INTERNAL BALLISTICS OF COMPOSITE CHARGES FOR POWER LAW OF BURNING BY THE EQUIVALENT CHARGE METHOD

by J. N. Kapur, *Hindu College, Delhi University, Delhi*

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1. Introduction

The equations of Internal Ballistics for the power law of burning were first solved for a single charge by Clemmow (1928, 1951). His method was recently extended to a composite charge consisting of two component charges by Patni (1955) for the general case and by Aggarwal and Mehta (1955) for the particular case of tubular component charges by using the 'Direct Method'. In this paper we have extended Clemmow's method to a composite charge consisting of n component charges by the use of a generalisation by the present author (Kapur, 1956*a*) of the equivalent charge method of Corner (1950) and Clemmow (1951).

As usual, the following assumptions are made in the present investigation:

(i) Co-volume correction terms have been neglected.

(ii) $\gamma_1 = \gamma_2 = \ldots . = \gamma_{n-1} = \gamma_n = \gamma$. However, we shall show in the last but one section how our theory can be adapted to the case when $\gamma_1, \gamma_2, \ldots, \gamma_n$ are unequal.

(iii) The pressure index is the same for all component charges.

The last is the only serious restriction, as most of the propellants in everyday use have different pressure indices. As a matter of fact only three pairs of propellants have the same pressure indices and in these three cases, the theory for $n=2$ can be strictly applied. Nevertheless if the composite charge has three or four component charges with different, though very nearly equal, pressure indices, it would be better to adjust the rates of burning constants of all of them to a common pressure index rather than to approximate the pressure index laws for all of them by the linear law. In this case our theory for general n would be useful. As a more practical example, the theory would be strictly applicable to the case when the n component charges have the same composition, and in particular the same pressure index. Another reason for our discussing the theory for general n is that, by the equivalent charge method, the discussion for a general n is in no way more complicated than for the particular case $n = 2$.

We shall, however, discuss in detail only the case $n = 2$ and compare the results obtained here with those obtained by Patni (1955) by the Direct Method.

2. The Basic Equations

Under the assumptions, the basic equations of Internal Ballistics for the composite charge, when all the component charges are burning, are the following:

$$F_1C_1z_1+F_2C_2z_2+ \ldots\ldots +F_nC_nz_n = Ap(x+l)+\tfrac{1}{2}(\gamma-1)w_1v^2, \quad \ldots \quad (1)$$

where

$$w_1 = 1{\cdot}05w+\frac{C_1}{3}+\frac{C_2}{3}+\ldots\ldots+\frac{C_n}{3} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

and

$$Al = K_0 - \left(\frac{C_1}{\delta_1}+\frac{C_2}{\delta_2}+\ldots\ldots+\frac{C_n}{\delta_n}\right) \quad \ldots \quad (3)$$

$$w_1\frac{dv}{dt} = Ap \quad \ldots \quad (4)$$

$$z_i = (1-f_i)(1+\theta_i f_i) \; [i = 1, 2, \ldots\ldots n] \quad \ldots \quad (5)$$

$$D_i\frac{df_i}{dt} = -\beta_i p^{\alpha} \; [i = 1, 2, \ldots\ldots, n] \quad \ldots \quad (6)$$

where F_i, C_i, D_i, β_i, z_i, f_i, θ_i, δ_i refer to the ith charge.

Without loss of generality, we can assume

$$\frac{D_1}{\beta_1} \leqslant \frac{D_2}{\beta_2} \leqslant \ldots\ldots \leqslant \frac{D_n}{\beta_n} \quad \ldots \quad (7)$$

We shall call $\frac{D_i}{\beta_i}$ as the '*effective ballistic size*' of the ith component charge, so that we have numbered the charges serially according to increasing effective ballistic sizes.

(7) can also be written as

$$\beta_i' \geqslant \beta_2' \geqslant \ldots\ldots \geqslant \beta_n' \quad \ldots \quad (8)$$

where

$$\beta_i' = \frac{\beta_i}{D_i} \quad \ldots \quad (9)$$

β_i' may be defined as the '*effective rate of burning constant*' for the ith component charge.

3. The Equivalent Charge and its Form-function

The equivalent charge is defined as that charge which would give the same ballistic equations as the composite charge both during and after burning. Let F, C, D, B, z, f, δ refer to the equivalent charge. Then the following results, which will be used later in the present investigation, have been established by the present author (Kapur, 1956*a*)

(i) $$C = C_1+C_2+\ldots\ldots+C_n \quad \ldots \quad (10)$$

(ii) $$CF = C_1F_1+C_2F_2+\ldots\ldots+C_nF_n \quad \ldots \quad (11)$$

(iii) $$\frac{D}{\beta} = \frac{D_n}{\beta_n} \text{ or } \beta' = \beta_n' \quad \ldots \quad (12)$$

(iv) $$\frac{C}{\delta} = \frac{C_1}{\delta_1}+\frac{C_2}{\delta_2}+\ldots\ldots+\frac{C_n}{\delta_n} \quad \ldots \quad (13)$$

(v) If $\frac{D_i}{\beta_i}$ $[i = 1, 2, \ldots\ldots n]$ are all distinct, the charge C_1 burns out first, then charge C_2, and the last charge to burn out is the nth, i.e. the charge with the largest effective web size. The burning consists of n distinct stages, and during the rth stage only rth, $(r+1)$th, ... and nth charges burn, the first $(r-1)$ charges having been burnt out earlier.

(vi) During the rth stage of burning the form-function for the equivalent charge is

$$\left.\begin{array}{l} z = A_r + B_r(1-f) - C_r(1-f)^2, \\ 1-\dfrac{1}{k_{r-1}} > f \geqslant 1-\dfrac{1}{k_r} \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

when

where

$$k_i = \frac{\beta_i'}{\beta'} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

$$\lambda_i = \frac{F_i C_i}{FC} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

$$\left.\begin{array}{l} A_r = \sum\limits_{i=1}^{r-1} \lambda_i \\ B_r = \sum\limits_{i=r}^{n} \lambda_i k_i (1+\theta_i) \\ C_r = \sum\limits_{i=r}^{n} \lambda_i k_i^2 \theta_i \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

(vii) The form-function for the equivalent charge is always continuous, but $\dfrac{dz}{df}$ would, in general, be discontinuous in crossing from rth stage to $(r+1)$th stage. It will be continuous here if and only if $\theta_r = 1$. If $\theta_r < 1$, the increment in $\dfrac{dz}{df}$ at the end of the rth stage is positive.

(viii) If p component charges have the same effective ballistic sizes, the number of distinct stages of burning is $n-p+1$ and these p charges behave as a single charge with mass equal to the sum of the masses of the p component charges; force constant equal to the weighted average of the force constants of the component charges, the weights being the corresponding charge masses; the effective ballistic size equal to the common effective ballistic size, and the form-factor equal to the weighted average of the form-factors of the component charges, the weights being the available energies in the component charges.

4. The Fundamental Differential Equation

From sections 2 and 3, the basic equations to be integrated for the rth stage of burning are

$$FCz = Ap(x+l) + \frac{1}{2}(\gamma-1)w_1 v^2 \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

$$w_1 \frac{dv}{dt} = Ap \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

$$z = A_r + B_r(1-f) - C_r(1-f)^2 \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

$$D\frac{df}{dt} = -\beta p^{\alpha} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

Making the substitutions

$$\xi = 1+\frac{x}{l} \qquad \ldots \qquad (22)$$

$$\eta = \frac{vAD}{FC\beta}\left(\frac{FC}{Al}\right)^{1-\alpha} \qquad \ldots \qquad (23)$$

$$\zeta = \frac{Al}{FC}p \qquad \ldots \qquad (24)$$

$$M = \frac{A^2D^2}{FC\beta^2 w_1}\left(\frac{FC}{Al}\right)^{2-2\alpha}, \qquad \ldots \qquad (25)$$

the basic equations become

$$z = \zeta\xi+\frac{1}{2}(\gamma-1)\frac{\eta^2}{M} \qquad \ldots \qquad (26)$$

$$\eta\frac{d\eta}{d\xi} = M\zeta \qquad \ldots \qquad (27)$$

$$z = A_r+B_r(1-f)-C_r(1-f)^2 \qquad \ldots \qquad (28)$$

$$\eta\frac{df}{d\xi} = -\zeta^\alpha \qquad \ldots \qquad (29)$$

It is easily seen that ξ, η, ζ are dimensionless variables corresponding respectively to shot-travel, velocity, and pressure and M is the modified dimensionless central ballistic parameter.

