SHOCK RELATIONS IN A FERMI-DIRAC GAS

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(*Received January 3; read May 4, 1956*)

1. The phenomenon of shock waves is getting more and more prominent in the astrophysical context. The passage of a shock wave through a perfect gas produces abrupt changes in the velocity, pressure, temperature, density and entropy. Using the principles of conservation of matter, momentum and energy, Rankine and Hugoniot obtained the equations to determine the jumps in these dynamical and physical variables of a perfect gas in terms of the Mach number of the incoming flow.

The matter of which the stars are composed varies from a perfect gas to a completely degenerate gas. Consequently in the present paper we have obtained for a Fermi-Dirac gas the relations corresponding to Rankine-Hugoniot relations for a perfect gas. Perfect gas and completely degenerate gas are the two limiting cases of a Fermi-Dirac gas. As usual it is possible to study the cases of weakly degenerate and strongly degenerate gases analytically, while the intermediate range is amiable only to the numerical methods.

2. In this section we collect the properties of a Fermi-Dirac gas which we shall use during the present investigation.

Let ρ, p, T, μ denote the density, pressure, temperature, mean molecular weight per free electron, while u, i and s denote the internal energy, enthalpy and entropy, per unit mass of the gas. Then ρ, p, u, s are given by the well-known relations (Chandrasekhar, 1938).

$$\rho = \frac{2\mu H}{h^3}(2\pi mkT)^{\frac{3}{2}} U_{\frac{1}{2}}(A) \qquad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

$$p = \frac{k}{\mu H}\rho T \frac{U_{\frac{3}{2}}(A)}{U_{\frac{1}{2}}(A)} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

$$u = \frac{3}{2}\frac{p}{\rho} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

and

$$s = -\frac{k}{\mu H}\log A + \frac{5}{2}\frac{k}{\mu H}\frac{U_{\frac{3}{2}}(A)}{U_{\frac{1}{2}}(A)} \qquad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

where m, H, k and h are mass of an electron, mass of an hydrogen atom, the Boltzmann constant and the Planck constant and

$$U_\nu(A) = \frac{1}{\Gamma(\nu+1)}\int_0^\infty \frac{t^\nu \, dt}{\frac{1}{A}l^t+1} \qquad \ldots \quad \ldots \quad \ldots \quad (2.5)$$

In the cases of weak degeneracy, $A \ll 1$, and strong degeneracy, $A \gg 1$, it is possible to obtain the value of (2.5) in the form of expansions in powers of A and $\frac{1}{\log A}$ respectively:

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case (i) $\Lambda \ll 1$,

$$U_{\frac{1}{2}}(\Lambda) = \sum_{n=0}^{\infty} (-1)^n \frac{\Lambda^{n+1}}{(n+1)^{\frac{3}{2}}} \qquad \ldots \quad \ldots \quad \ldots \quad (2.6)$$

$$U_{\frac{3}{2}}(\Lambda) = \sum_{n=0}^{\infty} (-1)^n \frac{\Lambda^{n+1}}{(n+1)^{\frac{5}{2}}} \qquad \ldots \quad \ldots \quad \ldots \quad (2.7)$$

case (ii) $\Lambda \gg 1$

$$U_{\frac{1}{2}}(\Lambda) = \frac{4}{3\pi^{\frac{1}{2}}} (\log_e \Lambda)^{\frac{3}{2}} \left[1+\frac{\pi^2}{8}\frac{1}{(\log \Lambda)^2}+\ldots\right] \quad \ldots \quad \ldots \quad (2.8)$$

$$U_{\frac{3}{2}}(\Lambda) = \frac{8}{15\pi^{\frac{1}{2}}} (\log_e \Lambda)^{\frac{5}{2}} \left[1+\frac{5\pi^2}{8}\frac{1}{(\log \Lambda)^2}+\ldots\right] \quad \ldots \quad \ldots \quad (2.9)$$

Eliminating T between (2.1) and (2.2) we get the pressure-density relation :—

$$\frac{p}{\rho^{\frac{3}{2}}} = \frac{h^2}{\pi m(2\mu H)^{\frac{5}{3}}} \frac{U_{\frac{3}{2}}(\Lambda)}{\{U_{\frac{1}{2}}(\Lambda)\}^{\frac{5}{3}}} \qquad \ldots \quad \ldots \quad \ldots \quad (2.10)$$

