METHOD OF TRIGONOMETRICAL SERIES IN CALCULATING THE MODIFICATION OF INTENSITY OF MONOCHROMATIC RADIATION DUE TO MULTIPLE COMPTON SCATTERING IN STELLAR ATMOSPHERE

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(Communicated by N. R. SEN, F.N.I.)

(*Received July 12 ; read October 5, 1956*)

1. The trigonometrical series method has been used by the author to solve the problem of softening of radiation (*Proc. Nat. Inst. of Sci. Ind.*, **20**, p. 530, 1954) in the first approximation. The intensity distribution at the lower boundary of the atmosphere of the star was taken to be of Gaussian type, and the calculations were carried through in the two orders of approximations, namely by retaining the first order and the second order terms in Compton wave-length in the Taylor's series expansion of scattered intensity. Change in wave-length was supposed to be due to Compton effect only. In the present paper, the same method has been extended to work out the problem of softening in a stellar atmosphere when the radiation emerging from the photospheric surface may be supposed to be monochromatic in nature. The first and the second order terms in the Taylor's series expansion of scattered intensity has been retained. The present method has been checked by obtaining the first order calculations by the method of trigonometrical series with only the first order terms of Taylor's series expansion of scattered intensity. It has been found that the results obtained are identical with those calculated by Chandrasekhar with the aid of Green's function. The second order calculations carried out by the present method substantially reduce the error which was noted by Chandrasekhar in the first order calculation.

2. The equation of transfer appropriate to the problem is given by (eq. (1), p. 530, *ibid.*)

$$\mu \frac{\partial I(\tau, \mu, \lambda)}{\partial \tau} = I(\tau, \mu, \lambda) - \frac{1}{4\pi} \int_{-1}^{+1} \int_{0}^{2\pi} I(\tau, \mu', \lambda - \gamma[1-\cos\theta])\, d\mu'\, d\phi' \quad \ldots \quad (1)$$

where γ, the Compton wave-length, is given by

$$\gamma = \frac{h}{mc} = 0{\cdot}0244^\circ \quad \ldots \quad (2)$$

τ, the optical thickness, is

$$\tau = \int_{z}^{\infty} \rho\sigma\, dz \quad \ldots \quad (3)$$

where ρ is the density and σ, the scattering coefficient, which is supposed to be independent of wave-length as in Thomson scattering. The Compton change in wave-length is given by

$$\delta\lambda = \gamma[1-\cos\theta] \quad \ldots \quad (4)$$

$I(\tau, \mu, \lambda)$ is the specific intensity of radiation of wave-length λ at an optical thickness τ and in a direction ϑ to the outward drawn normal and $\mu = \text{Cos}\,\vartheta$. θ is the angle of scattering, and

$$\cos\theta = \mu\mu' + (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos\phi' \quad \ldots \quad (5)$$

The source function represented by the right hand side of equation (1) means that a radiation of wave-length $\lambda-\gamma[1-\cos\theta]$ in the direction (μ', ϕ') when scattered in the direction $(\mu, 0)$ will have the wave-length λ.

It is supposed that $I(\tau, \mu', \lambda-\gamma[1-\cos\theta])$ can be expanded into a Taylor's series and in subsequent calculations, terms up to the second order in γ have been retained.

$$I(\tau, \mu', \lambda-\gamma[1-\cos\theta]) = I(\tau, \mu', \lambda)-\gamma(1-\cos\theta)\frac{\partial I(\tau, \mu', \lambda)}{\partial\lambda} + \frac{\gamma^2}{2}(1-\cos\theta)^2\frac{\partial^2 I(\tau, \mu', \lambda)}{\partial\lambda^2} - \quad \ldots \quad (6)$$

