STOCHASTIC PROCESSES ASSOCIATED WITH INTEGRALS OF A CLASS OF RANDOM FUNCTIONS

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1. Introduction

Many stochastic problems arise in physics where we have to deal with integrals or iterated integrals of random functions. The normal interpretation of such integrals of random functions as put forward by Ramakrishnan (1955) is an extension of the idea of Riemann integration to random functions. Let $x(t)$ be the random variable representing a process and in any experiment if we 'plot' the value of $x(t)$ against t, we obtain a 'realised trajectory' of the process. The realised value of the stochastic variable $y(\tau) = \int_0^\tau x(\tau')\, d\tau'$ corresponding to the realised curve of $x(\tau')$ in the interval $0 < \tau' < \tau$ is defined as the area enclosed by the trajectory, the τ-axis and the ordinates 0 and t. Iterated integrals of $x(\tau)$ can be defined similarly. If we wish to obtain the probability frequency function of $y(\tau)$ or of iterated integrals of $x(\tau)$ we have at first to assign a probability measure to the trajectory of $x(\tau)$ in the interval $(0, t)$ and this is known to be a very difficult problem (see for example Doob, 1953) in all except simple cases. However, if we confine ourselves to a random function $x(\tau)$ which represents a 'basic random process'—a Markoff process, homogeneous with respect to τ, whose typical trajectory is characterised by a finite number of discrete transitions, the trajectory remaining parallel to the τ-axis between two transitions—it is quite easy to assign a measure to its trajectory. In this paper we shall deal with the Poisson process, the simplest B.R.P. (basic random process) available to us.

2. Statement of the problem

In his recent work Ramakrishnan considered a class of stochastic processes represented by the random variable $y_m(t)$ defined symbolically as

$$y_m(t) = \int_0^t \phi_m(\tau_m)d\tau_m \int_0^{\tau_m} \phi_{m-1}(\tau_{m-1})d\tau_{m-1} \cdots \int_0^{\tau_2} \phi_1(\tau_1)x(\tau_1)d\tau_1 \quad \ldots \quad (1)$$

where $\phi_1(\tau)$, $\phi_2(\tau)$, . . ., $\phi_m(\tau)$ are deterministic functions of τ and $x(\tau)$ represents a basic random process. Equation (1) can be written in the form

$$\left.\begin{aligned} y_m(t) &= \int_0^t \phi_m(\tau)y_{m-1}(\tau)d\tau \\ y_1(t) &= \int_0^t \phi_1(\tau)x(\tau)d\tau \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

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showing that the p.f.f. (probability frequency function) of $y_m(t)$ can be obtained if the joint p.f.f. of $y_m(t)$ and $y_{m-1}(t)$ is obtained, and this in turn depends on the joint p.f.f. of $y_{m-1}(t)$ and $y_{m-2}(t)$ and so on.

In this paper we shall consider the case when $x(t)$ has a Poisson distribution. In this case we deal with the more general integral

$$y_m(t) = \int_0^t \phi_m(\tau_m)d\tau_m \int_0^{\tau_m} \phi_{m-1}(\tau_{m-1})d\tau_{m-1} \dots \int_0^{\tau_1} \phi_0(\tau_0) \frac{dx(\tau_0)}{d\tau_0} d\tau_0 \quad .. \quad (3)$$

$$y_0(t) = \int_0^t \phi_0(\tau) \frac{dx(\tau)}{d\tau} d\tau$$

where the random function $\frac{dx(\tau)}{d\tau}$ will be interpreted in Section 3. We recognise (1) at once as the particular case of (3) when in the latter we put $\phi_0(\tau) \equiv 1$. Our object is to find out the probability distribution of $y_m(t)$. In the following, we derive an explicit expression for the Laplace transform of the p.f.f. of $y_m(t)$ and go on to discuss particular cases of physical interest.

3. Laplace transform solution of the problem

If $x(\tau)$ is the random variable representing the number of 'events' occurring in a Poissonian manner, in the interval $(0, t)$, the probability that $x(t) = x$ at t ($x(t) = 0$ at $t = 0$) is

$$\pi(x\,;\,t) = e^{-\lambda t} \frac{(\lambda t)^x}{x\,!} \quad .. \quad .. \quad .. \quad .. \quad (4^*)$$

where λ is the probability per unit t that an event occurs. x can assume values 0, 1, 2, ... The characteristic feature of the trajectory or curve of growth of $x(\tau)$ is that its ordinate jumps up by unity when an event occurs and then remains constant till the occurrence of the next event. The realised value of $x(\tau)$ is given by

$$x^R(\tau) = \sum_{i=1}^{x} H(\tau - t_i) \quad .. \quad .. \quad .. \quad .. \quad (5)$$

where $t_1, t_2, \dots, t_x$ are the points at which the events have occurred and $H(x)$ is the Heaviside unit function. The following interpretation can now be given to the random variable $\frac{dx(\tau)}{d\tau}$. The realised value of $\frac{dx(\tau)}{d\tau}$ in the interval $(0, t)$ is given by

$$\frac{dx^R(\tau)}{d\tau} = \sum_{i=1}^{x} \delta(\tau - t_i) \quad .. \quad .. \quad .. \quad .. \quad (6)$$

δ being the Dirac delta function.

