PROPERTIES OF THE INTRINSIC DERIVATIVES OF THE FIRST AND HIGHER ORDERS OF THE UNIT NORMAL VECTOR FOR A CURVE IN A RIEMANNIAN SPACE

by (MISS) H. S. NAGARATHNAMMA, *Central College, Bangalore*

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Let V_n be a hypersurface of a Riemannian space V_{n+1}. Let $(x^1, x^2, \ldots x^n)$ denote the co-ordinates of any point of V_n and let $ds^2 = g_{ij}dx^i dx^j$ $(i, j = 1, 2, \ldots, n)$ be the corresponding metric. Similarly let $ds^2 = a_{\alpha\beta}dy^\alpha dy^\beta$ $(\alpha, \beta = 1, 2, \ldots, n+1)$ for V_{n+1}, the co-ordinates of any point being denoted by $(y^1, y^2, \ldots, y^{n+1})$.

At any point of V_n, $y^\alpha = y^\alpha(x^1, x^2, \ldots, x^n)$, $\alpha = 1, 2, \ldots, n+1$, and $g_{ij} = a_{\alpha\beta}y^\alpha_{;i}y^\beta_{;j}$, where the semicolon denotes tensor derivative with respect to the x's.

NOTATIONS AND DEFINITIONS

1. $C =$ a curve in V_n.
2. If $\mathbf{u}$ and $\mathbf{v}$ are two unit vectors, $\mathbf{u} \cdot \mathbf{v} = g_{ij}u^i v^j = g^{ij}u_i v_j$.
3. $e^i =$ components of the unit tangent vector at any point of C in V_n.

$E^\alpha = y^\alpha_{;i}e^i =$ components of the unit tangent vector to C considered as a curve in V_{n+1}.

4. $N^\alpha =$ components of the unit normal vector at any point of V_n, lying in V_{n+1}.

$$\therefore \quad a_{\alpha\beta}N^\alpha N^\beta = 1, \; a_{\alpha\beta}y^\alpha_{;i}N^\beta = 0, \; \alpha, \beta = 1, 2, \ldots, n+1, \; i = 1, 2, \ldots, n.$$

5. $\mathbf{q}^\alpha_1 = N^\alpha_{;i}e^i =$ components of the intrinsic derivative of the unit normal w.r.t. the curve C in V_n. The intrinsic derivative of $\mathbf{q}_1$ is called the *second intrinsic derivative* of the unit normal. In general, $\mathbf{q}^\alpha_r = \mathbf{q}^\alpha_{r-1;i}e^i$ will be called the components of the rth *intrinsic derivative* of the unit normal w.r.t. the curve, $r = 2, 3, \ldots$
6. $\Omega_{ij} =$ the second fundamental tensor of V_n defined by the relations $y^\alpha_{;ij} = \Omega_{ij}N^\alpha$.
7. $\dfrac{1}{R} = \Omega_{ij}e^i e^j =$ the normal curvature at any point of C in V_n.
8. (i) $k_1 = \dfrac{1}{\rho}, k_2, k_3, \ldots, k_n$, the n curvatures of C in V_{n+1}, defined by the Serret-Frenet formulae.

The second curvature k_2 is called the *torsion* of C.

(ii) $\kappa_1 = \dfrac{1}{\rho_g}, \kappa_2, \kappa_3, \ldots, \kappa_{n-1}$, the $(n-1)$ curvatures of C in V_n, defined by the Serret-Frenet formulae.

9. $s =$ arc of C, $\frac{\delta}{\delta s} =$ intrinsic derivative operator along C.

10. $G =$ geodesic which is tangent at a point P on C in V_n.

$S =$ arc of G, $\frac{\delta}{\delta S} =$ intrinsic derivative operator along G.

11. $\tau_1 = \frac{1}{R}, \tau_2, \ldots, \tau_n =$ the n curvatures of G in V_{n+1}. τ_2 is called the *geodesic torsion* of C at P.

12. $\theta^\alpha_1, \theta^\alpha_2, \ldots, \theta^\alpha_n$ are the components of the principal normal unit vector, the first binormal unit vector, the second binormal unit vector,... to C in V_{n+1} at the point P.