Using (27) and (29)

$$\frac{d}{df}(\eta) = \frac{d}{df}\left(-\xi^\alpha\frac{d\xi}{df}\right)$$

or

$$\frac{d}{df}\left(\zeta^\alpha\frac{d\xi}{df}\right) = M\zeta^{1-\alpha}. \qquad \ldots \qquad (30)$$

From (28)

$$\frac{dz}{df} = -B_r+2C_r(1-f)$$

$$= -\nu_r\sqrt{1-q_r z} \qquad \ldots \qquad (31)$$

where

$$\left.\begin{aligned} \nu_r &= \sqrt{B_r^2+4A_rC_r} \\ q_r &= \frac{4C_r}{B_r^2+4A_rC_r} \end{aligned}\right\} \qquad \ldots \qquad (32)$$

From (30) and (31)

$$(1-q_r z)\frac{d}{dz}\left(\zeta^\alpha\frac{d\xi}{dz}\right)-\tfrac{1}{2}q_r\zeta^\alpha\frac{d\xi}{dz} = \frac{M}{\nu_r^2}\zeta^{1-\alpha} \qquad \ldots \qquad (33)$$

Now let

$$Y = q_r\zeta\xi^\gamma \qquad \ldots \qquad (34)$$

be taken as dependent variable, and

$$Z = q_r z \qquad \ldots \qquad (35)$$

be taken as independent variable.

From (26) and (27)

$$dz = \xi^{1-\gamma} d(\zeta \xi^{\gamma})$$

$$\therefore \quad dZ = \xi^{1-\gamma} dY \quad \ldots \quad (36)$$

or

$$Y' = \xi^{\gamma-1} \quad \ldots \quad (37)$$

Now, on using Z instead of z as independent variable in (33) we get

$$[1-Z]\left[\alpha \frac{\zeta'}{\zeta} + \frac{\xi''}{\xi'}\right] - \frac{1}{2} = \frac{M}{q_r^2 \nu_r^2} \frac{\zeta^{1-2\alpha}}{\xi'} \quad \ldots \quad (38)$$

Also from (34), (35) and (37)

$$\frac{Y'}{Y} = \gamma \frac{\xi'}{\xi} + \frac{\zeta'}{\zeta}, \quad \ldots \quad (39)$$

and

$$\frac{Y''}{Y'} = (\gamma - 1) \frac{\xi'}{\xi} \quad \ldots \quad (40)$$

Also

$$\xi = (Y')^{\frac{1}{\gamma-1}}, \; \zeta = \frac{Y}{q_r (Y')^{\frac{-\gamma}{\gamma-1}}}$$

$$\therefore \quad \frac{\zeta'}{\zeta} = \frac{Y'}{Y} - \frac{\gamma}{\gamma-1} \frac{Y''}{Y'} \quad \ldots \quad (41)$$

From (40)

$$\frac{\xi''}{\xi'} - \frac{\xi'}{\xi} = \frac{Y'''}{Y''} - \frac{Y''}{Y'}$$

Substituting in (38) and simplifying, we get

$$\{1-Z\}\left\{\alpha \frac{Y'}{Y} + \frac{Y'''}{Y''} + \frac{Y''}{Y'}\left[-\frac{\alpha\gamma}{\gamma-1} - 1 + \frac{1}{\gamma-1}\right]\right\} - \frac{1}{2}$$

$$= \frac{M}{q_r^2 \nu_r^2} \frac{\gamma - 1}{Y''} \frac{Y^{1-2\alpha}}{q_r^{1-2\alpha} (Y')^{\frac{\gamma}{\gamma-1}(1-2\alpha) + \frac{1}{\gamma-1} - 1}} \quad \ldots \quad (42)$$

or

$$[1-Z]\left[\frac{Y'''}{Y''} + \frac{Y''}{Y'}(m-2) + \frac{\alpha Y'}{Y}\right] - \frac{1}{2} = \frac{Q_r (Y')^{2-2m}}{Y'' Y^{2\alpha-1}}, \quad \ldots \quad (43)$$

on setting

$$m = \frac{\gamma(1-\alpha)}{\gamma - 1} \quad \ldots \quad (44)$$

and

$$Q_r = \frac{(\gamma-1)M}{q_r^{3-2\alpha} \nu_r^2} \quad \ldots \quad (45)$$

(43) is a non-linear differential equation of the third order which can, in general, be integrated only numerically. With proper values of Q_r and appropriate initial conditions, it holds equally for any stage of burning.

We may here also note the important fact that the definitions of the variables Y and Z are different in the various stages in view of the constant q_r occurring in (34) and (35) and consequently dashes do not denote differentiation with respect to the same independent variable. If we want the same variables for all the stages, we can use

$$\bar{Y} = \zeta \xi^{\gamma}, \quad \bar{Z} = z$$

and then (43) becomes

$$[1 - q_r \bar{Z}]\left[\frac{\bar{Y}'''}{\bar{Y}''} + (m-2)\frac{\bar{Y}''}{\bar{Y}'} + \alpha \frac{\bar{Y}'}{\bar{Y}}\right] = \frac{Q_r q_r^2 (\bar{Y}')^{2-2m}}{\bar{Y}''(q_r \bar{Y})^{2\alpha-1}} \quad \ldots \quad \ldots \quad (46)$$

It will be better, however, to use (43) and adjust the initial conditions.

5. Initial Conditions for the First and the rth Stages. The stage in which the shot moves

The initial circumstance of band engraving will be represented by a finite shot-start pressure p_0, so that assuming that the shot starts moving in the first stage of motion, the initial conditions are

$$x = 0, \qquad v = 0, \qquad p = p_0$$

$$\xi = 1, \qquad \eta = 0, \qquad \zeta = \frac{Al}{FC} p_0 = \zeta_0$$

$$\left.\begin{aligned} &Y = q_1 \zeta_0 \quad Y' = 1 \quad Y'' = 0 \\ &z_0 = \zeta_0 \qquad Z = q_1 z_0 = q_1 \zeta_0 = q_1 \frac{Al}{FC} p_0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (47)$$

The first stage terminates when

$$f = 1 - \frac{1}{k_1}, \quad z = A_1 + \frac{B_1}{k_1} - \frac{C_1}{k_1^2}$$

i.e. when

$$Z = q_1 \left(A_1 + \frac{B_1}{k_1} - \frac{C_1}{k_1^2}\right) \quad \ldots \quad \ldots \quad (48)$$

If

$$A_1 + \frac{B_1}{k_1} - \frac{C_1}{k_1^2} < z_0 (= \zeta_0)$$

the shot will not start moving in the first stage of burning. In fact, the shot will start moving in the rth stage if

$$A_{r-1} + \frac{B_{r-1}}{k_{r-1}} - \frac{C_{r-1}}{k_{r-1}^2} < \zeta_0 < A_r + \frac{B_r}{k_r} - \frac{C_r}{k_r^2} \quad \ldots \quad \ldots \quad (49)$$

To determine the initial conditions for the rth stage, we note that during this stage

(i) $Y = q_r \zeta \xi^{\gamma}$

(ii) $Y' = \xi^{\gamma - 1}$

(iii) $Y'' = (\gamma-1)Y'\dfrac{\xi'}{\xi}$ [from (40)]

$$= (\gamma-1)\xi^{\gamma-2}\frac{d\xi}{df}\frac{df}{dz}\frac{dz}{dZ} \quad \text{[using (37)]}$$

$$= (\gamma-1)\xi^{\gamma-2}\left(-\frac{\eta}{\zeta^{\alpha}}\right)\frac{1}{\frac{dz}{df}}\frac{1}{q_r} \quad \text{[using (29) and (55)]}$$

or

$$q_r Y'' = -(\gamma-1)\frac{\xi^{\gamma-2}\eta}{\zeta^{\alpha}}\frac{1}{\frac{dz}{df}} \quad \ldots \quad (50)$$

Now ξ, η, ζ and hence $\dfrac{Y}{q_r}$ and Y' are continuous variables even at the end of a stage. However $q_r\,Y''$, on account of the discontinuity in $\dfrac{dz}{df}$ while crossing from the $(r-1)$th stage to the rth, changes abruptly in doing so unless $\theta_{r-1} = 1$, i.e. unless the $(r-1)$th charge is a cord. Accordingly in calculating the value of $q_r\,Y''$ at the beginning of any stage, we should take the value of $\dfrac{dz}{df}$ at the beginning of that stage and not its value at the end of the previous stage.