This value of $I(\tau, \mu', \lambda-\gamma[1-\cos\theta])$ is put in equation (1), and the resulting integro-differential equation of transfer is replaced by 2nd linear equations by the method of Gaussian quadrature. Restricting ourselves to the first approximation, we get two such equations, which when combined lead to the following partial differential equation with constant coefficients (cf. eq. (19), *ibid.*)

$$\frac{\partial^2 F}{\partial x^2}+\frac{3}{2}\frac{\partial^4 F}{\partial y^4}-\frac{5}{2}\frac{\partial^3 F}{\partial y^3}+3\frac{\partial^2 F}{\partial y^2}-2\frac{\partial F}{\partial y} = 0.. \quad \ldots \quad (7)$$

where

$$x = \frac{3}{2}\tau \text{ and } y = \frac{3}{2\gamma}(\lambda-\lambda_0) \quad \ldots \quad (8)$$

λ_0 being some suitably chosen wave-length of constant value.

$$K(x, y) = I_{+1}(x, y)+I_{-1}(x, y) = \frac{\partial F(x, y)}{\partial x},$$

$$H(x, y) = I_{+1}(x, y)-I_{-1}(x, y) = \sqrt{3}\left(\frac{\partial F(x, y)}{\partial y}-\frac{\partial^2 F(x, y)}{\partial y^2}\right) \quad \ldots \quad (9)$$

where $I_{+1}(x, y)$ and $I_{-1}(x, y)$ are the outward and inward intensities in the first approximation.

This equation (7) is to be solved under the following boundary conditions:

(1) existence of a known spectral distribution at the lower boundary denoted by $\tau = \tau_1$ or $x = x_1$. The intensity distribution in the present case is taken to be monochromatic and represented by Dirac's δ-function.

(2) absence of inward radiation at the outer boundary denoted by $\tau = 0$ or $x = 0$.

The boundary conditions are equivalent to

$$\frac{1}{2}\left[\frac{\partial F}{\partial x}+\sqrt{3}\left(\frac{\partial F}{\partial y}-\frac{\partial^2 F}{\partial y^2}\right)\right]_{x=x_1} = \psi(y) = \delta(y) \text{ in the present case} \quad \ldots \quad (10)$$

and

$$\left[\frac{\partial F}{\partial x}-\sqrt{3}\left(\frac{\partial F}{\partial y}-\frac{\partial^2 F}{\partial y^2}\right)\right] = 0 \quad \ldots \quad (11)$$

The general solution of the equation (7) is given by (cf. eq. (32), p. 534, *ibid.*)

$$F(x, y) = A_0(x^2+y)+B_0x+\sum_{n=1}^{\infty}\left[e^{\alpha_n x}\{a_n\cos(\beta_n x+ny)+b_n\sin(\beta_n x+ny)\}\right.$$

$$\left.+e^{-\alpha_n x}\{c_n\cos(\beta_n x-ny)+d_n\sin(\beta_n x-ny)\}\right] \quad .. \quad (12)$$

where

$$\alpha_n = \sqrt{\frac{3}{2}n^2\left(1-\frac{1}{2}n^2\right)+\frac{1}{2}\sqrt{\frac{9}{4}n^8-\frac{11}{4}n^6-n^4+4n^2}} \quad .. \quad .. \quad (13)$$

$$\beta_n = -\frac{1}{2}\sqrt{\frac{1}{2}\sqrt{\frac{9}{4}n^8-\frac{11}{4}n^6-n^4+4n^2}-\frac{3}{2}n^2\left(1-\frac{1}{2}n^2\right)} \quad .. \quad (14)$$

$$B_0 = \sqrt{3}A_0,\ A_0 = \frac{A_0'}{x_1+\sqrt{3}} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (15)$$

$$a_n = \frac{(\alpha_n^2+\beta_n^2-3n^2-3n^4)c_n+2\sqrt{3}n(n\beta_n-\alpha_n)\,dn}{(\beta_n-\sqrt{3}n)^2+(\alpha_n-\sqrt{3}n^2)^2} \quad .. \quad .. \quad (16)$$