13. $\mathbf{N}^\alpha_1 = N^\alpha, \mathbf{N}^\alpha_2, \mathbf{N}^\alpha_3, \ldots,$ are the components of the principal normal unit vector, the first binormal unit vector, the second binormal unit vector,... to the geodesic tangent G in V_{n+1}, at the point P.

14. $\mathbf{p}^i_1, \mathbf{p}^i_2, \ldots$ are the components of the principal normal unit vector, the first binormal unit vector,... to C considered as a curve in V_n.

The Serret-Frenet formulae are

(i) at a point of G in V_{n+1},

$$\frac{\delta \mathbf{N}^\alpha_r}{\delta S} = \mathbf{N}^\alpha_{r+1}\tau_{r+1} - \mathbf{N}^\alpha_{r-1}\tau_r, \quad r = 0, 1, 2, \ldots, n.$$

and

$$\mathbf{N}^\alpha_0 = E^\alpha, \quad \tau_0 = 0, \quad \tau_{n+1} = 0.$$

(ii) at a point of C in V_{n+1},

$$\frac{\delta \theta^\alpha_r}{\delta s} = \theta^\alpha_{r+1}k_{r+1} - \theta^\alpha_{r-1}k_r, \quad \text{for } r = 0, 1, 2, \ldots n, .$$

$$\theta^\alpha_0 = E^\alpha, \quad k_0 = 0, \quad k_{n+1} = 0.$$

(iii) at a point of C in V_n,

$$\frac{\delta \mathbf{p}^i_r}{\delta s} = \mathbf{p}^i_{r+1}\kappa_{r+1} - \mathbf{p}^i_{r-1}\kappa_r, \quad r = 0, 1, 2, \ldots, n-1.$$

$$\mathbf{p}^i_0 = e^i, \quad \kappa_0 = 0, \quad \kappa_n = 0.$$

THEOREM 1. *A necessary and sufficient condition that a curve C in V_n has constant torsion in V_{n+1} is that the second intrinsic derivative of the unit principal normal vector to C in V_{n+1} should be orthogonal to the first binormal unit vector.*

Proof: At a point P of C in V_{n+1}, we have,

$$\frac{\delta \theta^\alpha_1}{\delta s} = k_2\theta^\alpha_2 - k_1 E^\alpha.$$

Differentiating intrinsically,

$$\frac{\delta}{\delta s}\left(\frac{\delta \theta^\alpha_1}{\delta s}\right) = \frac{\delta k_2}{\delta s}\theta^\alpha_2 + k_2\left(\frac{\delta \theta^\alpha_2}{\delta s}\right) - \frac{\delta k_1}{\delta s}E^\alpha - k_1\frac{\delta E^\alpha}{\delta s}$$

$$= \frac{\delta k_2}{\delta s}\theta^\alpha_2 + k_2(k_3\theta^\alpha_3 - k_2\theta^\alpha_1) - \frac{\delta k_1}{\delta s}E^\alpha - k_1\frac{\theta^\alpha_1}{\rho} \qquad (1.1)$$

$$\therefore \qquad a_{\alpha\beta}\theta_2^\beta \frac{\delta}{\delta s}\left(\frac{\delta\theta_1^\alpha}{\delta s}\right) = a_{\alpha\beta}\theta_2^\alpha\theta_2^\beta \frac{\delta k_2}{\delta s}$$

i.e.
$$\theta_2 \cdot \frac{\delta}{\delta s}\left(\frac{\delta\theta_1}{\delta s}\right) = \frac{\delta k_2}{\delta s} \tag{1.2}$$

The left side of (1.2) vanishes if and only if the right side vanishes.

i.e. θ_2 is orthogonal to $\frac{\delta}{\delta s}\left(\frac{\delta\theta_1}{\delta s}\right)$ if and only if k_2 is constant.

COR. 1. *A necessary and sufficient condition that a curve C in V_n has constant curvature in V_{n+1} is that the unit tangent vector to C in V_{n+1} is orthogonal to the second intrinsic derivative of the unit principal normal vector to C in V_{n+1}.*

Proof: Taking scalar products with E^α on either side of equation (1.1), we have

$$E \cdot \frac{\delta}{\delta s}\left(\frac{\delta\theta_1}{\delta s}\right) = -\frac{\delta k_1}{\delta s}.$$

The L.H.S. $= 0$ if and only if the R.H.S. $= 0$.
Hence the result.