Let ξ_{r-1}, η_{r-1}, ζ_{r-1}; Y_{r-1}, Y'_{r-1}, Y''_{r-1}; Z_{-1} denote the values of the corresponding variables at the end of the $(r-1)$th stage. Then the initial conditions for the rth stage are

$$f = 1-\frac{1}{k_{r-1}}, \quad z = A_r+\frac{B_r}{k_r}-\frac{C_r}{k^2_{r-1}}$$

$$Z = q_r z = \frac{q_r}{q_{r-1}}Z_{r-1} \quad \ldots \quad (51a)$$

$$Y = q_r\zeta_{r-1}\xi^{\gamma}_{r-1} = \frac{q_r}{q_{r-1}}Y_{r-1} \quad \ldots \quad (51b)$$

$$Y' = \xi^{\gamma}_{r-1} = Y'_{r-1} \quad \ldots \quad (51c)$$

$$Y'' = -\frac{\gamma-1}{q_r}\frac{\xi^{\gamma-1}_{r-1}}{\zeta^{\alpha}_{r-1}}\eta_{r-1}\frac{1}{-\frac{B_r}{k_{r-1}}+\frac{2C_r}{k^2_{r-1}}}$$

or

$$Y'' = \frac{q_{r-1}}{q_r}\frac{-\frac{B_{r-1}}{k_{r-1}}+\frac{2C_{r-1}}{k^2_{r-1}}}{-\frac{B_r}{k_{r-1}}+\frac{2C_r}{k^2_{r-1}}}Y''_{r-1} \quad \ldots \quad (51d)$$

In case $\theta_{r-1} = 1$, (51*d*) reduces to

$$Y'' = \frac{q_{r-1}}{q_r}Y''_{r-1} \quad \ldots \quad (51e)$$

6. All-burnt Position. Muzzle Velocity

For any given value of f in the rth stage, i.e. for any value of f lying between $1-\frac{1}{k_{r-1}}$ and $1-\frac{1}{k_r}$, we can find x, v, and p as follows:

(i) We find z from (28), then Z from (35), remembering that

$$q_r = \frac{4C_r}{B_r^2+4A_rC_r}$$

and using (17) for calculating A_r, B_r, C_r.

(ii) We find Y, Y', Y'' by solving numerically equation (43) subject to initial conditions (51).

(iii) We find ξ from (37), ζ from (34) and η from (26).

(iv) Finally we find x from (22), p from (24) and v from (23).

Thus the complete solution can be tabulated against f.

The shot-travel $x_{\bar{2}}$, the velocity $v_{\bar{2}}$ and the pressure $p_{\bar{2}}$ at all-burnt are respectively given by x_n, v_n and p_n, the shot-travel, velocity, and pressure at the end of the nth stage of burning.

For the motion of the shot, after all-burnt, $z = 1$ and from (36)

$$d(\zeta\xi^{\gamma}) = 0$$

or

$$\zeta\xi^{\gamma} = \zeta_{\bar{2}}\xi_{\bar{2}}^{\gamma}$$

If suffix $\bar{3}$ denotes the muzzle position

$$\zeta_{\bar{3}}\xi_{\bar{3}}^{\gamma} = \zeta_{\bar{2}}\xi_{\bar{2}}^{\gamma} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (52)$$

Where

$$\xi_{\bar{3}} = 1+\frac{x_{\bar{3}}}{l}$$

(52) determines the pressure at the muzzle.

From (26), on putting $z = 1$

$$\eta_{\bar{3}}^2 = \frac{2M}{\gamma-1}(1-\zeta_{\bar{3}}\xi_{\bar{3}})$$

$\therefore$ the muzzle velocity is given by

$$\frac{A^2D^2}{F^2C^2\beta^2}v_{\bar{3}}^2\left(\frac{FC}{Al}\right)^{2-2\alpha} = \eta_{\bar{3}}^2 = \frac{2M}{\gamma-1}\left(1-\zeta_{\bar{2}}\xi_{\bar{2}}^{\gamma}\xi_{\bar{3}}^{1-\gamma}\right) \quad \ldots \quad \ldots \quad (53)$$

7. Maximum Pressure

Substituting for Y, Y', Y'' from (34), (37) and (50) in (41) and simplifying, we get

$$\frac{d\zeta}{dZ} = \frac{1}{\xi q_r}\left[1+\gamma\eta\zeta^{1-\alpha}\frac{1}{\frac{dz}{df}}\right]$$

$$\therefore \quad \frac{d\zeta}{dz} = \frac{1}{\xi}\left[1+\gamma\zeta^{1-\alpha}\eta\frac{1}{\frac{dz}{df}}\right]$$

and
$$\frac{d\zeta}{df}=\frac{1}{\xi}\left[\frac{dz}{df}+\gamma\eta\zeta^{1-\alpha}\right] \qquad (54)$$

Using (27) and (29), this can also be written as
$$\frac{dz}{df}=\frac{1}{\xi}\left[\frac{dz}{df}-\frac{\gamma}{2M}\frac{d}{df}(\eta^2)\right] \qquad (55)$$

From (54) $\frac{dz}{df}$ and hence $\frac{d\zeta}{df}$ will be continuous at the end of the rth stage if and only if $\theta_r = 1$. If $\theta_r < 1$, $\frac{dz}{df}$ and hence $\frac{d\zeta}{df}$ receives a positive increment at the end of this stage.

Therefore at the end of a stage $\frac{d\zeta}{df}$ can change sign from negative to positive, but not vice versa.

Now for a linear law ($\alpha = 1$), from (54) and (14).
$$\frac{d\zeta}{df}=\frac{1}{\xi}[-B_r+2C_r(1-f)+\gamma\eta] \qquad (56)$$

But in this case from (27) and (29)
$$\eta = M(f_0-f)$$
where f_0 is the value of f at the shot-start. Substituting in (56)
$$\frac{d\zeta}{df}=\frac{1}{\xi}[-B_r+(\gamma M+2C_r)(1-f)-\gamma M(1-f_0)] \qquad (57)$$

Throughout the rth stage, f goes on decreasing and consequently

(i) if $\gamma M+2C_r > 0$, $\frac{d\zeta}{df}$ can change sign from negative to positive, but not vice versa.

(ii) if $\gamma M+2C_r \leqslant 0$, $\frac{d\zeta}{df}$ is negative throughout the rth stage (except in the case when $f_0 = 1$, $\gamma M+2C_r = 0$ and each $\theta_r = -1$, when $\frac{d\zeta}{df}$ is zero throughout the rth stage. This possibility we neglect. In all other cases it can easily be shown that $B_r>0$) and again it cannot change sign from positive to negative.

Thus we see that for linear law, once $\frac{d\zeta}{df}$ becomes positive, it cannot change to negative in any stage or at the end of any stage. The uniqueness of maximum pressure for linear law is thus established and we can talk of 'the Maximum Pressure'.

For the power law of burning when $\alpha \neq 1$, the increment in $\frac{d\zeta}{df}$ at the end of a stage is still positive or zero and, therefore, $\frac{d\zeta}{df}$ can change sign from negative to positive and pressure maximum can occur here. During the rth stage, if C_r is positive, $\frac{dz}{df}$ goes on increasing as f decreases. η also increases since $\frac{d\eta}{dt}$ is proportional to pressure and is always positive. In this case so long as ζ is increasing $\frac{dz}{df}+\gamma\eta\zeta^{1-\alpha}$

goes on increasing and $\frac{d\zeta}{df}$ can change sign from negative to positive and not vice versa. After the pressure has reached maximum, ζ decreases. If, however, $\eta\zeta^{1-\alpha}$ continues to increase till all-burnt position, the pressure cannot become a minimum at any instant. If, however, $\eta\zeta^{1-\alpha}$ decreases, $\frac{d\zeta}{df}$ may change sign from positive to negative and a pressure minimum may arise. The above arguments have to be modified if C_r is negative.

Since η and ζ cannot be expressed as simple functions of f, it is not easy to establish the uniqueness of maximum pressure analytically as we could do in the case of linear law of burning. The problem can, however, be solved numerically in certain typical cases and this will be the subject of subsequent investigations. If, however, α is very near unity, as is the case in most modern gun propellants, the power law can be approximated very nearly by the linear law and therefore, for this case, we may, with some justification, speak of the maximum pressure. It will still be safer to talk of 'a pressure maximum'.

We can investigate the conditions for the occurrence of a pressure maximum in any stage in two alternative ways.

In the First Method, we prepare a table of values of the function

$$P(Z) = \frac{Y'}{Y} - \frac{\gamma}{\gamma-1}\frac{Y''}{Y'} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (58)$$

from the tables of Y, Y', Y'' already prepared. Wherever $P(Z)$ changes sign from positive to negative a pressure maximum occurs. If it changes sign from positive to negative only once and does not change sign from negative to positive, the maximum pressure is unique.

Let $\bar{Z}$ be the value of Z for which this change of sign occurs. Then a pressure maximum occurs in the rth stage, if

$$A_r + \frac{B_r}{k_{r-1}} - \frac{C_r}{k_{r-1}^2} < \frac{\bar{Z}}{q_r} < A_r + \frac{B_r}{q_r} - \frac{C_r}{k_r^2} \quad \ldots \quad \ldots \quad \ldots \quad (59)$$

It will occur at the end of the rth stage if

$$\bar{Z} = q_r\left(A_r + \frac{B_r}{q_r} - \frac{C_r}{k_r^2}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (60)$$

and will occur at all-burnt if

$$\bar{Z} = q_n(A_n + B_n - C_n) = q_n \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (61)$$

It may be noted that for a pressure maximum to occur at the end of a stage or at all-burnt, it is not necessary that $P(Z)$ should vanish at these instants. All that is required is that $P(Z)$ should change sign from positive to negative.