$$b_n = \frac{2\sqrt{3}n(n\beta_n-\alpha_n)c_n-(\alpha_n^2+\beta_n^2-3n^2-3n^4)\,dn}{(\beta_n-\sqrt{3}n)^2+(\alpha_n-\sqrt{3}n^2)^2} \quad .. \quad .. \quad (17)$$

$$c_n = \frac{M_nA_n'+N_nB_n'}{M_n^2+N_n^2} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (18)$$

$$d_n = \frac{N_nA_n'-M_nB_n'}{M_n^2+N_n^2} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (19)$$

$$M_n = \tfrac{1}{2}e^{\alpha_n x}\left\{\frac{(\alpha_n^2+\beta_n^2-3n^2-3n^4)(\alpha_n+\sqrt{3}n^2)+2\sqrt{3}n(n\beta_n-\alpha_n)(\beta_n+\sqrt{3}n)}{(\beta_n-\sqrt{3}n)^2+(\alpha_n-\sqrt{3}n^2)^2}\cos\beta_n x_1\right.$$

$$\left.-\frac{(\alpha_n^2+\beta_n^2-3n^2-3n^4)(\beta_n+\sqrt{3}n)-2\sqrt{3}n(n\beta_n-\alpha_n)(\alpha_n+\sqrt{3}n^2)}{(\beta_n-\sqrt{3}n)^2+(\alpha_n-\sqrt{3}n^2)^2}\sin\beta_n x_1\right.$$

$$-\tfrac{1}{2}e^{-\alpha_n x_1}\{(\alpha_n-\sqrt{3}n^2)\cos\beta_n x_1+(\beta_n-\sqrt{3}n)\sin\beta_n x_1\} \quad .. \quad .. \quad (20)$$

$$N_n = \tfrac{1}{2}e^{\alpha_n x_1}\left\{\frac{2\sqrt{3}n(n\beta_n-\alpha_n)(\alpha_n+\sqrt{3}n^2)-(\alpha_n^2+\beta_n^2-3n^2-3n^4)(\beta_n+\sqrt{3}n)}{(\beta_n-\sqrt{3}n)^2+(\alpha_n-\sqrt{3}n^2)^2}\cos\beta_n x_1\right.$$

$$\left.-\frac{(\alpha_n^2+\beta_n^2-3n^2-3n^4)(\alpha_n+\sqrt{3}n^2)+2\sqrt{3}n(n\beta_n-\alpha_n)(\beta_n+\sqrt{3}n)}{(\beta_n-\sqrt{3}n)^2+(\alpha_n-\sqrt{3}n^2)^2}\sin\beta_n x_1\right.$$

$$+\tfrac{1}{2}e^{-\alpha_n x_1}\{(\beta_n-\sqrt{3}n)\cos\beta_n x_1-(\alpha_n-\sqrt{3}n^2)\sin\beta_n x_1\} \quad .. \quad .. \quad (21)$$

A_0, A_n' and B_n' are the Fourier coefficients in the expansion of in a Fourier series,

$$\psi(y) = A_0'+\sum_{n=1}^{\infty}A_n'\cos ny+\sum_{n=1}^{\infty}B_n'\sin ny \quad .. \quad .. \quad (22)$$

Thus the constants a_n, b_n, c_n, d_n, A_0 can be known for a particular value of x_1 or τ_1, if the values of the Fourier coefficients are known. The emergent intensity at the outer boundary is given by the following relation (cf. eq. (45), *ibid.*)

$$I_{+1}(0, y) = \sqrt{3}A_0 + \sum_{n=1}^{\infty} \big[\{(a_n - c_n)\alpha_n + (b_n + d_n)\beta_n\} \cos ny + \\ + \{(b_n + d_n)\alpha_n - (a_n - c_n)\beta_n\} \sin ny\big] \quad .. \quad (23)$$

3. In the case under consideration, the distribution of intensity at the photospheric surface has been considered to be monochromatic, i.e. this distribution can be represented by the δ-function. If we want to apply the method developed in the former case to this problem, we require a Fourier series expansion of the δ-function. The recent theory of distribution by Schwartz (Introduction to the theory of distribution by Israel Halperin, Canadian Mathematical Congress, University of Toronto Press, 1952) classifies δ-function not as a point function but as a distribution and the expansibility of the distribution function δ has been rigorously established by Schwartz. Below we formally obtain the expansion in the usual way.