THEOREM 2. *For an asymptotic line $\tau_1 = 0$; for a line of curvature $\tau_2 = 0$, and conversely.*

The proof is straightforward, by using the relation between the tangent to the curve and the intrinsic derivative of the unit normal.

THEOREM 3. *A necessary and sufficient condition for a curve in a Riemannian space V_n of a V_{n+1} to be of constant normal curvature is that*

$$\mathbf{q}_2 \cdot E = -\frac{(\mathbf{q}_1 \cdot \mathbf{p}_1)}{\rho_g} = -\frac{(\mathbf{q}_1 \cdot \theta_1)}{\rho}$$

Proof: Let C be a curve of constant normal curvature.
Using the well-known formula (Weatherburn, 1950, Art. 75)

$$\mathbf{q}_1^\alpha = N^\alpha_{;i} e^i = -g^{l\kappa}\Omega_{\kappa i} y^\alpha_{;l} e^i, \tag{3.1}$$

$$a_{\alpha\beta}\mathbf{q}_1^\alpha y^\beta_{;j} e^j = -\Omega_{ij}e^i e^j = -\frac{1}{R} = \text{constant} \tag{3.2}$$

Differentiating (3.2) intrinsically w.r.t. C, we have

$$a_{\alpha\beta}\mathbf{q}_2^\alpha y^\beta_{;j} e^j + a_{\alpha\beta}\mathbf{q}_1^\alpha N^\beta \Omega_{ij} e^i e^j + a_{\alpha\beta}\mathbf{q}_1^\alpha y^\beta_{;j} e^j_{,\kappa} e^\kappa = 0. \tag{3.3}$$

Observing that the unit vector $\mathbf{N}$ is orthogonal to $\mathbf{q}_1$ we conclude that

$$a_{\alpha\beta}\mathbf{q}_2^\alpha y^\beta_{;j} e^j = -a_{\alpha\beta}\mathbf{q}_1^\alpha y^\beta_{;j} e^j_{,\kappa} e^\kappa \tag{3.4}$$

i.e.
$$\mathbf{q}_2 \cdot E = -\frac{(\mathbf{q}_1 \cdot \mathbf{p}_1)}{\rho_g}.$$

Conversely, when (3.4) holds,

$$\left(a_{\alpha\beta}\mathbf{q}_1^\alpha y^\beta_{;j} e^j\right)_{;i} e^i = 0$$

i.e.
$$\left(\frac{1}{R}\right)_{;i} e^i = 0$$

Therefore, $\frac{1}{R}=$ constant.

This proves the theorem considering C as a curve in V_n.

But using the well-known formula (Weatherburn, 1950, Art. 71)

$$\frac{\theta_1^\alpha}{\rho}=\frac{N^\alpha}{R}+\frac{y^\alpha_{;i}\mathbf{p}_1^i}{\rho_g},$$

and noting that $\mathbf{q}_1^\alpha$ is a vector orthogonal to N^α, it is evident that

$$\frac{(\mathbf{p}_1\,.\,\mathbf{q}_1)}{\rho_g}=\frac{(\theta_1\,.\,\mathbf{q}_1)}{\rho}.$$

We observe that for a curve of constant normal curvature,

$$\begin{aligned}(\mathbf{q}_2 \cdot E) &= -\frac{1}{\rho_g}(\mathbf{p}_1.\mathbf{q}_1)\\ &= -\frac{1}{\rho_g}\left(a_{\alpha\beta}\, y^\alpha_{;i}\mathbf{p}_1^i N^\beta_{;j}e^j\right)\\ &= -\frac{1}{\rho_g}\left(a_{\alpha\beta}\, y^\alpha_{;i}N^\beta_{;j}e^i\,\mathbf{p}_1^j\right)\\ &= -\frac{1}{\rho_g}\Omega_{ij}e^i\mathbf{p}_1^j \qquad (3.5)\end{aligned}$$

COR. 1. *A geodesic of* V_n *is of constant normal curvature if and only if the tangent vector to the curve is orthogonal to the second intrinsic derivative of the unit normal vector.*

This can also be obtained as a Cor. to Theorem 1. The result holds good for a geodesic of V_{n+1}, in particular.