Now consider equation (1) and the aggregate of random variables $y_i (i = 0, 1, 2, \dots m)$. We increase t by Δ. If an event occurs between t and $t+\Delta$, the random variables y_i $(i = 1, 2, \dots m)$ increase only by infinitesimal quantities of $O(\Delta)$. The increase in y_0 is $\phi_0(t)+O(\Delta)$. If no event occurs, the value of y_0 is

* We shall use the same symbols π and p to denote any p.f.f. and its Laplace transform respectively. The distinction between two functions will be apparent from the context.

unaltered, while every other random variable y_i ($i = 1, 2, \ldots m$) gets an increment equal to $\phi_i\, y_{i-1}\, \Delta + o(\Delta)$. Hence, if $\pi(y_m, y_{m-1}, \ldots, y_0 ; t)$ be the joint p.f.f. of $y_m, y_{m-1}, \ldots y_1, y_0$, it satisfies the equation

$$\frac{\partial \pi(y_m, y_{m-1}, \ldots, y_0 ; t)}{\partial t} = -\lambda \pi(y_m, y_{m-1}, \ldots, y_0 ; t) + \lambda \pi(y_m, y_{m-1}, \ldots, y_0 - \phi_0(t) ; t) - \sum_{i=1}^{m} y_{i-1}\phi_i(t) \frac{\partial \pi(y_m, y_{m-1}, \ldots, y_0 ; t)}{\partial y_i} \quad \ldots \quad (7)$$

with the initial condition

$$\pi(y_m, y_{m-1}, \ldots, y_0 ; 0) = \delta(y_m)\delta(y_{m-1}) \ldots \delta(y_0) \quad \ldots \quad \ldots \quad (8)$$

Defining the m-fold Laplace transform of $\pi(y_m, y_{m-1}, \ldots, y_0 ; t)$ as

$$p(s_m, s_{m-1}, \ldots, s_0 ; t) = \int_0^\infty \int_0^\infty \ldots \int_0^\infty \pi(y_m, y_{m-1}, \ldots, y_0 ; t) \exp\left[-s_m y_m - s_{m-1} y_{m-1} \ldots - s_0 y_0\right] . dy_m dy_{m-1} \ldots dy_0 \quad \ldots \quad (9)$$

we obtain

$$\frac{\partial p(s_m, s_{m-1}, \ldots, s_0 ; t)}{\partial t} = \sum_{i=1}^{m} s_i \phi_i(t) \frac{\partial p(s_m, s_{m-1}, \ldots, s_0 ; t)}{\partial s_{i-1}} - \lambda\left(1 - e^{-s_0 \phi_0(t)}\right) p(s_m, s_{m-1}, \ldots, s_0 ; t) \quad \ldots \quad (10)$$

(10) is a linear partial differential equation and therefore can be solved by the standard method of characteristics. The subsidiary equations are given by

$$\frac{dt}{1} = -\frac{ds_{m-1}}{s_m \phi_m(t)} = -\frac{ds_{m-2}}{s_{m-1}\phi_{m-1}(t)} = \ldots = -\frac{ds_0}{s_1 \phi_1(t)} \quad \ldots \quad (11)$$

The characteristics are given by

$$\int_0^t \phi_m(t_m) dt_m + \frac{s_{m-1}}{s_m} = \psi_{m-1}$$

$$\int_0^t \phi_{m-1}(t_{m-1}) \left\{ \psi_{m-1} - \int_0^{t_{m-1}} \phi_m(t_m) dt_m \right\} dt_{m-1} + \frac{s_{m-2}}{s_m} = \psi_{m-2}$$

$$\int_0^t \phi_{m-2}(t_{m-2}) \left\{ \psi_{m-2} - \int_0^{t_{m-2}} \phi_{m-1}(t_{m-1}) \left\{ \psi_{m-1} - \int_0^{t_{m-1}} \phi_m(t_m) dt_m \right\} dt_{m-1} \right\} dt_{m-2} + \frac{s_{m-3}}{s_m} = \psi_{m-3} \quad \ldots \quad (12)$$

