FLOW OF A COMPRESSIBLE FLUID THROUGH A CIRCULAR PIPE

by M. RAY, *Professor of Mathematics, Agra College*

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1. Introduction

The problems of heat transfer to a fluid flowing with the Poiseuille velocity distribution through a circular pipe when the walls are kept at constant temperature as well as when at a uniform temperature gradient have been solved (Howarth, 1953) under very restricted conditions. Firstly for the Poiseuille flow, the assumptions are that the pressure is uniform over each section of the pipe and that the longitudinal pressure gradient is constant. Then in the equation of energy the terms due to the pressure gradient and the dissipation have been neglected.

In the present paper the problem of the flow of a perfect gas through a circular pipe has been studied under non-adiabatic conditions, conductivity and viscosity being taken into account. Only restriction is that μ, the coefficient of viscosity, is constant and σ, the Prandtl number, is unity, otherwise the flow is perfectly general.

2. Equations of Motion

Take the axis of the pipe as z—axis, the mouth of the pipe being the section $z = 0$. In cylindrical co-ordinates (r, θ, z), if u, v, w be the components of velocity, the components of strain are

$$\left.\begin{aligned} e_{rr} &= 2\frac{\partial u}{\partial r},\ e_{\theta\theta} = \frac{2}{r}\frac{\partial v}{\partial \theta}+\frac{2u}{r},\ e_{zz} = 2\frac{\partial w}{\partial z},\\ e_{\theta z} &= \frac{1}{r}\frac{\partial w}{\partial \theta}+\frac{\partial v}{\partial z},\ e_{zr} = \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r},\ e_{r\theta} = \frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial \theta}. \end{aligned}\right\} \quad \ldots \quad (1)$$

The divergence is

$$\Delta = \frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}. \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

The stress components are

$$\left.\begin{aligned} p_{rr} &= -p+\lambda\Delta+\mu e_{rr},\ p_{\theta\theta} = -p+\lambda\Delta+\mu e_{\theta\theta},\ p_{zz} = -p+\lambda\Delta+\mu e_{zz},\\ p_{r\theta} &= \mu e_{r\theta},\ p_{\theta z} = \mu e_{\theta z},\ p_{zr} = \mu e_{zr}, \end{aligned}\right\} \quad (3)$$

where p is the pressure.

Taking all quantities independent of θ, we have

$$\left.\begin{aligned} e_{rr} &= 2\frac{\partial u}{\partial r},\ e_{\theta\theta} = \frac{2u}{r},\ e_{zz} = 2\frac{\partial w}{\partial z},\\ e_{\theta z} &= \frac{\partial v}{\partial z},\ e_{zr} = \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r},\ e_{r\theta} = \frac{\partial v}{\partial r}-\frac{v}{r},\\ \Delta &= \frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (4)$$

The equations of motion are

$$\left.\begin{aligned} \rho\left(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}\right) &= \frac{\partial}{\partial r}p_{rr}+\frac{\partial}{\partial z}p_{zr}+\frac{1}{r}\left(p_{rr}-p_{\theta\theta}\right), \\ \rho\left(u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}\right) &= \frac{\partial}{\partial r}p_{r\theta}+\frac{\partial}{\partial z}p_{\theta z}-\frac{2}{r}p_{r\theta}, \\ \text{and} \qquad \rho\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right) &= \frac{\partial}{\partial r}p_{zr}+\frac{\partial}{\partial z}p_{zz}+\frac{1}{r}p_{zr}. \end{aligned}\right\} \quad \ldots \quad (5)$$

Taking u and v both zero, the equations (5) with the help of equations (4) and (3) reduce to

$$\left.\begin{aligned} 0 &= -\frac{\partial p}{\partial r}+\frac{\partial}{\partial r}\left(\lambda\frac{\partial w}{\partial z}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial r}\right) \\ \rho w\frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z}+\frac{\partial}{\partial r}\left(\mu\frac{\partial w}{\partial r}\right)+\frac{\partial}{\partial z}\left\{(\lambda+2\mu)\frac{\partial w}{\partial z}\right\}+\frac{\mu}{r}\frac{\partial w}{\partial r} \end{aligned}\right\} \quad \ldots \quad (6)$$

The equation of continuity under the above conditions is

$$\frac{\partial}{\partial z}(\rho w) = 0. \quad \ldots \quad (7)$$

Also the equation of energy takes the form

$$\rho w\frac{\partial i}{\partial z}-w\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu}{\sigma}r\frac{\partial i}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{\mu}{\sigma}\frac{\partial i}{\partial z}\right)+\Phi, \quad \ldots \quad (8)$$

where i, the enthalpy, is the heat content, σ the Prandtl number and Φ, the dissipation function given by

$$4\Phi = \lambda\left(e_{rr}+e_{\theta\theta}+e_{zz}\right)^2+2\mu\left(e_{rr}^2+e_{\theta\theta}^2+e_{zz}^2+2e_{r\theta}^2+2e_{\theta z}^2+2e_{zr}^2\right)$$

so that in the present case, it becomes

$$\Phi = (\lambda+2\mu)\left(\frac{\partial w}{\partial z}\right)^2+\mu\left(\frac{\partial w}{\partial r}\right)^2 \quad \ldots \quad (9)$$