COR. 2. *If the normal curvature of a line of curvature of* V_n *is constant, then the tangent vector to the curve in* V_{n+1} *is orthogonal to the second intrinsic derivative of the unit normal vector.*

Proof: For a line of curvature,

$$(\Omega_{ij}-\lambda g_{ij})e^i=0$$

i.e.

$$\Omega_{ij}e^i=\lambda g_{ij}e^i$$

$$\therefore\quad \Omega_{ij}e^i\mathbf{p}_1^j=\lambda g_{ij}e^i\mathbf{p}_1^j=0$$

$$\therefore\quad \text{by (3.5)},\ (E.\mathbf{q}_2)=0.$$

THEOREM 4. *At every point of* C *in a hypersurface* V_n *of a Riemannian space* V_{n+1}, *the second intrinsic derivative w.r.t. the curve of the unit normal vector to* V_n *is orthogonal to the normal vector if and only if* C *is an asymptotic line as well as a line of curvature.*

Proof:

$$a_{\alpha\beta}\mathbf{q}_1^\alpha N^\beta=0, \text{ since } N \text{ is a unit vector.}$$

Differentiating intrinsically,

$$a_{\alpha\beta}\mathbf{q}_2^\alpha N^\beta=-a_{\alpha\beta}\mathbf{q}_1^\alpha\mathbf{q}_1^\beta=-g^{pq}\Omega_{pi}\Omega_{qj}e^ie^j, \text{ from (3.1).}$$

Also,

$$\mathbf{q}_1^\alpha = \frac{\delta N^\alpha}{\delta S} = \tau_2 N_2^\alpha - \tau_1 E^\alpha \text{ by the Serret-Frenet formulae.}$$

$$\therefore \quad -a_{\alpha\beta}\mathbf{q}_1^\alpha \mathbf{q}_1^\beta = -(\tau_1^2+\tau_2^2)$$

$\therefore$ $\mathbf{q}_2$ is orthogonal to N if and only if $\mathbf{q}_1$ is a null vector.

i.e. if and only if $\tau_1 = 0$, and $\tau_2 = 0$, i.e. by Theorem 2, if and only if C is an asymptotic line as well as a line of curvature.

THEOREM 5. *If $\mathbf{q}_3$ is orthogonal to $\mathbf{N}$, then $\mathbf{q}_2$ is orthogonal to $\mathbf{q}_1$ and conversely.*

For

$$a_{\alpha\beta}\mathbf{q}_1^\alpha N^\beta = 0 \tag{5.1}$$

$$\therefore \quad a_{\alpha\beta}\mathbf{q}_2^\alpha N^\beta + a_{\alpha\beta}\mathbf{q}_1^\alpha \mathbf{q}_1^\beta = 0 \tag{5.2}$$

$$\therefore \quad a_{\alpha\beta}\mathbf{q}_3^\alpha N^\beta + 2a_{\alpha\beta}\mathbf{q}_1^\alpha \mathbf{q}_1^\beta = 0 \tag{5.3}$$

The vanishing of one term in (5.3) implies the vanishing of the other. Hence the result.

Differentiate (5.3) intrinsically w.r.t. the curve,

$$a_{\alpha\beta}\mathbf{q}_4^\alpha N^\beta + 4a_{\alpha\beta}\mathbf{q}_3^\alpha \mathbf{q}_1^\beta + 3a_{\alpha\beta}\mathbf{q}_2^\alpha \mathbf{q}_2^\beta = 0 \tag{5.4}$$

Any two of the properties

(i) $\mathbf{q}_4$ orthogonal to N
(ii) $\mathbf{q}_3$ orthogonal to $\mathbf{q}_1$
(iii) $\mathbf{q}_2$ is a null vector

implies the other.