In the Second Method, we use (54).

A pressure maximum can occur in the rth stage if $\frac{d\zeta}{df}$ is negative at the beginning and positive at the end of this stage, i.e. if

$$-B_r + \frac{2C_r}{k_{r-1}} + \gamma\eta_{r-1}\zeta_{r-1}^{1-\alpha} < 0 \quad \ldots \quad \ldots \quad \ldots \quad (62a)$$

and

$$-B_r + \frac{2C_r}{k_r} + \gamma\eta_r\zeta_r^{1-\alpha} > 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (62b)$$

A pressure maximum can occur at the end of the rth stage if

either (i) $\dfrac{d\zeta}{df}$ is negative at the end of the rth stage and positive at the beginning of the $(r+1)$th,

or (ii) $\dfrac{d\zeta}{df}$ is zero at the end of the rth stage,

or (iii) $\dfrac{d\zeta}{df}$ is zero at the beginning of the $(r+1)$th stage.

The corresponding conditions are:

$$\text{either (i)} \quad -B_r+\frac{2C_r}{k_r}+\gamma\eta_r\zeta_r^{1-\alpha}<0 \quad \ldots \quad (63a)$$

$$\text{and} \quad -B_{r+1}+\frac{2C_{r+1}}{k_r}+\gamma\eta_r\zeta_r^{1-\alpha}>0 \quad \ldots \quad (63b)$$

$$\text{or (ii)} \quad -B_r+\frac{2C_r}{k_r}+\gamma\eta_r\zeta_r^{1-\alpha}=0 \quad \ldots \quad (63c)$$

$$\text{or (iii)} \quad -B_{r+1}+\frac{2C_{r+1}}{k_r}+\gamma\eta_r\zeta_r^{1-\alpha}=0 \quad \ldots \quad (63d)$$

For a pressure maximum to occur at all-burnt

$$-B_n+2C_n+\gamma\eta_n\zeta_n^{1-\alpha}\leqslant 0. \quad \ldots \quad (64)$$

8. Particular Case of Two Component Charges

In this section, we consider the case $n = 2$ on account of its importance in practical use. This case has been discussed also by Patni (1955) by the Direct Method. We shall compare the results of our present investigation with his and shall try to explain why in some cases our results are different from his. To make the comparison easier, in the first sub-section, we explain the connections between his notation and ours.

8.1. *Notations*

The following constants occur in Patni's investigations:

$$\lambda=\frac{\beta_1'}{\beta_2'}\frac{1+\theta_2}{1+\theta_1}\frac{\theta_1}{\theta_2}, \qquad \mu=1-\lambda \quad \ldots \quad (65)$$

$$K_1=\frac{4\theta_1}{(1+\theta_1)^2}, \qquad K_2=\frac{4\theta_2}{(1+\theta_2)^2} \quad \ldots \quad (66)$$

$$\lambda'=\frac{F_1C_1}{F_2C_2}\frac{K_2}{K_1}, \qquad K_2\mu'=(1-\mu^2-\lambda^2)\lambda' \quad \ldots \quad (67)$$

$$\nu'=1+\lambda^2\lambda', \qquad a=\lambda^2\lambda'^2\mu^2+\nu'^2+K_2\mu'\nu' \quad \ldots \quad (68)$$

$$b=\frac{K_2}{a}\nu', \qquad b'=\frac{K_2}{a}, \qquad a'=\frac{F_2C_2+F_1C_1K_2}{F_1C_1} \quad \ldots \quad (69)$$

$$M_2=\frac{A^2D_2^2}{F_2C_2\beta_2^2w}\left(\frac{F_2C_2}{Al}\right)^{2-2\alpha}, \quad Q=\frac{M_2(\gamma-1)}{(1+\theta_2)^2ab^{3-2\alpha}}, \quad Q'=\frac{M_2(\gamma-1)}{(1+\theta_2)^2a'b'^{3-2\alpha}} \quad \ldots \quad (70)$$

He also writes

$$\zeta_2 = \frac{Al}{F_2C_2}p, \quad \eta_2 = \frac{AD_2}{F_2C_2\beta_2}v\left(\frac{F_2C_2}{Al}\right)^{1-\alpha} \quad \ldots \quad \ldots \quad (71)$$

The variables used for the first stage are

$$Y = b\zeta_2\xi^\gamma, \quad Z = b\left(z_2 + \frac{F_1C_1}{F_2C_2}z_1\right) \quad \ldots \quad \ldots \quad \ldots \quad (72)$$

and for the second stage

$$Y = b'\zeta_2\xi^\gamma, \quad Z = b'\left(z_2 + \frac{F_1C_1}{F_2C_2}\right) \quad \ldots \quad \ldots \quad \ldots \quad (73)$$

Since from (12) $\frac{D}{\beta} = \frac{D_2}{\beta_2}$, we easily see that

$$\text{(i)} \quad \frac{\zeta}{\zeta_2} = \frac{F_2C_2}{FC} = \lambda_2 \,; \quad \frac{\eta}{\eta_2} = \lambda_2^\alpha \,; \quad \frac{\eta}{\zeta^\alpha} = \frac{\eta_2}{\zeta_2^\alpha} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (74)$$

$$\text{(ii)} \quad \frac{M}{M_2} = \left(\frac{F_2C_2}{FC}\right)^{2\alpha-1} = (\lambda_2)^{2\alpha-1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (75)$$

$$\text{(iii)} \quad a = \lambda^2\lambda'^2(1-\lambda)^2 + (1+\lambda^2\lambda')^2 + [1-\lambda^2-(1-\lambda)^2](1+\lambda^2\lambda')$$

$$= (1+\lambda\lambda')^2 = \left(1 + \frac{\beta_1'}{\beta_2'}\,\frac{1+\theta_1}{1+\theta_2}\,\frac{F_1C_1}{F_2C_2}\right)^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (76)$$

$$\text{(iv)} \quad \nu_1 = \sqrt{B_1^2 + 4A_1C_1} = B_1 = \lambda_1k_1(1+\theta_1) + \lambda_2k_2(1+\theta_2)$$

$$= \lambda_2(1+\theta_2)\left[1 + \frac{F_1C_1}{F_2C_2}\,\frac{\beta_1'}{\beta_2'}\,\frac{1+\theta_1}{1+\theta_2}\right]$$

$$= \lambda_2(1+\theta_2)(1+\lambda\lambda') \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (77)$$

$$\text{(v)} \quad \nu_2 = \sqrt{B_2^2 + 4A_2C_2} = \sqrt{\lambda_2^2k_2^2(1+\theta_2)^2 + 4\lambda_1\lambda_2k_2^2\theta_2}$$

$$= \lambda_2(1+\theta_2)\sqrt{1 + k_2\frac{F_1C_1}{F_2C_2}} = \lambda_2(1+\theta_2)\sqrt{a'} \quad \ldots \quad \ldots \quad \ldots \quad (78)$$

$$\text{(vi)} \quad q_1 = \frac{4C_1}{B_1^2 + 4A_1C_1} = \frac{4(\lambda_1k_1^2\theta_1 + \lambda_2k_2^2\theta_2)}{\lambda_2^2(1+\theta_2)^2(1+\lambda\lambda^1)^2}$$

$$= \frac{4\theta_2}{1+\theta_2}\,\frac{1}{\lambda_2}\,\frac{1 + \frac{F_1C_1}{F_2C_2}\,\frac{\beta_1'^2}{\beta_2'^2}\,\frac{\theta_1}{\theta_2}}{(1+\lambda\lambda')^2}$$

$$= \frac{K_2}{\lambda_2}\,\frac{1+\lambda^2\lambda'}{(1+\lambda\lambda')^2} = \frac{K_2\nu'}{\lambda_2 a} = \frac{b}{\lambda_2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (79)$$

$$\text{(vii)} \quad q_2 = \frac{4C_2}{B_2^2 + 4A_2C_2} = \frac{4\lambda_2k_2^2\theta_2}{\lambda_2^2(1+\theta_2)^2a'} = \frac{K_2}{a'\lambda_2} = \frac{b'}{\lambda_2} \quad \ldots \quad \ldots \quad \ldots \quad (80)$$

(viii) In the first stage

$$Y = q_1\zeta\xi^{\gamma} = \frac{b}{\lambda_2}\lambda_2\zeta_2\xi^{\gamma} = b\zeta_2\xi^{\gamma} \quad \ldots \quad (81)$$

$$Z = q_1 z = \frac{b_1}{\lambda_2}\,\frac{F_1C_1z_1+F_2C_2z_2}{FC} = b\left(z_2+\frac{F_1C_1}{F_2C_2}z_1\right) \quad \ldots \quad (82)$$