The equation of state for a perfect gas is

$$i\rho = \frac{\gamma}{\gamma-1}p, \quad \ldots \quad (10)$$

where γ is the ratio of specific heats at constant pressure and at constant volume. Further we take the Stoke's assumption, namely,

$$3\lambda+2\mu = 0. \quad \ldots \quad (11)$$

These are all the equations. To solve these we make the following assumptions

$$\left.\begin{aligned} \mu &= \text{constant}, \\ \sigma &= 1. \end{aligned}\right\} \quad \ldots \quad (12)$$

3. Solutions of the Equations

From (7) we see that ρw may be a function of r but we assume it to be an absolute constant, so we put

$$\rho w = \rho_0 w_0 \quad \ldots \quad (13)$$

where the suffix zero indicates values at the centre of the mouth of the pipe. We further assume

$$w = e^{-\alpha z} w_1 \quad \ldots \quad (14)$$

where w_1 is a function of r only, and α some constant. Now the first equation of (6) gives

$$\frac{\partial p}{\partial r} = -\frac{2}{3}\mu \frac{\partial^2 w}{\partial r \partial z} + \mu \frac{\partial^2 w}{\partial r \partial z} = \frac{1}{3}\mu \frac{\partial^2 w}{\partial r \partial z},$$

so that

$$p = p_1 + \frac{1}{3}\mu \frac{\partial w}{\partial z} = p_1 - \frac{1}{3}\mu\alpha e^{-\alpha z} w_1, \quad \ldots \quad (15)$$

where p_1, a constant, is the hydrostatic pressure.

The second equation of (6) gives

$$\rho w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \mu \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{dw}{\partial r} + \frac{4}{3}\frac{\partial^2 w}{\partial z^2}\right],$$

hence with (13), (14) and (15) becomes

$$\frac{d^2 w_1}{dr^2} + \frac{1}{r}\frac{dw_1}{dr} + k^2 w_1 = 0, \quad \ldots \quad (16)$$

where

$$k^2 = \alpha^2 + \frac{\alpha}{\mu}\rho_0 w_0. \quad \ldots \quad (17)$$

The solution of equation (16) is

$$w_1 = w_0 J_0(kr) \quad \ldots \quad (18)$$

where J_0 is the Bessel function of order zero.

Now from (10) and (13) we have

$$\rho_0 w_0 i = \frac{\gamma}{\gamma - 1} p w$$

Therefore

$$\rho_0 w_0 \frac{\partial i}{\partial z} = \frac{\gamma}{\gamma-1}\left(w\frac{\partial p}{\partial z} + p\frac{\partial w}{\partial z}\right) = \frac{\gamma}{\gamma-1}\left(\frac{2}{3}\mu\alpha^2 e^{-2\alpha z} w_1{}^2 - p_1 \alpha e^{-\alpha z} w_1\right) \quad \ldots \quad (19)$$

from (14) and (15).

Now from (9), (11) and (12), the equation (8) becomes

$$\rho w \frac{\partial i}{\partial z} = w\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 i}{\partial r^2} + \frac{1}{r}\frac{\partial i}{\partial r} + \frac{\partial^2 i}{\partial z^2} + \frac{4}{3}\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2\right] \quad \ldots \quad (20)$$

so that with (19) we get

$$\frac{\gamma}{\gamma-1}\left(\frac{2}{3}\mu\alpha^2 e^{-2\alpha z} w_1{}^2 - p_1\alpha e^{-\alpha z} w_1\right)$$

$$= \frac{1}{3}\mu\alpha^2 e^{-2\alpha z} w_1^2 + \mu\left[\frac{\partial^2 i}{\partial r^2} + \frac{1}{r}\frac{\partial i}{\partial r} + \frac{\partial^2 i}{\partial z^2} + \frac{4}{3}\alpha^2 e^{-2\alpha z} w_1^2 + e^{-2\alpha z}\left(\frac{dw_1}{dr}\right)^2\right] \quad \ldots \quad (21)$$

To solve the equation (21), we put

$$i = i_1 + I_1 e^{-2\alpha z} + I_2 e^{-\alpha z}, \quad \ldots \quad (22)$$

where i_1, a constant, corresponds to hydrostatic temperature, and I_1, I_2 are functions of r only.