Let C denote the velocity of sound, then

$$C^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\text{Entropy}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.11)$$

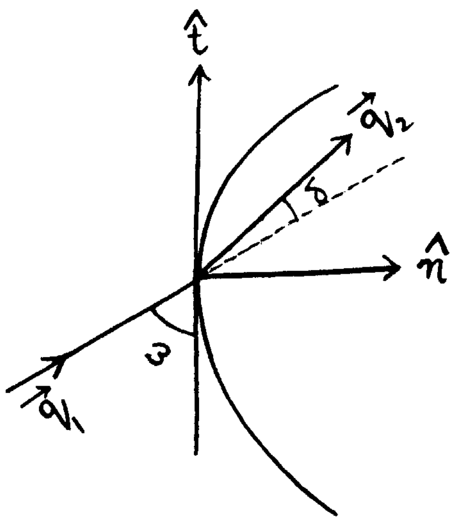
From (2.4) entropy depends on Λ alone. Hence s is constant when Λ is constant, so that

$$C^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\Lambda} = \frac{5}{3}\frac{p}{\rho} \qquad \ldots \quad \ldots \quad \ldots \quad (2.12)$$

Also, from (2.3) enthalpy

$$i = u+\frac{p}{\rho} = \frac{5}{2}\frac{p}{\rho} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.13)$$

3. We shall consider a curved shock. To obtain the equations connecting the dynamical and physical variables on the two sides of a shock wave, it is convenient



to consider the motion relative to the wave. Let suffix 1 refer to conditions in front of the shock wave and suffix 2 refer to conditions behind the shock wave. Thus $\vec{q}_1$ is the velocity with which the elements of fluid which cross the wave at P enter it and $\vec{q}_2$ is the velocity with which they leave it. Let $\hat{n}$ and $\hat{t}$ be the unit vectors in the direction of the normal to the shock surface at P and in the direction of the tangent at P which lies in the plane of $\hat{n}$ and $\vec{q}_1$. We shall denote the velocity components in the direction of $\hat{n}$ and $\hat{t}$ by suffixes n and t. Further let ω be the angle which the oncoming stream makes with the tangent $\hat{t}$ and δ be the angle of deflection of the stream, i.e. the angle between the directions of the incident and emergent streams.

We now write the equations ensuring the conservation of mass, momentum and energy:

$$q_{2t} = q_{1t} \qquad (3.1)$$

$$\rho_2 q_{2n} = \rho_1 q_{1n} = m \text{ (say)} \qquad (3.2)$$

$$p_2 - p_1 = m(q_{1n} - q_{2n}) \qquad (3.3)$$

$$i_2 + \frac{1}{2}\vec{q}_2^{\,2} = i_1 + \frac{1}{2}\vec{q}_1^{\,2} \qquad (3.4)$$

or from (2.13) and (3.1)

$$5\frac{p_2}{\rho_2} + q_{2n}^2 = 5\frac{p_1}{\rho_1} + q_{1n}^2 \qquad (3.5)$$

The equations (3.1), (3.2), (3.3) and (3.4) are same as the equations for a perfect gas with γ, the ratio of specific heats, equal to $\frac{5}{3}$. This results from the fact that the enthalpy for a Fermi-Dirac gas can also be put in the same form, namely $\frac{\gamma}{\gamma - 1}\frac{p}{\rho}$, as for a polytropic gas. Hence we can solve these equations in the usual manner (Howarth, 1953) to give us

$$\frac{q_{2n}}{q_{1n}} = \frac{\sigma^2 + 3}{4\sigma^2} \qquad (3.6)$$

$$\frac{\rho_2}{\rho_1} = \frac{4\sigma^2}{\sigma^2 + 3} \qquad (3.7)$$

and

$$\frac{p_2}{p_1} = \frac{1}{4}(5\sigma^2 - 1) \qquad (3.8)$$

where

$$\sigma = \frac{q_{1n}}{C_1} = \frac{q_1 \sin w}{C_1} = M_1 \sin w \qquad (3.9)$$

M_1 being to Mach number of the oncoming stream.