(ix) In the second stage

$$Y = q_2\zeta\xi^{\gamma} = \frac{b_1}{\lambda_2}\lambda_2\zeta_2\xi^{\gamma} = b'\zeta_2\xi^{\gamma} \quad \ldots \quad (83)$$

$$Z = q_2 z = \frac{b'}{\lambda_2}\,\frac{F_1C_1+F_2C_2z_2}{FC} = b'\left(z_2+\frac{F_1C_1}{F_2C_2}\right) \quad \ldots \quad (84)$$

(x) $$Q_1 = \frac{M(\gamma-1)}{q_1^{3-2\alpha}v_1^2} = \frac{M(\gamma-1)}{\left(\frac{b}{\lambda_2}\right)^{3-2\alpha}\lambda_2^2(1+\theta_2)^2(1+\lambda\lambda')^2}$$

$$= \frac{M_2(\gamma-1)(\lambda_2)^{-1+2\alpha}(\lambda_2)^{3-2\alpha}}{b^{3-2\alpha}\lambda_2^2(1+\theta_2)^2 a}$$

$$= \frac{M_2(\gamma-1)}{ab^{3-2\alpha}(1+\theta_2)^2} = Q \quad \ldots \quad (85)$$

(xi) $$Q_2 = \frac{M(\gamma-1)}{q_2^{3-2\alpha}v_2^2} = \frac{M(\gamma-1)}{\left(\frac{b'}{\lambda_2}\right)^{3-2\alpha}\lambda_2^2(1+\theta_2)^2 a'}$$

$$= \frac{M(\gamma-1)}{a'b'^{3-2\alpha}(1+\theta_2)^2} = Q' \quad \ldots \quad (86)$$

From (81), (82), (83), and (84), we find that the variables used by Patni are the same as ours and using (85), (86) we see that the differential equations for the two stages are also the same in the two investigations.

In the succeeding sub-section, we examine the results which are different. In some cases the present treatment gives results which are simpler or more general or physically more significant; in other cases the differences are more fundamental.

8.2. *Conditions for the simultaneous and non-simultaneous burning out of component charges*

From (6) $\frac{D_1}{\beta_1}\frac{df_1}{dt} = \frac{D_2}{\beta_2}\frac{df_2}{dt}$. Integrating and remembering that when ignition starts $f_1 = 1, f_2 = 1$, we get

$$\frac{1-f_1}{\beta_1'} = \frac{1-f_2}{\beta_2'} \quad \ldots \quad (87)$$

(i) If $\beta_1' > \beta_2'$; $f_1 < f_2$: charge C_1 burns out first.

(ii) If $\beta_1' = \beta_2'$; $f_1 = f_2$: both the charges burn out simultaneously.

(iii) If $\beta_1' < \beta_2'$; $f_2 < f_1$: charge C_2 burns out first.

The corresponding conditions used by Patni (1955) and first deduced by Venkatesan and Patni (1953) are

(i) If $\beta_1' f_{20} > \beta_2' f_{10}$, charge C_1 burns out first.

(ii) If $\beta_1' f_{20} = \beta_2' f_{10}$, charges C_1 and C_2 burn out simultaneously.

(iii) If $\beta_1' f_{20} < \beta_2' f_{10}$, charge C_2 burns out first. Where f_{10}, f_{20} are the values of f_1, f_2 when motion starts.

Now from (87)

$$\frac{1-f_{10}}{\beta_1'} = \frac{1-f_{20}}{\beta_2'}$$

or

$$\beta_1' f_{20} - \beta_2' f_{10} = \beta_1' - \beta_2' \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (88)$$

Hence Venkatesan and Patni's conditions reduce to our conditions and the latter are simpler in the sense that they involve only the parameters of the component charges and not f_{10}, f_{20} which are functions of the shot-start pressure.

8.3. *Initial conditions for the first and the second stages*

The initial value of Z given by Patni for the first stage is

$$\begin{aligned} Z &= b\left(z_{20} + \frac{F_1 C_1}{F_2 C_2} z_{10}\right) \\ &= \frac{q_1 \lambda_2}{F_2 C_2}(F_1 C_1 z_{10} + F_2 C_2 z_{20}) \quad [\text{using } (79)] \\ &= q_1 z_0 = q_1 \zeta_0 \\ &= q_1 \frac{Al}{FC} p_0 \end{aligned}$$

which is the same as the value obtained by us in (47).

It is obvious that the initial conditions should be expressed in terms of the shot-start pressure, which in practice we are likely to be given rather than in terms of z_{10}, z_{20} which have to be later determined in terms of p_0 and, therefore, Patni's conditions should be replaced by

$$Z = b \frac{Al}{F_2 C_2} p_0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (89)$$

Again from (51), the initial conditions for the second stage are

$$Z = \frac{q_2}{q_1} Z_1 = \frac{q_2(F_1 C_1 + F_2 C_2 Z_{2;1})}{FC} = b'\left(\frac{F_1 C_1}{F_2 C_2} + Z_{2;2,1}\right)$$

$$Y = \frac{q_2}{q_1} Y_1 = q_2 \zeta_{2;1} \xi_2^{\gamma}; \qquad 1 = b' \zeta_{2;2,1} \xi_{2,1}^{\gamma},$$

$$Y' = \quad Y_1' = \xi_1^{\gamma-1} \qquad = \xi_{2,1}^{\gamma-1}$$

$$Y'' = \frac{q_1}{q_2} \frac{-\frac{B_1}{k_1} + \frac{2C_1}{k_1^2}}{-\frac{B_2}{k_1} + \frac{2C_2}{k_1^2}} Y_1'' \qquad = \frac{q_1}{q_2} \frac{-\frac{B_1}{k_1} + \frac{2C_1}{k_1^2}}{-\frac{B_2}{k_1} + \frac{2C_2}{k_1^2}} Y_{2,1}''$$

where we denote the end of the first stage by suffix 1 and Patni denotes it by (2, 1).

Comparison with Patni's conditions $[Z = Z_{2,1}\,;\ Y = Y_{2,1}\,;\ Y' = Y'_{2,1}\,;\ Y'' = Y''_{2,1}]$ shows that the first, second and fourth conditions differ. In all the three, one cause of discrepancy is the overlooking of the fact that Y and Z do not represent the same variables in the two stages.

In the first stage

$$Y = b\zeta_2\xi^{\gamma},\ \ Z = b\left(z_2+\frac{F_1C_1}{F_2C_2}z_1\right)$$

and in the second stage

$$Y = b'\zeta_2\xi^{\gamma},\ Z = b'\left(z_2+\frac{F_1C_1}{F_2C_2}\right)$$

and so although z_1, z_2, ζ_2, ξ are continuous at the end of the first stage, Y and Z are not.

In the fourth condition, an additional source of discrepancy when $\theta_1 \neq 1$ is that while $\frac{Z}{b}$ at the end of the first stage is equal to $\frac{Z}{b'}$ at the beginning of the second stage, their derivatives are not necessarily continuous at this instant. Thus in the first stage, from (5), (6), (29), (66), (72) and (87)

$$\xi' = \frac{d\xi}{dz} = \frac{1}{b}\,\frac{d\xi}{\frac{dz_2}{df_2}df_2+\frac{F_1C_1}{F_2C_2}\frac{dz_1}{df_1}df_1}$$

$$= -\frac{1}{b}\frac{d\xi}{df}\,\frac{1}{(1+\theta_2)\sqrt{1-K_2z_2}+\frac{F_1C_1}{F_2C_2}\frac{\beta_1'}{\beta_2'}(1+\theta_1)\sqrt{1-K_1z_1}}$$

$$= \frac{1}{b}\frac{\eta}{\zeta^{\alpha}}\,\frac{1}{(1+\theta_2)\sqrt{1-K_2z_2}+\frac{F_1C_1}{F_2C_2}\frac{\beta_1'}{\beta_2'}(1+\theta_1)\sqrt{1-K_1z_1}}$$

Similarly in the second stage

$$\xi' = \frac{1}{b'}\frac{\eta}{\zeta^{\alpha}}\,\frac{1}{(1+\theta_2)\sqrt{1-K_2z_2}}$$

$\therefore$ ξ' is discontinuous at the end of the first stage. But from (37) and (40)

$$Y'' = (\gamma-1)\frac{Y'\xi'}{\xi} = (\gamma-1)\xi^{\gamma-2}\xi'$$

Since ξ' is discontinuous at the end of the first stage, its value should be taken at the beginning of the second stage.