Differentiate (5.4) intrinsically,

$$a_{\alpha\beta}\mathbf{q}_5^\alpha N^\beta + a_{\alpha\beta}\mathbf{q}_4^\alpha \mathbf{q}_1^\beta + 4a_{\alpha\beta}\mathbf{q}_4^\alpha \mathbf{q}_1^\beta + 4a_{\alpha\beta}\mathbf{q}_3^\alpha \mathbf{q}_2^\beta + 6a_{\alpha\beta}\mathbf{q}_2^\alpha \mathbf{q}_3^\beta = 0$$

i.e.

$$10a_{\alpha\beta}\mathbf{q}_2^\alpha \mathbf{q}_3^\beta + 5a_{\alpha\beta}\mathbf{q}_1^\alpha \mathbf{q}_4^\beta + a_{\alpha\beta}N^\alpha \mathbf{q}_5^\beta = 0 \tag{5.5}$$

$\therefore$ The orthogonality of any two of the pairs ($\mathbf{q}_2$, $\mathbf{q}_3$); ($\mathbf{q}_1$, $\mathbf{q}_4$); (N, $\mathbf{q}_5$) implies that of the other.

THEOREM 6. *The sum of the squares of magnitudes of the intrinsic derivative of the unit normal vector at a given point w.r.t. any orthogonal ennuple of directions at the point is an invariant.*

Proof: Square of the magnitude of $\mathbf{q}_1 = a_{\alpha\beta}\mathbf{q}_1^\alpha \mathbf{q}_1^\beta$

$$= a_{\alpha\beta}N^\alpha_{;i}e^i N^\beta_{;j}e^j$$

$$= a_{\alpha\beta}\left(\tau_2\mathbf{N}_2^\alpha - \tau_1 E^\alpha\right)\left(\tau_2\mathbf{N}_2^\beta - \tau_1 E^\beta\right) = \tau_2^2+\tau_1^2$$

Also,

$$\mathbf{N}^{\alpha}_{;i}e^i = -g^{l\kappa}\Omega_{\kappa i}y^{\alpha}_{;l}e^i$$

$$\therefore \quad a_{\alpha\beta}N^{\alpha}_{;i}e^iN^{\beta}_{;j}e^j = a_{\alpha\beta}\left(-g^{l\kappa}\Omega_{\kappa i}y^{\alpha}_{;l}e^i\right)\left(-g^{m\sigma}\Omega_{\sigma j}y^{\beta}_{;m}e^j\right)$$

$$= g^{lm}\Omega_{mj}\Omega_{li}e^ie^j$$

Thus the square of the magnitude of the intrinsic derivative of N^{α} w.r.t. $e^i_{h|}$, where $e^i_{h|}$, $(h = 1, 2, \ldots, n)$ define an orthogonal ennuple at the point considered, is equal to

$$g^{lm}\Omega_{li}\Omega_{mj}e^i_{h|}e^j_{h|} = \tau^2_{1/h}+\tau^2_{2/h}$$

$$\therefore \quad \sum_h \left(\tau^2_{1/h}+\tau^2_{2/h}\right) = \sum_h g^{lm}\Omega_{li}\Omega_{mj}e^i_{h|}e^j_{h|} = g^{lm}\Omega_{li}\Omega_{mj}g^{ij} \tag{6.1}$$

Expressed in words, (6.1) means

The sum of the squares of the normal curvature and the geodesic torsion of n mutually orthogonal curves at any point of V_n *is an invariant.*

Hence in particular,

The sum of the squares of normal curvatures in the principal directions at any point of $V_n = g^{lm}\Omega_{li}\Omega_{mj}g^{ij}$.

THEOREM 7. *If C is a geodesic as well as a line of curvature of* V_n, *the torsion of C is zero and conversely a geodesic of zero torsion is a line of curvature.*

Since C is a geodesic, $\theta^{\alpha}_1 = N^{\alpha}$ and $\tau_1 = k_1$

$$\therefore \quad \theta^{\alpha}_{1;i}e^i = N^{\alpha}_{,i}e^i$$

i.e.

$$k_2\theta^{\alpha}_2 - k_1E^{\alpha} = N^{\alpha}_{;i}e^i = -\tau_1E^{\alpha}, \tag{7.1}$$

using Theorem 2, and the Serret-Frenet formula for $N^{\alpha}_{;i}e^i$.

$$\therefore \quad k_2 = 0.$$

Conversely, if $k_2 = 0$ and $\theta^{\alpha}_1 = N^{\alpha}$, we have $\tau_2 = 0$

$\therefore$ By Theorem 2, C is a line of curvature.

In ordinary three-dimensional geometry, this theorem means that if C is a geodesic as well as a line of curvature, then it is a plane curve, conversely a geodesic of zero torsion is a line of curvature.