Substituting (22) in (21) and equating coefficients of $e^{-\alpha z}$ and $e^{-2\alpha z}$ from both sides, we get

$$\frac{d^2I_2}{dr^2}+\frac{1}{r}\frac{dI_2}{dr}+\alpha^2 I_2+\frac{\gamma}{\gamma-1}\frac{p_1\alpha}{\mu}w_1 = 0 \quad \ldots \quad (23)$$

and

$$\frac{d^2I_1}{dr^2}+\frac{1}{r}\frac{dI_1}{dr}+4\alpha^2 I_1 = \frac{5-3\gamma}{3(\gamma-1)}\alpha^2 w_1^2-\left(\frac{dw_1}{dr}\right)^2 \quad \ldots \quad (24)$$

To solve (23), we take $I_2 = Aw_1$, where A is a constant so that (23) gives

$$\frac{d^2w_1}{dr^2}+\frac{1}{r}\frac{dw_1}{dr}+\alpha^2 w_1+\frac{\gamma}{\gamma-1}\frac{p_1\alpha}{\mu A}w_1 = 0$$

hence comparing with (16) and (17), we get

$$\rho_0 w_0 = \frac{\gamma}{\gamma-1}\frac{p_1}{A} \quad \ldots \quad (25)$$

Now from (15) and (18) we have at the centre of the mouth ($r = 0$, $z = 0$),

$$p_0 = p_1-\frac{1}{3}\mu\alpha w_0$$

since

$$J_0(kr) = 1 \quad \text{when} \quad r = 0.$$

Therefore

$$p_1 = p_0+\frac{1}{3}\mu\alpha w_0, \quad \ldots \quad (26)$$

hence from (25),

$$A = \frac{i_0}{w_0}+\frac{1}{3}\frac{\gamma}{\gamma-1}\frac{\mu\alpha}{\rho_0}, \text{ using } i_0\rho_0 = \frac{\gamma}{\gamma-1}p_0,$$

Thus

$$I_2 = \left(i_0+\frac{1}{3}\frac{\gamma}{\gamma-1}\frac{\mu\alpha w_0}{\rho_0}\right)J_0(kr). \quad \ldots \quad (27)$$

To solve the equation (24), we note that it is not however possible to find an exact solution. We find a series solution. The equation (24) with the help of (18) reduces to

$$\frac{d^2I_1}{dr^2}+\frac{1}{r}\frac{dI_1}{dr}+4\alpha^2 I_1 = \frac{5-3\gamma}{3(\gamma-1)}\alpha^2 w_0^2 J_0^2(kr)-k^2 w_0^2 J_1^2(kr). \quad \ldots \quad (28)$$

Now

$$\left.\begin{aligned} J_0(kr) &= 1-\frac{k^2r^2}{4}+\frac{k^4r^4}{64}-\ldots, \\ J_1(kr) &= \tfrac{1}{2}kr\left(1-\frac{k^2r^2}{8}+\ldots\right), \end{aligned}\right\} \quad \ldots \quad (29)$$

and

so that (28) is, for small values of kr,

$$\frac{d^2I_1}{dr^2}+\frac{1}{r}\frac{dI_1}{dr}+4\alpha^2 I_1 = \frac{5-3\gamma}{3(\gamma-1)}\alpha^2 w_0^2\left(1-\frac{k^2r^2}{2}+\frac{3}{32}k^4r^4-\ldots\right)$$

$$-\frac{1}{4}k^4 w_0^2 r^2\left(1-\frac{k^2r^2}{4}+\ldots\right). \quad \ldots \quad (30)$$

Let us put

$$I_1 = B_0+B_1 r+B_2 r^2+B_3 r^3+\ldots \quad \ldots \quad (31)$$

Substituting this in (30) and equating coefficients of different powers of r from both sides, we get the following relations.

$$B_1 = 0 = B_3 = B_5 = \text{etc.}$$

$$\left.\begin{aligned} B_2 &= \frac{5-3\gamma}{12(\gamma-1)}\alpha^2 w_0^2 - B_0\alpha^2 \\ B_4 &= -\frac{5-3\gamma}{48(\gamma-1)}(\alpha^2+\tfrac{1}{2}k^2)\alpha^2 w_0^2 + \frac{1}{64}k^4 w_0^2 + \frac{1}{4}B_0\alpha^4 \\ B_6 &= \frac{5-3\gamma}{144(\gamma-1)}\left(\frac{1}{3}\alpha^4+\frac{1}{6}\alpha^2k^2+\frac{1}{8}k^4\right)\alpha^2 w_0^2 - \frac{1}{576}(\alpha^2-k^2)k^4 w_0^2 - \frac{1}{36}B_0\alpha^6 \end{aligned}\right\} \quad (32)$$