Therefore, the first, second and fourth conditions for the second stage given by Patni should be replaced by

$$\frac{Z}{b'} = \frac{Z_{2,1}}{b} = \frac{F_1C_1}{F_2C_2}+z_{2;\,2,1} \qquad \text{(90}a\text{)}$$

$$\frac{Y}{b'} = \frac{Y_{2,1}}{b} = \zeta_{2;\,2,1}\xi^{\gamma}_{2,1} \qquad \text{(90}b\text{)}$$

$$Y'' = (\gamma-1)\xi^{\gamma-2}_{2,1}\frac{1}{b'}\frac{\eta_{2,1}}{\zeta^{\alpha}_{2,1}}\,\frac{1}{(1+\theta_2)\sqrt{1-K_2z_{2;\,2,1}}}$$

$$= \frac{\gamma-1}{b'}\,\xi^{\gamma-2}_{2,1}\frac{\eta_{2;\,2,1}}{\zeta^{\alpha}_{2;\,2,1}}\,\frac{1}{(1+\theta_2)\sqrt{1-K_2z_{2;\,2,1}}} \qquad \text{(90c)}$$

as from (74) at any instant $\frac{\eta}{\zeta^{\alpha}} = \frac{\eta_2}{\zeta_2^{\alpha}}$. When first charge is cord, the last condition can be replaced by

$$\frac{Y''}{b} = \frac{Y''_{2,1}}{b'} \qquad \text{(90}d\text{)}$$

8.4. *The conditions for a pressure maximum in the first stage*

At the beginning of the first stage $z = z_0 = \zeta_0$, and at the end of the first stage

$$\lambda_1 z_1 + \lambda_2 z_2 = z = A_1 + \frac{B_1}{k_1} - \frac{C_1}{k_1^2}$$

$$= \lambda_1 + \lambda_2 \frac{\beta_2'}{\beta_1'}\left(1+\theta_2 - \frac{\beta_2'}{\beta_1'}\theta_2\right)$$

Since at the end of the first stage $z_1 = 1$

$$z_{2;\,2,1} = \frac{\beta_2'}{\beta_1'}\left(1+\theta_2 - \frac{\beta_2'}{\beta_1'}\theta_2\right) \qquad (91)$$

At the end of the second stage

$$z_1 = z_2 = 1;\ z = 1. \qquad (92)$$

A pressure maximum will occur in the first stage if

$$z_0 < \bar{z} < \lambda_1 + \lambda_2 \frac{\beta_2'}{\beta_1'}\left(1+\theta_2 - \frac{\beta_2'}{\beta_1'}\theta_2\right) \qquad (93)$$

If the shot starts in the first stage, $\bar{z} > z_0$, as the argument in our tables starts from z_0. If the shot does not start in the first stage, $\bar{z} > \lambda_1 + \lambda_2 z_{2;\,2,1}$ and the possibility of a pressure maximum in the first stage does not arise.

Assuming that the shot starts moving in the first stage, the condition becomes

$$\frac{\bar{Z}}{q_1} < \lambda_1 + \lambda_2 \frac{\beta_2'}{\beta_1'}\left(1+\theta_2 - \frac{\beta_2'}{\beta_1'}\theta_2\right) \qquad (94)$$

Instead of (94), Patni gives the following two conditions:

$$\left[\frac{\nu' + \lambda\lambda'\mu}{\sqrt{a}} - \sqrt{1-\bar{\bar{Z}}}\right]\left[\frac{\nu' - \lambda\lambda'\mu}{\sqrt{a}} + \sqrt{1-\bar{\bar{Z}}}\right] < \frac{\nu'^2 K_2}{a} \qquad (95)$$

and

$$\left[\frac{\nu' - \mu}{\lambda\sqrt{a}} - \sqrt{1-\bar{\bar{Z}}}\right]\left[\frac{\nu' + \mu}{\lambda\sqrt{a}} + \sqrt{1-\bar{\bar{Z}}}\right] < \frac{\nu'^2 K_1}{\lambda^2 a} \qquad (96)$$

Since from (76) $a > 0$, (95) gives

$$\nu'^2 - \left(\lambda\lambda'\mu - \sqrt{a}\sqrt{1-\bar{\bar{Z}}}\right)^2 < \nu'^2 K_2$$

or

$$\nu'^2 - \nu'^2(1 - K_2\bar{z}_2) < \nu'^2 K_2$$

or

$$K_2\bar{z}_2 < K_2 \qquad (97)$$

Similarly (96) gives

$$\nu'^2-\left(\mu+\lambda\sqrt{a}\sqrt{1-\bar{Z}}\right)^2 < \nu'^2 K_1$$

or

$$\nu'^2-\left(\mu\nu'+\nu'\lambda\sqrt{1-K_2\bar{z}_2}\right)^2 < \nu'^2 K_1$$

or

$$1-\left(\sqrt{1-K_1\bar{z}_1}\right)^2 < K_1$$

or

$$K_1\bar{z}_1 < K_1 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (98)$$

(95) is equivalent to the condition $\bar{z}_2 < 1$ if and only if $K_2 > 0$, i.e. if $\theta_2 > 0$. If θ_2 can be negative also (95) should be replaced by

$$\left[\frac{\nu'+\lambda\lambda'\mu}{\sqrt{a}}-\sqrt{1-\bar{Z}}\right]\left[\frac{\nu'-\lambda\lambda'\mu}{\sqrt{a}}+\sqrt{1-\bar{Z}}\right] \lessgtr \frac{\nu'^2 K_2}{a} \quad \text{according as } \theta_2 \gtrless 0. \quad \ldots \quad \ldots \quad (99)$$

Similarly (96) can be replaced by

$$\left[\frac{\nu'-\mu}{\lambda\sqrt{a}}-\sqrt{1-\bar{Z}}\right]\left[\frac{\nu'+\mu}{\lambda\sqrt{a}}+\sqrt{1-\bar{Z}}\right] \lessgtr \frac{\nu'^2 K_1}{\lambda^2 a} \quad \text{according as } \theta_1 \gtrless 0. \quad \ldots \quad \ldots \quad (100)$$

Now (95) and (96) (or (99) and (100)) have been deduced from the consideration that $\bar{z}_1$, $\bar{z}_2$ should be both less than unity, if maximum pressure is to occur in the first stage. In other words both $\bar{f}_1$ and $\bar{f}_2$ should be greater than zero.

Now from (87), if $\beta_1' > \beta_2'$, $f_2 < f_1$ so that if $f_1 > 0$, f_2 would be automatically greater than zero and, therefore, condition (100) would imply (99). Similarly, if $\beta_1' < \beta_2'$ (99) would imply (100), and if $\beta_1' = \beta_2'$ (99) and (100) would become identical.

Thus (99) and (100) improve the condition of Patni in as much as they hold for progressively burning surfaces as well. Further, only one of the conditions is independent. Besides, condition (94) is simpler than (99) or (100).

8.5. *The condition for a pressure maximum to occur in the second stage*

From (59), the condition for a pressure maximum to occur in the second stage is

$$A_2+\frac{B_2}{k_1}-\frac{C_2}{k_1^2} < \frac{\bar{Z}}{q_2} < A_2+\frac{B_2}{k_2}-\frac{C_2}{k_2^2}$$

This simplifies to

$$\lambda_1+\frac{\beta_2'}{\beta_1'}\left(1+\theta_2-\frac{\beta_2'}{\beta_1'}\theta_2\right) < \bar{z} < 1$$

or

$$\lambda_1+\frac{\beta_2'}{\beta_1'}\left(1+\theta_2-\frac{\beta_2'}{\beta_1'}\theta_2\right) < \lambda_1+\lambda_2\bar{z}_2 < 1$$

or

$$\frac{\beta_2'}{\beta_1'}\left(1+\theta_2-\frac{\beta_2'}{\beta_1'}\theta_2\right) < \bar{z}_2 < 1 \quad \ldots \quad \ldots \quad \ldots \quad (101)$$

or

$$z_{2;\,2,\,1} < \bar{z}_2 < 1 \quad \ldots \quad \ldots \quad \ldots \quad (102)$$

Patni gives the following three conditions for a pressure maximum to occur in the second stage

$$\beta_1' f_{20} > \beta_2' f_{10} \quad \ldots \quad (103a)$$

$$\mu + \lambda\sqrt{1-K_2} < \sqrt{1-K_1} \quad \ldots \quad (103b)$$

and

$$\frac{F_2 C_2(\bar{Z}) - F_1 C_1}{b' F_2 C_2} < 1. \quad \ldots \quad (103c)$$

The first condition simply implies that charge C_1 burns out first. The second condition expresses the same fact if $\theta_1, \theta_2 > 0$. If, however, these can be negative, this should be replaced by

$$\left.\begin{array}{ll} \mu+\lambda\sqrt{1-K_2} < \sqrt{1-K_1}, & \text{if } \theta_1, \theta_2 > 0 \\ & \text{or if } \theta_1, \theta_2 < 0 \\ \mu+\lambda\sqrt{1-K_2} > \sqrt{1-K_1}, & \text{if } \theta_1 > 0, \theta_2 < 0 \\ & \text{or if } \theta_1 < 0, \theta_2 > 0 \end{array}\right\} \quad \ldots \quad (104)$$

The third condition is equivalent to

$$\bar{z}_2 < 1$$

But this is not sufficient to ensure a pressure maximum in the second stage. The complete condition is (102) which is expressed as (101).