THEOREM 8. *If C is an asymptotic line of* V_n, *the ratio of the torsion of C in* V_n *to that of C in* V_{n+1} *is numerically equal to the ratio of the cosine of the angle between the principal normal to C in* V_n *and the first binormal to C in* V_{n+1}, *to the cosine of the angle between the principal normal to C in* V_{n+1} *and the first binormal to C in* V_n.

Proof: It is well known (Weatherburn, 1950, Art. 71) that

$$\frac{\theta^{\alpha}_1}{\rho} = \frac{N^{\alpha}}{R} + \frac{y^{\alpha}_{;i}\mathbf{p}^i_1}{\rho_g} \tag{8.1}$$

For an asymptotic line,

$$\frac{1}{\rho}=\frac{1}{\rho_g}, \quad a_{\alpha\beta}\theta_1^{\alpha}y^{\beta}_{;i}\mathbf{p}_1^i=1,$$

and

$$a_{\alpha\beta}\theta_1^{\alpha}N^{\beta}=0 \tag{8.2}$$

$$\therefore \quad a_{\alpha\beta}\frac{\delta\theta_1^{\alpha}}{\delta s}y^{\beta}_{;i}\mathbf{p}_1^i+a_{\alpha\beta}\theta_1^{\alpha}y^{\beta}_{;ij}e^i\mathbf{p}_1^j+a_{\alpha\beta}\theta_1^{\alpha}y^{\beta}_{;i}\frac{\delta\mathbf{p}_1^i}{\delta s}=0$$

i.e.

$$a_{\alpha\beta}[k_2\theta_2^{\alpha}-k_1E^{\alpha}]y^{\beta}_{;i}\mathbf{p}_1^i+a_{\alpha\beta}\theta_1^{\alpha}N^{\beta}(\Omega_{ij}e^i\mathbf{p}_1^j) + a_{\alpha\beta}\theta_1^{\alpha}y^{\beta}_{;i}[\kappa_2\mathbf{p}_2^i-\kappa_1e^i] \tag{8.3}$$

where κ_1, κ_2 are the first and second curvatures of C in V_n.

In virtue of (8.2), and the relations $a_{\alpha\beta}\theta_1^{\alpha}y^{\beta}_{;i}e^i=0$ and $a_{\alpha\beta}y^{\alpha}_{;i}\mathbf{p}_1^iE^{\beta}=0$, (8.3) becomes

$$k_2\left(a_{\alpha\beta}\theta_2^{\alpha}y^{\beta}_{;i}\mathbf{p}_1^i\right)+\kappa_2\left(a_{\alpha\beta}y^{\alpha}_{;i}\mathbf{p}_2^i\theta_1^{\beta}\right)=0$$

i.e.

$$\frac{\kappa_2}{k_2}=-\frac{(\theta_2\,.\,\mathbf{p}_1)}{(\theta_1\,.\,\mathbf{p}_2)},$$

which is the result of the theorem.

THEOREM 9. *For any two conjugate directions, the quotients of the normal curvature by the torsion of the geodesic tangent are in the ratio of the cosine of the angle between the tangent to the first direction and the first binormal to the geodesic tangent to the second, to the cosine of the angle between the tangent to the second direction and the first binormal to the geodesic tangent to the first.*

Proof: For conjugate lines we have the known property that the intrinsic derivative of the unit normal vector w.r.t. one of the directions is orthogonal to the other (Weatherburn, 1950, Art. 75).

So that

$$a_{\alpha\beta}\frac{\delta N^{\alpha}}{\delta s}\epsilon^{\beta}=0, \quad a_{\alpha\beta}\frac{\delta N^{\alpha}}{\delta\bar{s}}E^{\beta}=0 \tag{9.1}$$

where ϵ^{α} and E^{α} are the components of the tangent vectors in V_{n+1} to the two directions, s, $\bar{s}$, the corresponding arcs, $\mathbf{N}_2$, $\mathcal{N}_2$ the components of the first binormals to the geodesic tangents.