and so on. Now from (22), (27) and (31) we have

$$i = i_1 + (B_0 + B_2 r^2 + \ldots)e^{-2\alpha z} + \left(i_0 + \frac{1}{3}\frac{\gamma}{\gamma-1}\frac{\mu\alpha w_0}{\rho_0}\right)e^{-\alpha z}J_0(kr), \quad \ldots \quad (33)$$

so that at $r = 0$, $z = 0$,

$$i_0 = i_1 + B_0 + i_0 + \frac{1}{3}\frac{\gamma}{\gamma-1}\frac{\mu\alpha w_0}{\rho_0}$$

giving

$$B_0 = -i_1 - \frac{1}{3}\frac{\gamma}{\gamma-1}\frac{\mu\alpha w_0}{\rho_0} \quad \ldots \quad (34)$$

With this value of B_0, we can find from (32) values of B_2, B_4, B_6, etc. Hence we have from (33),

$$\begin{aligned} i = {} & i_1 + \left(i_0 + \frac{1}{3}\frac{\gamma}{\gamma-1}\frac{\mu\alpha w_0}{\rho_0}\right)e^{-\alpha z}J_0(kr) \\ & - \left(i_1 + \frac{1}{3}\frac{\gamma}{\gamma-1}\frac{\mu\alpha w_0}{\rho_0}\right)e^{-2\alpha z}\left[1 - \alpha^2 r^2 + \frac{1}{4}\alpha^4 r^4 - \frac{1}{36}\alpha^6 r^6 + \ldots\right] \\ & + \frac{5-3\gamma}{12(\gamma-1)}e^{-2\alpha z}\alpha^2 w_0^2 r^2\left[1 - \frac{1}{4}\left(\alpha^2+\frac{1}{2}k^2\right)r^2 + \frac{1}{12}\left(\frac{1}{3}\alpha^4+\frac{1}{6}\alpha^2k^2+\frac{1}{8}k^4\right)r^4 + \ldots\right] \\ & + \frac{1}{64}e^{-2\alpha z}k^4 w_0^2 r^4\left[1 + \frac{1}{9}(k^2-\alpha^2)r^2 + \ldots\right]. \quad \ldots \quad (35) \end{aligned}$$

Now the boundary condition to be satisfied at the surface of the pipe is

when $\qquad r = a, \quad w = 0$

giving $\qquad J_0(ka) = 0, \quad \ldots \quad (36)$

whose roots are (Lamb, 1932)

$$ka = \cdot 766\pi,\ 1\cdot 76\pi,\ 2\cdot 75\pi, \text{ etc.}$$

Taking the first root we have from (17)

$$\left(\alpha^2 + \frac{\alpha}{\mu}\rho_0 w_0\right)a^2 = \cdot 6\pi^2 \quad \ldots \quad (37)$$

giving α.

Thus k and α are both known, hence the solution is found completely.

4. Approximate Result

Putting $\rho_0 w_0 = \beta\mu\alpha$ we have from (17), $k = \alpha\sqrt{1+\beta}$, therefore with $ka = \cdot 776\pi$, we get the equation to determine β in the form

$$\cdot 6\pi^2\mu^2\beta^2 - a^2\rho_0^2 w_0^2\beta - a^2\rho_0^2 w_0^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

giving

$$\beta = \frac{a^2\rho_0^2 w_0^2}{1\cdot 2\pi^2\mu^2}\left[1+\left(1+\frac{2\cdot 4\pi^2\mu^2}{a^2\rho_0^2 w_0^2}\right)^{\frac{1}{2}}\right]$$

$$= \delta\left[1+\left(1+\frac{2}{\delta}\right)^{\frac{1}{2}}\right]$$

where

$$\delta = \frac{a^2\rho_0^2 w_0^2}{1\cdot 2\pi^2\mu^2}.$$

Since μ is usually small, δ is large so that

$$\beta = 1+2\delta.$$

Further since k is known and β is large, α is usually small, also μ is small but $\frac{\alpha}{\mu}$ is something definite, hence we can approximately put

$$k^2 = \frac{\alpha}{\mu}\rho_0 w_0 \text{ so that } \alpha = \frac{\mu}{\rho_0 w_0}\frac{\cdot 6\pi^2}{a^2}. \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

Thus as an approximate result we get from (35), neglecting terms containing α^2 and $\mu\alpha$,

$$i = i_1 + i_0 e^{-\alpha z} J_0(kr) - i_1 e^{-2\alpha z} + \frac{1}{64} e^{-2\alpha z} k^4 w_0^2 r^4 \left(1+\frac{1}{9}k^2r^2+\ldots\right) \quad \ldots \quad (40)$$

where α is given by (39).

References

Howarth, L. (1953). Modern Developments in Fluid Dynamics, 2, 774, 778.
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