Suppose we want to expand $\psi(y) = \delta(y)$ in the interval $-\pi$ and π. We have

$$\left.\begin{aligned} \psi(y) &= 0 \text{ for } -\pi < y \leqslant \epsilon \\ \psi(y) &= 0 \text{ for } \epsilon \leqslant y < +\pi \end{aligned}\right\} \quad .. \quad .. \quad .. \quad (24)$$

and

$$\underset{\epsilon \to 0}{\mathrm{Lt}} \int_{-\epsilon}^{\epsilon} \psi(y)\, dy = 1 \quad .. \quad .. \quad .. \quad .. \quad (25)$$

Considering this as a symmetric function, let us assume the expansion

$$\psi(y) = c_0 + \sum_{n=1}^{\infty} c_n \cos ny \quad .. \quad .. \quad .. \quad .. \quad (26)$$

Then multiplying by dy and integrating between the limits $-\pi$ and π, we have

$$\underset{\epsilon \to 0}{\mathrm{Lt}} \int_{-\epsilon}^{\epsilon} \psi(y)\, dy = 2\pi c_0 \quad .. \quad .. \quad .. \quad .. \quad (27)$$

Hence

$$c_0 = \frac{1}{2\pi} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (28)$$

Similarly multiplying (26) by cos $ny\, dy$ and integrating between the same limits, we have

$$\underset{\epsilon \to 0}{\mathrm{Lt}} \int_{-\epsilon}^{\epsilon} \psi(y) \cos ny\, dy = \pi c_n \quad .. \quad .. \quad .. \quad (29)$$

As the left hand side is unity, $c_n = \dfrac{1}{\pi}$

$$\psi(y) = \delta(y) = \frac{1}{2\pi}\left[1 + 2\sum_{n=1}^{\infty} \cos ny\right] \quad .. \quad .. \quad .. \quad (30)$$

Therefore

$$\left.\begin{aligned} A_0' &= \frac{1}{2\pi} \\ A_n' &= 2 \\ B_n' &= 0 \end{aligned}\right\} \qquad \ldots \quad (31)$$

Thus we can evaluate the constants a_n, b_n, c_n, d_n and A_0 for this distribution at the lower boundary of the atmosphere and the emergent intensity can be calculated from equation (23).

4. When only the first order terms in γ in the Taylor's series expansion of $I(\tau, \mu', \lambda-\gamma[1-\cos\theta])$ is retained and the calculations are followed through, the equation corresponding to equation (7) becomes (cf. eq. (51), p. 537, *ibid.*)

$$\frac{\partial^2 s(x, y)}{\partial x^2}+\frac{\partial^2 s(x, y)}{\partial y^2} = 2\frac{\partial s(x, y)}{\partial y} \qquad \ldots \quad (32)$$

where $I_{+1}(x, y)+I_{-1}(x, y) = \dfrac{\partial s}{\partial x}$ and $I_{+1}(x, y)-I_{-1}(x, y) = \sqrt{3}\dfrac{\partial s}{\partial y}$ .. (33)

This equation (32) is to be solved under the boundary conditions similar to those given in art. 2, which in the present case can be expressed as follows :—

$$\left.\begin{aligned} \frac{1}{2}\left[\frac{\partial s}{\partial x}+\sqrt{3}\frac{\partial s}{\partial y}\right]_{x=x_1} &= \psi(y) = \delta(y) \\ \left[\frac{\partial s}{\partial x}-\sqrt{3}\frac{\partial s}{\partial y}\right] &= 0 \end{aligned}\right\} \qquad \ldots \quad (34)$$