(9.1) gives

$$\left.\begin{aligned} a_{\alpha\beta}(\tau_2\mathbf{N}_2^{\alpha}-\tau_1E^{\alpha})\epsilon^{\beta}=0 \\ a_{\alpha\beta}(\mathcal{T}_2\mathcal{N}_2^{\alpha}-\mathcal{T}_1\epsilon^{\alpha})E^{\beta}=0 \end{aligned}\right\} \tag{9.2}$$

where $\tau_2=$ geodesic torsion, $\tau_1=$ normal curvature in the direction E^{α}, $\mathcal{T}_2=$geodesic torsion, $\mathcal{T}_1=$ normal curvature in the direction ϵ^{α}.

Hence, we get

$$\frac{\dfrac{\tau_2}{\tau_1}}{\dfrac{\mathcal{T}_2}{\mathcal{T}}}=\frac{\mathcal{N}_2\,.\,E}{\mathbf{N}_2\,.\,\epsilon}.$$

Hence the result.

THEOREM 10. *A curve for which the rth principal normal vector at any point is codirectional with the unit normal vector, is an asymptotic line, $r \geq 2$.*

$$\theta^{\alpha}_{r} = N^{\alpha}, \quad r \geqslant 2 \tag{10.1}$$

$$\therefore \quad \theta^{\alpha}_{r;i} e^{i} = N^{\alpha}_{;i} e^{i}$$

$$k_{r+1}\theta^{\alpha}_{r+1} - k_{r}\theta^{\alpha}_{r-1} = \tau_{2}\mathbf{N}^{\alpha}_{2} - \tau_{1}E^{\alpha} \tag{10.2}$$

Taking scalar products with E^{α} on either side of (10.2), $\tau_1 = 0$ and hence by Theorem 2, C is an asymptotic line.

Also,

$$k^{2}_{r+1} + k^{2}_{r} = \tau^{2}_{2} \tag{10.3}$$

In ordinary geometry for $r = 2$, this theorem gives the familiar definition of an asymptotic line that the binormal to the curve is the normal to the surface and $\tau_2 = k_2$.

For a general V_n, the converse theorem does not however hold for any asymptotic line.

From (10.2) we get

$$\left.\begin{aligned} (\mathbf{N}_{2} \,.\, \theta_{r-1}) &= -\frac{k_{r}}{\tau_{2}} \\ (\mathbf{N}_{2} \,.\, \theta_{r}) &= 0 \\ (\mathbf{N}_{2} \,.\, \theta_{r+1}) &= \frac{k_{r+1}}{\tau_{2}} \end{aligned}\right\} \tag{10.4}$$

From (10.3) and (10.4) we get

$$(\mathbf{N}_{2} \,.\, \theta_{r-1})^{2} + (\mathbf{N}_{2} \,.\, \theta_{r+1})^{2} = 1 \tag{10.5}$$

i.e. if $\mathbf{N}^{\alpha}_{2}$ makes an angle ϕ with θ^{α}_{r-1}, then it makes an angle $\frac{\pi}{2} - \phi$ with θ_{r+1}.

For $r = 2$, $\mathbf{N}_2 \,.\, \theta_r = 0$ gives the property that the binormal of C is orthogonal to the binormal of G.

THEOREM 11. *A curve for which*

$$\theta^{\alpha}_{r} = \lambda \frac{\delta N^{\alpha}}{\delta s}, \quad r \geqslant 1$$

is an asymptotic line of V_n.

We have

$$\theta^{\alpha}_{r} = \lambda \frac{\delta N^{\alpha}}{\delta s} = \lambda\left(\tau_{2}\mathbf{N}^{\alpha}_{2} - \tau_{1}E^{\alpha}\right).$$

Taking scalar products w.r.t. E, we get $\tau_1 = 0$, since E is orthogonal to N_2 and to θ_r.

Hence C is an asymptotic line.