Hence Patni's condition for this case should be replaced by

$$\frac{\beta_2'}{\beta_1'}\left(1+\theta_2-\frac{\beta_2'}{\beta_1'}\theta_2\right) < \frac{F_2 C_2 \bar{Z} - F_1 C_1}{b' F_2 C_2} < 1 \quad \ldots \quad (105)$$

8.6. *The conditions for a pressure maximum to occur at the end of the first stage or at all-burnt*

We have seen that $\frac{d\zeta}{df}$ receives a positive or zero increment at the end of the first stage and, therefore, it can change sign from negative to positive here and a pressure maximum can occur at this instant. The condition for this is obtained from (60) and (63) by putting $r = 1$. This possibility has not been considered in the earlier papers and so no corresponding conditions exist there.

A pressure maximum can also occur at all-burnt. The conditions are obtained from (61) or (64) by putting $n = 2$. This possibility was also not considered earlier except for the case when the charges burn out simultaneously. This case we deal in the next sub-section.

8.7. *The case when both the component charges burn out simultaneously*

If $\beta_1' = \beta_2'$, it has been proved (Kapur, 1956*a*) that the composite charge behaves as a single charge with mass

$$C_1 + C_2,$$

force constant

$$\frac{C_1 F_1 + C_2 F_2}{C_1 + C_2},$$

form factor

$$\frac{C_1F_1\theta_1+C_2F_2\theta_2}{C_1F_1+C_2F_2},$$

effective ballistic size

$$\frac{D_1}{\beta_1}=\frac{D_2}{\beta_2}$$

and propellant density

$$\frac{C_1+C_2}{\frac{C_1}{\delta_1}+\frac{C_2}{\delta_2}}$$

This case, therefore, needs no separate treatment.

9. Some Further Extensions of Earlier Results

In this section we shall consider two further generalisations :

(i) So far the form-function for the ith component charge has been taken in the standard form

$$z_i=(1-f_i)(1+\theta_i f_i)$$

In sub-section 9.1, we consider the case when the ith component charge can have more general form-function

$$z_i=\phi_i(f_i) \quad \ldots \quad (106)$$

and in particular when it has the cubic form-function

$$z_i=(1-f_i)\left(1+\theta_i f_i+\psi_i f_i^2\right) \quad \ldots \quad (107)$$

(ii) In sub-section 9.2, we consider the case when $\gamma_1, \gamma_2, \ldots, \gamma_n$ are not necessarily equal.

9.1. *General form-function for component charges*

When (106) gives the form-function for the ith component charge, the form-function for the rth stage for the equivalent charge is (Kapur, 1956*a*)

$$z=\lambda_1+\lambda_2+\ldots+\lambda_{r-1}+\sum_{i=r}^{n}\lambda_i\phi_i[1-k_i(1-f)] \quad \ldots \quad (108)$$

$$=P_r(f) \text{ [say]} \quad \ldots \quad (109)$$

Now we take, for this section,

$$Y=\zeta\,\xi^{\gamma} \quad \ldots \quad (110)$$

as dependent variable and f as independent variable.

From (109) and (36)

$$P_r'(f)\,df=dz=\xi^{1-\gamma}d(\zeta\,\xi^{\gamma})$$

or

$$P_r'(f)\,df=\xi^{1-\gamma}dY$$

or

$$Y'=P_r'(f)\,\xi^{\gamma-1}$$

or

$$\xi = \left[\frac{Y'}{P'_r(f)}\right]^{\frac{1}{\gamma-1}} = \left[\frac{Y'}{P'_r}\right]^{\frac{1}{\gamma-1}} \quad \ldots \ldots \ldots (111)$$

From (110)

$$\zeta = Y\xi^{-\gamma} = Y\left(\frac{Y'}{P'_r}\right)^{-\frac{\gamma}{\gamma-1}} \quad \ldots \ldots \ldots (112)$$

Substituting from (111), (112) in (30)

$$\frac{d}{df}\left\{Y^{\alpha}\left(\frac{Y'}{P'_r}\right)^{-\frac{\gamma\alpha}{\gamma-1}}\frac{1}{\gamma-1}\left(\frac{Y'}{P'_r}\right)^{\frac{1}{\gamma-1}-1}\frac{P'_rY''-Y'P''_r}{P'^2_r}\right\} = MY^{1-\alpha}\left(\frac{Y'}{P'_r}\right)^{-\frac{\gamma(1-\alpha)}{\gamma-1}}$$

or

$$\frac{1}{\gamma-1}\frac{d}{df}\left[Y^{\alpha}Y'^{-\frac{\gamma\alpha-\gamma+2}{\gamma-1}}P_r'^{\frac{\gamma\alpha-\gamma}{\gamma-1}}(P'_rY''-P''_rY')\right] = \frac{MY^{1-\alpha}(Y')^{-\frac{\gamma(1-\alpha)}{\gamma-1}}}{(P'_r)^{-\frac{\gamma(1-\alpha)}{\gamma-1}}} \quad \ldots (113)$$

(113) is a non-linear differential equation of the third order between Y and f and by integrating it we can tabulate Y, Y', Y'' as functions of f.

In the particular case, when (107) represents the form-function for the ith component charge

$$P_r(f) = A'_r+B'_r(1-f)-C'_r(1-f)^2+D'_r(1-f)^3 \quad \ldots \ldots (114)$$

where

$$A'_r = \sum_{r=1}^{r-1}\lambda_i \quad \ldots \ldots \ldots \ldots \ldots \ldots (115a)$$

$$B'_r = \sum_{i=r}^{n}\lambda_ik_i(1+\theta_i+\psi_i) \quad \ldots \ldots \ldots \ldots \ldots (115b)$$

$$C'_r = \sum_{i=r}^{n}\lambda_ik_i^2(\theta_i+2\psi_i) \quad \ldots \ldots \ldots \ldots \ldots (115c)$$

$$D'_r = \sum_{i=r}^{n}\lambda_ik_i^3\psi_i \quad \ldots \ldots \ldots \ldots \ldots \ldots (115d)$$

Substituting from (114) in (113) we get the fundamental differential equation.

During the rth stage

$$Y = \zeta\xi^{\gamma}, \quad Y' = P'_r(f)\xi^{\gamma-1}$$

$$Y'' = P''_r(f)\xi^{\gamma-1}-(\gamma-1)P'_r(f)\frac{\eta}{\zeta^{\alpha}} \quad \text{[using (29)]}$$

Unless $\theta_r = 1$, $P'_r(f)$ and hence Y' is discontinuous at the end of the rth stage, and unless $\theta_r = \psi_r = 1$ Y'' will also be discontinuous at the end of the rth stage. Therefore in writing initial conditions for any stage, the values of $P'_r(f)$ and $P''_r(f)$ should be taken at the beginning of that stage and not at the end of the previous stage, while the values of ξ, η, ζ can be taken at the end of the previous stage.

Making use of (110), (111), (112) and (29)

$$\frac{\zeta'}{\zeta} = \frac{Y'}{Y} - \frac{\gamma}{\gamma-1}\frac{Y''}{Y'} + \frac{\gamma}{\gamma-1}\frac{P_r''}{P_r'} \qquad \ldots \quad \ldots \quad \ldots \quad (116)$$

and

$$\zeta' = \frac{1}{\xi}[P_r' + \gamma\eta\zeta^{1-\alpha}] \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (117)$$

A pressure maximum occurs when ζ' changes sign from negative to positive.

The solution corresponding to $f = 0$ determines the shot-travel, pressure and velocity at all-burnt and then (52), (53) determines the corresponding quantities for the muzzle.

Two recent papers by Aggarwal (1955 ; another in press) are entirely devoted to a discussion of the solution for the general form-function discussed by us in section 9.1. He, however, uses z as independent variable and later solves for f in terms of z either from a cubic equation of the more general form-function $z = \phi(f)$. This is not necessary in our method, as we make a direct use of f as the independent variable.