Further the Mach number M_2 of the emergent stream and the angle of deflection δ are given by

$$M_2^2 = (M_1^2 + 3)\left[\frac{1}{5\sigma^2 - 1} + \frac{3}{\sigma^2 + 3}\right] - 3 \qquad (3.10)$$

and

$$\cot \delta = \tan w \left[\frac{4}{3}\frac{M_1^2}{\sigma^2 - 1} - 1\right] \qquad (3.11)$$

We shall now determine T_2 and s_2.
From (2.10) we have

$$\frac{U_{\frac{3}{2}}(A_2)}{\{U_{\frac{1}{2}}(A_2)\}^{\frac{5}{3}}} = \frac{U_{\frac{3}{2}}(A_1)}{\{U_{\frac{1}{2}}(A_1)\}^{\frac{5}{3}}} \cdot \frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^{\frac{5}{3}} \quad \ldots \quad \ldots \quad \ldots \quad (3.12)$$

knowing p_1 and p_2 for the oncoming stream just before entering the shock wave we determine A_1 from (2.10). Then knowing $\frac{\rho_2}{\rho_1}$ and $\frac{p_2}{p_1}$ from (3.7) and (3.8) in terms of σ, we determine A_2 from (3.12).

Further from (2.2) and (2.4) we get

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}\frac{U_{\frac{3}{2}}(A_1)}{U_{\frac{1}{2}}(A_1)}\bigg/\frac{U_{\frac{3}{2}}(A_2)}{U_{\frac{1}{2}}(A_2)} \quad \ldots \quad \ldots \quad \ldots \quad (3.13)$$

and

$$\frac{\mu H}{k}(s_2 - s_1) = (\log_e A_1 - \log_e A_2) + \frac{5}{2}\left[\frac{U_{\frac{3}{2}}(A_2)}{U_{\frac{1}{2}}(A_2)} - \frac{U_{\frac{3}{2}}(A_1)}{U_{\frac{1}{2}}(A_1)}\right] \quad \ldots \quad (3.14)$$

The solution of (3.12) for A_2 in an analytical form is possible in the two limiting cases mentioned before, while for the intermediate range we can solve it only by numerical methods. We can effect the solution of (3.12) by plotting $\frac{U_{\frac{3}{2}}(A)}{\{U_{\frac{1}{2}}(A)\}^{\frac{5}{3}}}$ against A with the help of tables prepared by McDougall and Stoner (1939). Our A and U_v are equal to their e^{η} and $\frac{F_v}{\Gamma_{(v+1)}}$ respectively. We have tabulated $\log_{10}\left\{\frac{U_{\frac{3}{2}}}{U_{\frac{1}{2}}}\right\}$ and $\log_{10}\left\{\frac{U_{\frac{3}{2}}}{U_{\frac{1}{2}}^{\frac{5}{3}}}\right\}$ against η in the table (Appendix).

4. In this section we shall consider the two limiting cases mentioned above case (i) $A \ll 1$.

From (2.6) and (2.7) we have

$$\frac{U_{\frac{3}{2}}(A)}{U_{\frac{1}{2}}(A)} \approx 1 + \frac{A}{4\sqrt{2}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

and

$$\frac{U_{\frac{3}{2}}(A)}{\{U_{\frac{1}{2}}(A)\}^{\frac{5}{3}}} \approx \frac{1}{A^{\frac{2}{3}}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

Hence from (2.10) and (3.12) we get

$$A_1 = \frac{1}{\epsilon_1^{\frac{3}{2}}}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.3)$$

where

$$\epsilon_1 = \frac{p_1}{\rho_1^{\frac{5}{3}}}\frac{\pi m(2\mu H)^{\frac{5}{3}}}{h^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.4)$$

so that

$$\frac{A_2}{A_1} = \left[\frac{p_1}{p_2}\cdot\left(\frac{\rho_2}{\rho_1}\right)^{\frac{5}{3}}\right]^{\frac{3}{2}}$$

$$= \frac{256\sigma^5}{(5\sigma^2-1)^{\frac{3}{2}}(\sigma^2+3)^{\frac{5}{2}}} \quad \ldots \quad \ldots \quad \ldots \quad (4.5)$$

From (3.13) and (3.14), we have, on retaining only the first power of A_1 and A_2,

$$\frac{T_2}{T_1} = \frac{p_2\,\rho_1}{p_1\,\rho_2}\left[1+\frac{A_1}{4\sqrt{2}}\left(1-\frac{A_2}{A_1}\right)\right] \quad \ldots \quad (4.6)$$

and

$$\frac{\mu H}{k}(s_2-s_1) = \log_e\frac{A_1}{A_2} - \frac{5A_1}{8\sqrt{2}}\left(1-\frac{A_2}{A_1}\right) \quad \ldots \quad (4.7)$$

$$= \frac{3}{2}\log_e\left[\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^{\frac{5}{3}}\right] - \frac{5A_1}{8\sqrt{2}}\left(1-\frac{A_2}{A_1}\right), \quad \ldots \quad (4.8)$$

on using (4.5).