Also,

$$\theta^{\alpha}_{r} = N^{\alpha}_{2} \text{ and } \lambda\tau_{2} = 1 \tag{11.1}$$

From (11.1) we have $\theta^{\alpha}_{r;i} e^{i} = \mathbf{N}^{\alpha}_{2;i} e^{i}$

i.e.

$$k_{r+1}\theta^{\alpha}_{r+1} - k_{r}\theta^{\alpha}_{r-1} = \tau_{3}\mathbf{N}^{\alpha}_{3} - \tau_{2}N^{\alpha} \tag{11.2}$$

$$\therefore \quad k^{2}_{r+1} + k^{2}_{r} = \tau^{2}_{3} + \tau^{2}_{2} \tag{11.3}$$

Taking scalar products with θ_p, $p \neq (r+1)$ or $(r-1)$, on either side of (11.2) we get

$$\frac{\tau_3}{\tau_2} = -\frac{(N \cdot \theta_p)}{(\mathbf{N}_3 \cdot \theta_p)} \tag{11.4}$$

Similarly taking scalar products with N_p, $p \neq 3$ or 1, on either side of (11.2) we get

$$\frac{k_{r+1}}{k_r} = -\frac{(\mathbf{N}_p \cdot \theta_{r-1})}{(\mathbf{N}_p \cdot \theta_{r+1})} \tag{11.5}$$

For $r = 2$, in ordinary geometry, (11.1) means $\theta_2^\alpha = \mathbf{N}_2^\alpha$ and from (11.2) and (11.3), since $\tau_3 = 0$, $k_3 = 0$, it follows that $\theta_1^\alpha = N^\alpha$ i.e. C is a geodesic of S_3, i.e. a straight line.

For

$$r = 1,\ k_2^2 + k_1^2 = \tau_2^2.$$

But since C is an asymptotic line, the torsion at any point is equal to the geodesic torsion there (Weatherburn, 1931).

$\therefore\ k_1 = 0$, i.e. C is a straight line.

THEOREM 12. *Properties of a curve for which* $\theta_r^\alpha = \mathbf{N}_s^\alpha$, $r, s \neq 1$.

Since

$$\theta_r^\alpha = \mathbf{N}_s^\alpha \tag{12.1}$$

$$\theta^\alpha_{r;\,i} e^i = \mathbf{N}^\alpha_{s;\,i} e^i$$

i.e.

$$k_{r+1}\theta^\alpha_{r+1} - k_r\theta^\alpha_{r-1} = \tau_{s+1}\mathbf{N}^\alpha_{s+1} - \tau_s \mathbf{N}^\alpha_{s-1} \tag{12.2}$$

$$\therefore\ k^2_{r+1} + k_r^2 = \tau^2_{s+1} + \tau_s^2 \tag{12.3}$$

Taking scalar products with θ_p, $p \neq (r+1)$ or $(r-1)$, on either side of (12.2),

$$\frac{\tau_{s+1}}{\tau_s} = -\frac{(\theta_p \cdot \mathbf{N}_{s-1})}{(\theta_p \cdot \mathbf{N}_{s+1})} \tag{12.4}$$

Taking scalar products with N_p, $p \neq s+1$, or $s-1$ *on either side of (12.2)*

$$\frac{k_{r+1}}{k_r} = -\frac{(\theta_{r-1} \cdot \mathbf{N}_p)}{(\theta_{r+1} \cdot \mathbf{N}_p)} \tag{12.5}$$

The cases $r = s = 1$, $r = 2$, $s = 1$ have already been considered.

THEOREM 13. *For a geodesic or an asymptotic line,*

$$\tau_2(\mathbf{N}_2 \cdot \theta_1) + k_2(\mathbf{N}_1 \cdot \theta_2) = 0$$

i.e.

$$\frac{\tau_2}{k_2} = -\frac{(\mathbf{N}_1 \cdot \theta_2)}{(\mathbf{N}_2 \cdot \theta_1)}$$

Proof:

$a_{\alpha\beta}\theta_1^\alpha N^\beta = 1$ or 0 according as C is a geodesic or an asymptotic line.

$$\therefore\ a_{\alpha\beta}\theta^\alpha_{1;\,i} e^i N^\beta + a_{\alpha\beta}\theta_1^\alpha N^\beta_{;\,i} e^i = 0$$

i.e.

$$a_{\alpha\beta}\left(k_2\theta_2^\alpha - k_1 E^\alpha\right) N^\beta + a_{\alpha\beta}\theta_1^\alpha\left(\tau_2 N_2^\beta - \tau_1 E^\beta\right) = 0$$

i.e.

$$\tau_2(\mathbf{N}_2 \cdot \theta_1) + k_2(\mathbf{N}_1 \cdot \theta_2) = 0.$$

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