of the prolate spheroid is given by

$$\frac{\frac{b^2+c^2}{c^2}(1-\mu^2) + \frac{2b^4}{c^4}\mu^2}{R\left(1+\frac{\epsilon}{R}P_2\right)\left[(1-\mu^2) + \frac{b^4}{c^4}\mu^2\right]^{\frac{3}{2}}}, \qquad \ldots \quad \ldots \quad \ldots \quad (19)$$

where $a = b = R - \frac{\epsilon}{2}$ are the two semi-minor axes of the prolate and $c = R + \epsilon$ is the semi-major axis. Using these values in (19) we have

$$\frac{2}{R}\left[1 + \frac{2\epsilon}{R}P_2 + \frac{\epsilon^2}{R^2}\left\{\frac{21}{8}(1+\mu^2) - P_2\left(\frac{3}{2}\mu^2 + 2P_2\right)\right\}\right]. \qquad \ldots \quad (20)$$

Therefore the change in the surface tension pressure due to deformation up to first order in $\frac{\epsilon}{R}$ is $\frac{4T}{R}\frac{\epsilon}{R}P_2$, which can be written as

$$\frac{2T}{R}\frac{\epsilon}{R}(3\mu^2-1). \quad \ldots \quad (21)$$

For stability the angular part $\frac{2T}{R}\frac{\epsilon}{R}3\mu^2$ of the surface tension pressure must be equal to the electric pressure at the surface which involves μ^2 and is given by (17). The remaining part of the surface tension pressure in (21) is balanced by the hydrostatic pressure. Therefore we have

$$\frac{2T}{R}\frac{\epsilon}{R}3\mu^2 = \frac{9E_0^2\mu^2}{8\pi}$$

or

$$\frac{\epsilon}{R} = \frac{3E_0^2R}{16\pi T}, \quad \ldots \quad (22)$$

which is the required expression for the ellipticity of a prolate spheroid for $\frac{\epsilon}{R} << 1$.

The expressions (14) and (22) are exactly identical and hold for small values of $\frac{\epsilon}{R}$. The above relation has also been experimentally verified, the details of which will be given in a subsequent paper.

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Summary

The problem of the equilibrium configuration of a small conducting liquid drop in a uniform external electric field has been investigated by two different methods, namely, the energy method and the equilibrium method. The results obtained by both the methods show that the equilibrium configuration is a prolate spheroid of ellipticity given by

$$\frac{\epsilon}{R} = \frac{3E_0^2R}{16\pi T} \left(\text{for } \frac{\epsilon}{R} << 1\right)$$

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