(cf. eqs. (54), (55), p. 537, *ibid.*)

The expression for the emergent intensity at the outer boundary is given by

$$I_{+1}(0, y) = \sqrt{3}A_0+\sum_{n=1}^{\infty}\left[\left\{(a_n'-c_n')\alpha_n'+(b_n'+d_n')\beta_n'\right\}\cos ny + \left\{(b_n'+d_n')\alpha_n'-(a_n'-c_n')\beta_n'\right\}\sin ny\right] \qquad \ldots \quad (35)$$

where

$$A_0 = \frac{A_0'}{x_1+\sqrt{3}} \qquad \ldots \quad (36)$$

$$a_n' = \frac{(\alpha_n'^2+\beta_n'^2-3n^2)c_n'-2\sqrt{3}n\,\alpha_n'd_n'}{\alpha_n'^2+(\beta_n'-\sqrt{3}n)^2} \qquad \ldots \quad (37)$$

$$b_n' = -\frac{2\sqrt{3}n\,\alpha_n'c_n'+(\alpha_n'^2+\beta_n'^2-3n^2)\,dn'}{\alpha_n'^2+(\beta_n'-\sqrt{3}n)^2} \qquad \ldots \quad (38)$$

$$c_n' = \frac{M_n'A_n'+N_n'B_n'}{M_n'^2+N_n'^2} \qquad \ldots \quad (39)$$

$$d_n' = \frac{N_n'A_n'-M_n'B_n'}{M_n'^2+N_n'^2} \qquad \ldots \quad (40)$$

in which

$$\alpha'_n = \sqrt{\tfrac{1}{2}(n^2+\sqrt{n^4+4n^2})} \quad \ldots \quad (41)$$

$$\beta'_n = \sqrt{\tfrac{1}{2}(\sqrt{n^4+4n^2}-n^2)} \quad \ldots \quad (42)$$

$$M'_n = \tfrac{1}{2}e^{\alpha'_n x_1}\left\{-\frac{2\sqrt{3}\alpha'^2_n+(\alpha'^2_n+\beta'^2_n-3n^2)(\beta'_n+\sqrt{3}n)}{\alpha'^2_n+(\beta'_n-\sqrt{3}n)^2}\sin\beta'_n x_1 + \frac{(\alpha'^2_n+\beta'^2_n-3n^2)\alpha'_n-2\sqrt{3}n\alpha'_n(\beta'_n+\sqrt{3}n)}{\alpha'^2_n+(\beta'_n-\sqrt{3}n)^2}\cos\beta'_n x_1\right\} - \tfrac{1}{2}e^{-\alpha'_n x_1}\left\{\alpha'_n\cos\beta'_n x_1+(\beta'_n-\sqrt{3}n)\sin\beta'_n x_1\right\} \quad \ldots \quad (43)$$

$$N'_n = \tfrac{1}{2}e^{\alpha'_n x_1}\left\{-\frac{(\alpha'^2_n+\beta'^2_n-3n^2)\alpha'_n-2\sqrt{3}n\alpha'_n(\beta'_n+\sqrt{3}n)}{\alpha'^2_n+(\beta'_n-\sqrt{3}n)^2}\sin\beta'_n x_1 - \frac{2\sqrt{3}\alpha'^2_n+(\alpha'^2_n+\beta'^2_n-3n^2)(\beta'_n+\sqrt{3}n)}{\alpha'^2_n+(\beta'_n-\sqrt{3}n)^2}\cos\beta'_n x_1\right\} + \tfrac{1}{2}e^{-\alpha'_n x_1}\left\{(\beta'_n-\sqrt{3}n)\cos\beta'_n x_1-\alpha'_n\sin\beta'_n x_1\right\} \quad \ldots \quad (44)$$

And A'_0, A'_n and B'_n are the Fourier coefficients in the Fourier series representation of δ-function (eq. (30)) and are given by eq. (31).