In principle we can solve for $\psi_{m-1}, \psi_{m-2}, \ldots, \psi_1, \psi_0$ successively but we will not need the explicit expressions here. Equation (10) may be solved now by making a transformation of the variables $s_0, s_1, s_2, \ldots s_{m-1}$ to $\psi_0, \psi_1, \psi_2, \ldots, \psi_{m-1}$. The solution is finally obtained as

$$p(s_m, s_{m-1}, \ldots, s_1, s_0\,; t) = \exp\left[-\lambda t+\lambda\int_0^t \exp\left\{-s_m\phi_0 F_m\right\} dt_0\right] \quad .. \quad (13)$$

where F_m is given by

$$F_m = \psi_0 - \int_0^{t_0}\phi_1(t_1)dt_1\left[\psi_1-\int_0^{t_1}\phi_2(t_2)dt_2\left\{\psi_2-\ldots\left(\psi_{m-1}-\int_0^{t_{m-1}}\phi_m(t_m)dt_m\right)\right\}\right] \quad (14)$$

This completes the formal solution of (10). However we are interested in $\pi(y_m\,; t)$, the p.f.f. of y_m alone and not the joint p.f.f. of the aggregate $y_0, y_1, \ldots, y_m$. We note that the Laplace transform of $\pi(y_m\,; t)$ is $p(s, 0, 0, \ldots 0; t)$ since

$$\pi(y_m\,; t) = \int_{y_{m-1}}\int_{y_{m-2}}\cdots\int_{y_0}\pi(y_m, y_{m-1}, \ldots, y_0\,; t)dy_{m-1}dy_{m-2}\ldots dy_0$$

A simple expression for $p(s, 0, 0, \ldots 0; t)$ can be obtained from (14) if we put $s_0 = s_1 = \ldots = s_{m-1} = 0$, $s_m = s$ in (12) and solve for the ψ's. In such a case we find that

$$[F_m]s_m = s\,; s_{m-1} = s_{m-2} = \ldots = s_0 = 0$$

$$= \int_{t_0}^{t}\phi_1(t_1)dt_1\int_{t_0}^{t}\phi_2(t_2)dt_2\ldots\int_{t_{m-1}}^{t}\phi_m(t_m)dt_m \quad .. \quad .. \quad (15)$$

which we shall denote by

$$f_m(t, t_0)$$

Hence the Laplace transform of $\pi(y_m\,; t)$ is given by

$$p(s\,; t) = \exp\left[-\lambda t+\lambda\int_0^t \exp\left\{-\phi_0(t_0)f_m(t, t_0)s\right\} dt_0\right] \quad .. \quad (16)$$

Moments of y_m are obtained by making use of the relation,

$$\mathcal{E}\left\{y_m^n(t) = \left[\left(-\frac{\partial}{\partial s}\right)^n p(s\,; t)\right]_{s=0}\right\} \quad .. \quad .. \quad .. \quad (17)$$

The first three moments are

$$\mathcal{E}\{y_m(t)\} = \lambda\int_0^t\phi_0(t_0)f_m(t, t_0)dt_0$$

$$= \lambda\int_0^t\phi_m(t_m)dt_m\int_0^{t_m}\phi_{m-1}(t_{m-1})dt_{m-1}\ldots\int_0^{t_1}\phi_0(t_0)dt_0 \quad .. \quad .. \quad (18)$$

$$\mathcal{E}\left\{y_m^2(t)\right\} = \left[\mathcal{E}\{y_m(t)\}\right]^2+\lambda\int_0^t\phi_0^2(t_0)f_m^2(t, t_0)dt_0 \quad .. \quad .. \quad .. \quad .. \quad (19)$$

$$\mathcal{E}\left\{y_m^3(t)\right\} = \left[\mathcal{E}\left\{y_m(t)\right\}\right]^3 + 3\lambda^2\left[\int_0^t \phi_0(t_0) f_m(t, t_0) dt_0\right].$$
$$\left[\int_0^t \phi_0^2(t_0) f_m^2(t, t_0) dt_0\right] + \lambda \int_0^t \phi_0^3(t_0) f_m^3(t, t_0) dt_0 \quad \ldots \quad (20)$$

4. Particular cases of the stochastic process defined by (3)

(1) $m = 0$; The random variable $y_0(t)$ we are interested in is given by

$$y_0(t) = \int_0^t \phi_0(\tau) \frac{dx(\tau)}{d\tau} d\tau \quad \ldots \quad (21)$$