In the limit $A \to 0$, we get from (4.6) and (4.8) the known results for a perfect gas with $\gamma = \frac{5}{3}$.

Using (3.7), (3.8), (4.3) and (4.5), the equations (4.6) and (4.8) reduce to

$$\frac{T_2}{T_1} = \frac{(5\sigma^2-1)(\sigma^2+3)}{16\sigma^2}\left[1+\frac{1}{4(2\epsilon^3)^{\frac{1}{2}}}\left\{1-\frac{256\sigma^2}{(5\sigma^2-1)^{\frac{3}{2}}(\sigma^2+3)^{\frac{5}{2}}}\right\}\right] \quad \ldots \quad (4.9)$$

and

$$\frac{\mu H}{k}(s_2-s_1) = \log_e\left\{\frac{(5\sigma^2-1)^{\frac{3}{2}}(\sigma^2+3)^{\frac{5}{2}}}{256\sigma^5}\right\} - \frac{5}{8(2\epsilon^3)^{\frac{1}{2}}}\left\{1-\frac{256\sigma^2}{(5\sigma^2-1)^{\frac{3}{2}}(\sigma^2+3)^{\frac{5}{2}}}\right\} \quad (4.10)$$

case (ii) $A \gg 1$.

From (2.8) and (2.9) we have

$$\frac{U_{\frac{3}{2}}(A)}{U_{\frac{1}{2}}(A)} = \frac{2}{5}\log_e A\left[1+\frac{\pi^2}{2}\frac{1}{(\log_e A)^2}+\ldots\right] \quad \ldots \quad (4.11)$$

and

$$\frac{U_{\frac{3}{2}}(A)}{\{U_{\frac{1}{2}}(A)\}^{\frac{5}{3}}} = \frac{1}{5}\left(\frac{9\pi}{2}\right)^{\frac{1}{3}}\left[1+\frac{5\pi^2}{12}\frac{1}{(\log_e A)^2}+\ldots\right] \quad \ldots \quad (4.12)$$

From (2.10) and (4.12) we have

$$\log_e A_1 = \left\{\frac{12}{5\pi^2}(\epsilon-1)\right\}^{-\frac{1}{2}} \quad \ldots \quad (4.13)$$

and

$$\frac{1+\frac{5\pi^2}{12}\frac{1}{(\log_e A_2)^2}}{1+\frac{5\pi^2}{12}\frac{1}{(\log_e A_1)^2}} = \frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^{\frac{5}{3}} \quad \ldots \quad (4.14)$$

so that

$$\frac{\log A_2}{\log A_1} = \left\{\frac{1-\frac{1}{\epsilon}}{\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^{\frac{5}{3}}-\frac{1}{\epsilon}}\right\}^{\frac{1}{2}}, \quad \ldots \quad (4.15)$$

where

$$\epsilon = \frac{p_1}{\rho_1^{\frac{5}{3}}}\,\frac{\pi m(2\mu H)^{\frac{5}{3}}}{h^2}\,.\,5\left(\frac{2}{9\pi}\right)^{\frac{1}{3}} \quad \ldots \quad (4.16)$$

$$= 5\left(\frac{2}{9\pi}\right)^{\frac{1}{3}}\epsilon_1 \quad \ldots \quad (4.17)$$

Finally from (3.13), (3.14) and (4.11) we have

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}\frac{\log_e A_1}{\log_e A_2} \qquad \text{(4.18)}$$

and

$$\frac{\mu H}{k}(s_2 - s_1) = \frac{\pi^2}{2}\left[\frac{1}{\log_e A_2} - \frac{1}{\log_e A_1}\right]. \qquad \text{(4.19)}$$