Thus evaluating a'_n, b'_n, c'_n, d'_n and A_0, the emergent intensity given by eq. (35) can be calculated.

The results of the first order calculations are shown in Table I, and the values obtained by this method are compared with those found by Chandrasekhar in the same approximation with the aid of Green's function. There is complete agreement between the two results. This shows that the present method is at least as accurate as the Green's function method used by Chandrasekhar.

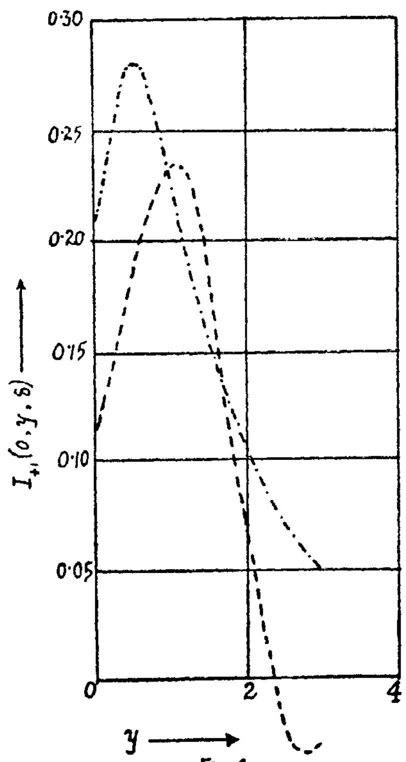
In Table II, the first order and the second order calculations are compared and the results are shown in Fig. 1.

TABLE I

y	$I_{+1}(0, y, \delta)$ First order calculation (trigonometrical series)	$I_{+1}(0, y, \delta)$ First order calculation (Chandrasekhar's method)
0	0·21	0·20
0·5	0·28	0·28
1·0	0·24	0·24
1·5	0·16	0·16
2·0	0·11	0·11
2·5	0·07	0·07
3·0	0·05	0·04

TABLE II

y	$I_{+1}(0, y, \delta)$ First order calculation (trigonometrical series)	$I_{+1}(0, y, \delta)$ Second order calculation (trigonometrical series method)
0	0·21	0·11
0·5	0·28	0·18
1·0	0·24	0·23
1·5	0·16	0·20
2·0	0·11	0·07
2·5	0·07	—0·03
3·0	0·05	—0·03



Compton shift and distribution of intensity of the emergent radiation from an electron atmosphere; the photospheric radiation is supposed to be monochromatic and is represented by δ-function; y is the shift in units of Compton wavelength.

— . — . — . — First order curve.
— — — — — Second order curve.

From the adjoining graph it appears that the second order calculations contribute important changes in the first order curve. For instance, (i) the shift of the maximum is increased and the maximum is lowered, (ii) in addition to the shift there is a lowering of the curve, which considerably reduces the violet displacement noticed by Chandrasekhar.

But in the extreme right the curve goes a little below the y-axis, though the amount of lowering below the y-axis is small as is apparent from Table II.

We may conclude from these results that the first order calculation is substantially modified in the calculation to the second order. It may also be remarked that the third order calculation will raise our curve very slightly.

ACKNOWLEDGEMENT

In conclusion, the author takes pleasure in recording gratitude to Prof. N. R. Sen for kind suggestions and encouragement during the preparation of this work.

Summary

The method of solutions by the use of trigonometrical series, developed by the author in previous papers for treatment of radiation scattering problems in electron atmosphere, has been further applied to work out the problem of modification of intensity up to the second order, in stellar atmosphere of slowly moving electron. The primary radiation at the photospheric level has been assumed to be monochromatic which is represented by δ-function. This case as is well known is of fundamental importance. The emergent intensity at the outer surface has been calculated for a particular value of optical thickness. The method has been checked by independent calculation of the first order intensity and comparison with Chandrasekhar's calculations made differently. It is shown that the second order calculations considerably modify those of the first order.

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Issued July 5, 1957.