This random variable is of considerable importance in physics. A simple example can be cited to illustrate (21). Let us consider the fluctuation of voltage at the anode of a valve due to fluctuations in the number of electrons emitted per unit time by the cathode (known as 'shot effect', cf. Schottky, 1918). Since the average rate of emission of electrons remains constant, we may expect the number $x(t)$ of electrons emitted in an interval of time t to form a homogeneous additive process with a Poisson distribution. The equation governing the variation of fluctuating voltage $V(t)$ at the anode is then given by

$$\frac{dV}{dt} + \frac{V}{RC} = \frac{E}{RC} - \frac{\epsilon}{C}\frac{dx(t)}{dt} \quad \ldots \quad (22)$$

when it is assumed that the circuit between anode and earth is equivalent to a resistance R in parallel with a capacity C. ϵ is the charge of the electron. The solution of (22) is

$$V(t)e^{t/RC} - V_0 - E(e^{t/RC} - 1) = -\int_0^t \frac{\epsilon}{C}\frac{dx(\tau)}{d\tau} e^{\tau/RC} d\tau \quad \ldots \quad (23)$$

where V_0 is the value of V at $t = 0$.

The moments of V of any order can immediately be obtained by the use of equations (16) and (17). The mean square deviation of V for example is equal to $\lambda\epsilon^2 R/2C$, a result obtained by Moyal (1949) by a different method.

(2) More generally, let us consider the random variable (21). If pulses occur at random in accordance with the Poissonian law and if $\phi_0(\tau)$ is the response of a system to a single pulse at a time τ after the occurrence of the pulse,

$$\int_0^t \phi_0(t-\tau) \frac{dx(\tau)}{d\tau} d\tau$$

is the total response of the system at time t to all the pulses occurring in the interval $(0, t)$. Ramakrishnan has proved that

$$\int_0^t \phi_0(t-\tau) \frac{dx(\tau)}{d\tau} d\tau$$

represents a process 'equivalent' to that defined by

$$y_0(t) = \int_0^t \phi_0(\tau) \frac{dx(\tau)}{d\tau} d\tau,$$

i.e. the two processes have the same p.f.f. at all t though their 'curves of growth' are different. Therefore the moments of our response function are the same as those of $y_0(t)$, and the first two moments may be obtained directly from the general expression (18) and (19).

$$\mathcal{E}\{y_0(t)\} = \lambda \int_0^t \phi_0(\tau)\, d\tau \quad \ldots \quad (24)$$

$$\mathcal{E}\{y_0^2(t)\} = [\mathcal{E}\{y_0(t)\}]^2 + \lambda \int_0^t \phi_0^2(\tau) d\tau \quad \ldots \quad (25)$$

If in the above formulae we make $t \to \infty$ we obtain what are known as Campbell's Theorems, proof of which have been given and discussed by several workers (see, for instance, Rowland, 1936; Whittaker, 1938; Bell, 1953). The method we have developed here gives the proof in a very simple manner, and moreover provides an extension of Campbell's results even to the case when the system has not reached a stationary state.

(3) Put $\phi_0(\tau) \equiv 1$. The random variable $y_m(t)$ now becomes

$$y_m(t) = \int_0^t \phi_m(t_m) dt_m \int_0^{t_m} \phi_{m-1}(t_{m-1}) dt_{m-1} \ldots \int_0^{t_2} \phi_1(t_1) x(t_1) dt_1 \quad \ldots \quad (26)$$

so that we obtain iterated integrals of $x(t)$ as a particular case of (1) and their moments etc. may be written down immediately from equation (16) to (20).

(4) A particular case of interest is when $m = 1$, $\phi_0(\tau) \equiv 1$.

If we take $\phi_1(t) \equiv 1$ also, $y_1(t)$ is given by

$$y_1(t) = \int_0^t x(t_1) dt_1 \quad \ldots \quad (27)$$

$p(s; t)$, the Laplace transform of $\pi(y_1; t)$ is given by

$$p(s\,;\,t) = \exp\left[-\lambda t + \frac{\lambda}{s}(1 - e^{-st})\right] \quad \ldots \quad (28)$$

Inverting, we have

$$\pi(y_1\,;\,t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \exp\left[-\lambda t + \frac{\lambda}{s}(1 - e^{-st}) + s y_1\right] ds \quad \ldots \quad (29)$$

i.e.

$$\pi(y_1\,;\,t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{-\lambda t + s y_1} \sum_{n=0}^{\infty} \frac{\lambda^n}{s^n} \frac{(1 - e^{-st})^n}{n!} ds \quad \ldots \quad (30)$$