(4.18) and (4.19), on using (4.13) and (4.15), reduce to

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}\left[\frac{\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^{\frac{5}{3}} - \frac{1}{\epsilon}}{1-\frac{1}{\epsilon}}\right]^{\frac{1}{2}}$$

$$= \frac{(5\sigma^2-1)(\sigma^2+3)}{16\sigma^2}\left\{\frac{\left(\frac{5\sigma^2-1}{4}\right)\left(\frac{\sigma^2+3}{4\sigma^2}\right)^{\frac{5}{3}} - \frac{1}{\epsilon}}{1-\frac{1}{\epsilon}}\right\}^{\frac{1}{2}}$$

and

$$\frac{\mu H}{k}(s_2 - s_1) = \pi\left\{\frac{3}{5}(\epsilon-1)\right\}^{\frac{1}{2}}\left[\left\{\frac{\frac{5\sigma^2-1}{4}\left(\frac{\sigma^2+3}{4\sigma^2}\right)^{\frac{5}{3}} - \frac{1}{\epsilon}}{1-\frac{1}{\epsilon}}\right\}^{\frac{1}{2}} - 1\right]$$

We know that the entropy of a mass element increases due to passage through a shock wave, hence

$$\log_e A_2 < \log_e A_1, \text{ i.e. } A_1 > A_2 .$$

Consequently the degree of degeneracy of the mass element is reduced as it passes through the shock wave.

Abstract

The study of shock waves is becoming more and more prominent in the astrophysical context. When an element of mass crosses a shock wave, discontinuous changes are produced in its physical and dynamical variables. Rankine and Hugoniot relations determine the jumps in these variables for a perfect gas. The matter of which stars are composed ranges from perfect gas to degenerate gas. Hence the relations corresponding to Rankine-Hugoniot relations for a perfect gas have been obtained to determine the jumps in the physical and dynamical variables for a Fermi-Dirac gas. The perfect gas and the degenerate gas form the two limiting cases of a Fermi-Dirac gas. For these two limiting cases these jumps have been obtained in an explicit form as functions of Mach number of the incoming flow while for the intermediate range a numerical method is indicated for determining them.

References

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Appendix

Table

$\eta = \log_e A$	$\log_{10} \frac{U_{\frac{3}{2}}}{U_{\frac{1}{2}}}$	$\log_{10} \frac{U_{\frac{3}{2}}}{U_{\frac{1}{2}}^{\frac{5}{3}}}$
−4·0	·0013712	1·1613475
−3·5	·0022901	1·0187081
−3·0	·0037393	·8773364
−2·5	·0060735	·7380590
−2·0	·0097885	·6020591
−1·5	·0155789	·4710420
−1·0	·0243088	·3472337
− ·5	·0369822	·2333371
0·0	·0543743	·1318777
·5	·0768398	·0447185
1·0	·1040626	$\bar{1}$·9724243
1·5	·1351349	$\bar{1}$·9142023
2·0	·1688363	$\bar{1}$·8682884
2·5	·2039484	$\bar{1}$·8325197
3·0	·2394509	$\bar{1}$·8047782
3·5	·2745931	$\bar{1}$·7832351
4·0	·3088739	$\bar{1}$·7664169
4·5	·3419922	$\bar{1}$·7531865
5·0	·3737880	$\bar{1}$·7426823
5·5	·4041998	$\bar{1}$·7342616
6·0	·4332280	$\bar{1}$·7274444
6·5	·4609074	$\bar{1}$·7218685
7·0	·4872988	$\bar{1}$·7172669
7·5	·5124718	$\bar{1}$·7134351
8·0	·5365000	$\bar{1}$·7102167
8·5	·5594570	$\bar{1}$·7074911
9·0	·5814153	$\bar{1}$·7051665
9·5	·6024426	$\bar{1}$·7031694
10·0	·6226028	$\bar{1}$·7014425
10·5	·6419557	$\bar{1}$·6999408
11·0	·6605551	$\bar{1}$·6986266
11·5	·6784516	$\bar{1}$·6974705
12·0	·6956920	$\bar{1}$·6964489
12·5	·7123185	$\bar{1}$·6955422
13·0	·7283700	$\bar{1}$·6947335
13·5	·7438823	$\bar{1}$·6940097
14·0	·7588878	$\bar{1}$·6933587
14·5	·7734173	$\bar{1}$·6927722
15·0	·7874980	$\bar{1}$·6922414
15·5	·8011555	$\bar{1}$·6917595
16·0	·8144138	$\bar{1}$·6913212

Issued June 27, 1957.