9.2. *The case when $\gamma_1, \gamma_2, \ldots. \gamma_n$ are not necessarily equal*

In this case (Kapur, 1956*a*, *b*), the only important change would be that in all our previous discussions, λ_i would be replaced by λ_i', where

$$\lambda_i' = \frac{\dfrac{C_iF_i}{\gamma_i-1}}{\dfrac{CF}{\gamma-1}} \; [i = 1, 2, \ldots, n] \qquad \ldots \quad \ldots \quad \ldots \quad (118)$$

and

$$\frac{CF}{\gamma-1} = \sum_{i=1}^{n} \frac{C_iF_i}{\gamma_i-1} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (119a)$$

with

$$\frac{1}{\gamma-1} = \frac{\displaystyle\sum_{i=1}^{n} \frac{m_iC_iz_i}{\gamma_i-1}}{\displaystyle\sum_{i=1}^{n} m_iC_iz_i} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (119b)$$

where m_i denotes the number of gram molecules per gram of the gases produced by the burning of the ith component charge. But from (119*b*), γ is not necessarily constant. A satisfactory average value of γ can however be found, following the method of Corner as illustrated in Kapur (1956*a*), and this average value has to be used in (119*a*). After all-burnt, however, γ is a constant and is given by

$$\frac{1}{\gamma-1} = \frac{\displaystyle\sum_{i=1}^{n} \frac{m_iC_i}{\gamma_i-1}}{\displaystyle\sum_{i=1}^{n} m_iC_i} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (119c)$$

In this case, we can solve for F and γ from (119*b*) and (119*c*).

10. Particular Case of Constant Burning Surfaces

When all the component charges are tubular ($\theta_1 = \theta_2 = \ldots\ldots = \theta_n = 0$), the form-function for the rth stage, from (16) and (18) is

$$z = A_r + B_r(1-f) \quad \ldots \quad (120)$$

where

$$A_r = \sum_{i=1}^{r-1} \lambda_i, \quad B_r = \sum_{i=r}^{n} \lambda_i k_i \quad \ldots \quad (121)$$

In this case (30) becomes

$$\frac{d}{dz}\left(\zeta^{\alpha}\frac{d\xi}{dz}\right) = \frac{M}{B_r^2}\zeta^{1-\alpha} \quad \ldots \quad (122)$$

From (121), for a single charge $B_r = 1$ and (122) reduces to

$$\frac{d}{dz}\left(\zeta^{\alpha}\frac{d\xi}{dz}\right) = M\zeta^{1-\alpha} \quad \ldots \quad (123)$$

From (122) and (123), we see that the differential equation for the rth stage for the composite charge is obtained from that of a single charge by replacing M by $\frac{M}{B_r^2}$ and therefore proceeding as in Clemmow (1951, page 119), the fundamental differential equation is

$$\frac{2XZ}{1+m}\frac{d^2X}{dZ^2} + \frac{2m(\gamma-1)Z}{(1+m)(\gamma-m)}\left(\frac{dX}{dZ}\right)^2 + X\frac{dX}{dZ} = 1 \quad \ldots \quad (124)$$

with

$$X = \xi^{\frac{\gamma-2m}{2m}}, \quad Z = \frac{m(\gamma-m)}{1+m}\frac{M}{B_r^2}(\zeta\xi^{\gamma})^{\frac{1}{m}}, \quad m = \frac{1}{3-2\alpha} \quad \ldots \quad (125)$$

If $k_1, k_2, \ldots, k_n = 1$, i.e. for the case of simultaneous burning out of all component charges, we see from (121) that

$$B_1 = \sum_{i=1}^{n} \lambda_i = 1$$

and the fundamental differential equation is the same as that for a single charge with form-factor zero.

For non-simultaneous burning out, making use of (29), (36), (120), (125) we get for the rth stage

$$\frac{dX}{dZ} = \frac{dX}{d\xi}\frac{d\xi}{df}\frac{df}{dz}\frac{d(z)}{d(\zeta\xi^{\gamma})}\frac{d(\zeta\xi^{\gamma})}{d(Z)}$$

$$= \frac{\gamma-m}{2m}\xi^{\frac{\gamma-m}{2m}-1}\left(-\frac{\eta}{\zeta^{\alpha}}\right)\left(-\frac{1}{B_r}\right)\xi^{1-\gamma}\frac{1+m}{\gamma-m}\frac{B_r^2}{M}(\zeta\xi^{\gamma})^{\frac{m-1}{m}}$$

$$= \frac{1+m}{2m}\frac{B_r}{M}\eta\xi^{-\frac{\gamma+m}{2m}}\zeta^{-2+\alpha} \quad \ldots \quad (126)$$

From (125) and (126), we see that in crossing from $(r-1)$th stage to the rth, while X remains continuous, Z and $\frac{dX}{dZ}$ do not, due to the presence of the constant

B_r in their expressions. In fact X, $B_r^2 Z$ and $\frac{1}{B_r}\frac{dX}{dZ}$ remain continuous in this process. Accordingly if X_{r-1}, Z_{r-1}, $\left(\frac{dX}{dZ}\right)_{r-1}$ are the values at the end of the $(r-1)$th stage, the initial conditions for the rth stage are

$$X = X_{r-1} \quad \cdots \quad (127a)$$

$$Z = \frac{B_{r-1}^2}{B_r^2} Z_{r-1} \quad \cdots \quad (127b)$$

$$\frac{dX}{dZ} = \frac{B_r}{B_{r-1}}\left(\frac{dX}{dZ}\right)_{r-1} \quad \cdots \quad (127c)$$

Initial conditions, for the first stage, obviously are

$$X = 1, \quad Z = \frac{m(\gamma-m)}{1+m}\frac{M}{B_1^2}\zeta_0^{\frac{1}{m}}, \quad \frac{dX}{dZ} = 0, \quad \cdots \quad (128)$$

assuming that the shot start moving in the first stage. Subject to initial conditions (127) and (128), equation (124) can be integrated numerically, and then maximum pressure, all-burnt position, and muzzle velocity can be found as for a single charge.

Alternatively, for the special case of tubular component charges (46) reduces to

$$\frac{\bar{Y}'''}{\bar{Y}''}+(m-2)\frac{\bar{Y}''}{\bar{Y}'}+\alpha\frac{\bar{Y}'}{\bar{Y}} = \frac{(\gamma-1)M}{B_r^2}\frac{(\bar{Y}')^{2-2m}}{\bar{Y}''(\bar{Y})^{2\alpha-1}} \quad \cdots \quad (129)$$

which reduces to the corresponding Clemmow's equation for a single charge when $B_r = 1$. This equation does not contain the independent variable explicitly and can, therefore, be integrated as a second order equation with $\bar{Y}'$ as dependent and $\bar{Y}$ as independent variable.

For the particular case $n = 2$, in the light of our discussion above for general θ_1, θ_2; the following remarks supplement Patni's treatment of this special case:

(i) The last condition $[Y'' = (Y'')_{2,1}]$ of his conditions (96) giving the initial conditions for the second stage should be modified to take into account the discontinuity in Y'', which is definitely present as $\theta_1 \neq 1$. This condition should, therefore, be replaced by using (51)

$$Y'' = \frac{B_1}{B_2}(Y'')_{2,1} \quad \cdots \quad (130)$$

(ii) The two conditions given in his (100) are not independent. If $\beta_1' > \beta_2'$, the second implies the first; if $\beta_1' < \beta_2'$, the two are identical (since in this case $f_{10} = f_{20}$, $z_{10} = z_{20}$, $\mu^* = 0$) and if $\beta_1' < \beta_2'$ the first would imply the second. These conclusions can be verified directly also.

(iii) From (105), we see that his conditions (101) can be modified to

$$\frac{\beta_2'}{\beta_1'}+\frac{F_1C_1}{F_2C_2} < (Z)_1 < 1+\frac{F_1C_1}{F_2C_2} \quad \cdots \quad (131)$$

(iv) In his (102), since $\beta_1' = \beta_2'$, $\mu^* = 0$, (102) simplifies to

$$(Z)_1 = 1+\frac{F_1C_1}{F_2C_2}$$

It is not necessary, however, for

$$\frac{\bar{\bar{Y}}'}{\bar{\bar{Y}}} - \frac{\gamma}{\gamma-1}\frac{\bar{\bar{Y}}''}{\bar{\bar{Y}}'}$$

to vanish at all-burnt. It should simply change sign from positive to negative.

SUMMARY

In this paper the equivalent charge method has been used for discussing the Internal Ballistics of composite charges consisting of n component charges with the same pressure index. In the particular case $n = 2$, the results have been compared with Patni (1955) and his (i) conditions for simultaneous and non-simultaneous burning out, (ii) initial conditions for the first and second stages, (iii) conditions for a pressure maximum to occur in the first or second stage, have been improved. It has also been shown here that a pressure maximum can occur at the end of a stage or at all-burnt. The uniqueness of maximum pressure has also been examined. In the last but one section, possible generalisations to the case when $\gamma_1, \gamma_2, \ldots . \gamma_n$ are not necessarily equal and to the case when component charges can have general form-function have been discussed.

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