Expanding $(1 - e^{-st})^n$ binomially and remembering that

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{sz}}{s^n} ds = \frac{z^{n-1}}{(n-1)!} \text{ for } z > 0$$
$$= 0 \text{ otherwise} \quad \ldots \quad (31)$$

we obtain

$$\pi(y_1\,;\,t) = e^{-\lambda t}\delta(y_1) + \sum_{n=1}^{\infty} e^{-\lambda t} \frac{\lambda^n}{n!} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{(y_1 - rt)^{n-1}}{(n-1)!} H(y_1 - rt) \quad (32)$$

$\pi(y_1; t)$ is continuous for all $y_1 > 0$ except at $y_1 = t$; at $y_1 = t$, $\pi(y_1; t)$ has a finite discontinuity.

A simple example of processes represented by $y_1(t)$ would be the total age of individuals generated in a Poissonian manner.

Ramakrishnan obtained the same probability distribution by a different method. Writing

$$\int_0^t x(\tau)d\tau \equiv \int_0^t \frac{dx(\tau)}{d\tau}(t-\tau)d\tau$$

he showed, by using the method of inverse trajectories, that the stochastic process defined by

$$\int_0^t x(\tau)d\tau$$

is equivalent to that defined by

$$\int_0^t \frac{dx(\tau)}{d\tau}\tau\, d\tau$$

whose p.f.f. satisfies the stochastic equation

$$\frac{\partial\pi(y_1\,;\,t)}{\partial t} = -\lambda\pi(y_1\,;\,t)+\lambda\pi(y_1-t\,;\,t) \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

The solution of (33) is identical with the expression (32).

(5) Another case of interest is when $m = 1$, $\phi_0(t_0) \equiv 1$, $\phi_1(t_1) \equiv e^{-t_1}$; $y_1(t)$ is then given by

$$y_1(t) = \int_0^t e^{-t_1}\, x(t_1)dt_1 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

The Laplace transform of $\pi(y_1; t)$ in this case is

$$p(s\,;\,t) = \exp\left[-\lambda t+\lambda\int_0^t e^{s(e^{-t}-e^{-\tau})}d\tau\right] \quad \ldots \quad \ldots \quad \ldots \quad (35)$$

This solution can be arrived at from the equation

$$\frac{\partial\pi(y_1\,;\,t)}{\partial t} = -\lambda\pi(y_1\,;\,t)+\lambda\pi(y_1-\overline{1-e^{-t}}\,;\,t)+\frac{\partial}{\partial y_1}\{y_1\pi(y_1\,;\,t)\} \quad \ldots \quad (36)$$

obtained by Ramakrishnan by considering the equivalent process represented by the symbolic integral

$$e^{-t}\int_0^t (e^{\tau}-1)\frac{dx(\tau)}{d\tau}d\tau$$

(6) Take $\phi_i(\tau) \equiv 1$, $i = 0, 1, 2, \ldots m$. The random variable $y_m(t)$ is then

$$y_m(t) = \int_0^t dt_m \int_0^{t_m} dt_{m-1} \ldots \int_0^{t_2} x(t_1)dt_1 \quad \ldots \quad \ldots \quad (37)$$

and the p.f.f. of $y_m(t)$ now has the Laplace transform.

$$p(s\,;t) = \exp\left[-\lambda t+\lambda\int_0^t \exp\left\{-\frac{s(t-t_1)^m}{m\,!}\right\}dt_1\right] \quad \cdots \quad \cdots \quad (38)$$

This is easily seen to be the Laplace transform solution of the equation

$$\frac{\partial\pi(y_m\,;t)}{\partial t} = -\lambda\pi(y_m\,;t)+\lambda\pi\left(y_m-\frac{t^m}{m\,!}\,;t\right) \quad \cdots \quad \cdots \quad (39)$$

obtained by Ramakrishnan by the method of inverse trajectories.

We observe that in (3) $\phi_0(\tau_0)$ can be replaced by $\phi_0(\tau_0)\,g\,(\tau_1-\tau_0)$ when g is either a polynomial or an exponential function, since the essential feature of the technique we have developed lies in the simple fact that we always find a *finite set* of random variables $y_i(t)$ whose changes for an infinitesimal increment of the parameter t may be expressed solely in terms of variables belonging to the set $y_i(t)$.

As has been stated earlier the results of the present paper will find applications in the solution of differential equations involving random functions of time, since any solution of a linear differential equation can be represented as an iterated integral. Previous methods for dealing with such problems were based on spectral theory, but we hope that the interpretation of integrals of random functions given by Ramakrishnan and the methods developed by us in this paper are more suitable for the understanding of the problems from a phenomenological point of view.

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