ON STRONG RIESZIAN SUMMABILITY OF INFINITE SERIES

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1. Strong Cesàro summability, generally termed as *strong summability*, has been applied with success to various problems in the theory of infinite series. In a recent paper Boyd and Hyslop (1952) introduced a definition for strong Rieszian summability, making use of Riesz's 'arithmetic mean' (Hardy and Riesz, 1915; Chandrasekharan and Minakshisundaram, 1952) and demonstrated its equivalence to strong Cesàro summability.

In the present paper we introduce a definition for strong Rieszian summability in general, and try to build up a general theory of strong Rieszian summability. Theorems 1 to 7 consist of various results analogous to those known for strong Cesàro summability. Corollary 1 to Theorem 8 suggests an alternative definition for strong Rieszian summability. This definition considered in conjunction with the result of Boyd and Hyslop (1952) leads to the conclusion that strong Rieszian summability of type n is equivalent to strong Cesàro summability. Theorem 9 deals with the relationship between absolute Riesz summability and strong Rieszian summability of the same order. Finally in Theorem 10 we show that strong Rieszian summability includes also strong logarithmic summability as a particular case.

2.1. Let $a_0+a_1+ \ldots .$, be a given infinite series.

We denote by s_n^r the nth Cesàro mean of order $r(r > -1)$ of the series Σa_n. The series Σa_n is said to be summable (C, r) to sum s, if

$$\lim_{n\to\infty} s_n^r = s.$$

The series Σa_n is said to be absolutely summable (C, r) or summable $|C, r|$ if the sequence s_n^r is of bounded variation, that is to say, if the infinite series

$$\sum |s_n^r - s_{n-1}^r|$$

is convergent.

Again, if

$$\sum_{v=1}^{n} |s_v^{r-1} - s| = o(n),$$

as $n \to \infty$, the series Σa_n is said to be *strongly summable* (C, r), or *summable* $[C, r]$. And if, instead,

$$\sum_{v=1}^{n} |s_v^{r-1} - s|^q = o(n),$$

as $n \to \infty$, the series Σa_n is said to be *strongly summable* (C, r) with index q, where $q > 0$.

If

$$\sum_{v=1}^{n} |s_v^{r-1}|^q = O(n),$$

as $n \to \infty$, Σa_n is said to be bounded $[C, k]$ with index q.

2.2. Let $\{\lambda_n\}$ be an arbitrary sequence of positive numbers, such that

$$0 < \lambda_0 < \lambda_1 \cdots\cdots < \lambda_n \to \infty,$$

as $n \to \infty$. We write

$$A_n = a_0 + a_1 + \cdots\cdots + a_n,$$

and, if

$$t > 0,\ \lambda_n < t \leqslant \lambda_n + 1,$$

$$A_\lambda(t) \equiv A_n = a_0 + a_1 + \cdots\cdots + a_n = \sum_{\lambda_v < t} a_v$$

and, for $k > 0$,

$$A_\lambda^k(t) = \sum_{\lambda_v < t} (t - \lambda_v)^k a_v \qquad \qquad (2.2.1)$$

$$= k \int_0^t (t-\tau)^{k-1} A_\lambda(\tau) d\tau$$

$$= \int_0^t (t-\tau)^k dA_\lambda(\tau).$$

We define

$$A_\lambda^0(t) = A_\lambda(t).$$

The definition (2.2.1) is still applicable for negative k and $A_\lambda^k(t)$ is clearly integrable (L) over any finite range so long as $k > -1$ (Kuttner, 1953).

If we write

$$C_\lambda^k(x) = x^{-k} A_\lambda^k(x),$$

then $C_\lambda^k(x)$ is called the Riesz mean of order k, type λ, associated with the series Σa_n.

If

$$\lim_{x \to \infty} C_\lambda^k(x) = s,$$

where s is finite, the series Σa_n is said to be summable (R, λ, k), $k \geqslant 0$, to sum s.

If $C_\lambda^k(x)$ is of bounded variation in (h, ∞), that is to say, if the integral

$$\int_h^\infty |dC_\lambda^k(x)|$$

converges, where h is a convenient positive number, the series Σa_n is said to be absolutely summable (R, λ, k) or summable $|R, \lambda, k|$, $k \geqslant 0$ (Obreschkoff, 1929).

It is known that summability (C, α) is equivalent to summability (R, n, α) (Riesz, 1911; Hobson, 1926) and summability $|C, \alpha|$ is equivalent to summability $|R, n, \alpha|$, $\alpha \geqslant 0$ (Hyslop, 1936).

2.3. The above analysis suggests: that it may be possible to introduce the concept of strong Rieszian summability in quite a consistent and straightforward manner and this strong Rieszian summability may have the same place in the theory of Rieszian summability as is indicated by the strong Cesàro summability in the field of Cesàro summability. Further this strong Rieszian summability may be a generalisation of strong Cesàro summability in the same sense as the summability processes (R, λ, α) and $|R, \lambda, \alpha|$ are those of (C, α) and $|C, \alpha|$ respectively.

We are naturally led to the following definitions:

DEFINITION 1

If

$$\int_{\lambda_0}^{x} |C_\lambda^{k-1}(t) - s|\, dt = o(x), \qquad \ldots \quad \ldots \quad \ldots \quad (2.3.1)$$

*as $x \to \infty$, we say that Σa_n is strongly summable to sum s, or simply summable $[R, \lambda, k]$ to sum s, $(k > 0)$.**

DEFINITION 2

If

$$\int_{\lambda_0}^{x} |C_\lambda^{k-1}(t) - s|^q dt = o(x), \qquad \ldots \quad \ldots \quad \ldots \quad (2.3.2)$$

as $x \to \infty$, we say that Σa_n is strongly summable by Riesz means of type λ, order k and index q, or simply summable $[R, \lambda, k, q]$ to sum s.

DEFINITION 3

If

$$\int_{\lambda_0}^{x} |C_\lambda^{k-1}(t)|^q dt = O(x), \qquad \ldots \quad \ldots \quad \ldots \quad (2.3.3)$$

as $x \to \infty$, we say that Σa_n is bounded $[R, \lambda, k, q]$.

When $q = 1$ boundedness $[R, \lambda, k, q]$ is denoted simply by $[R, \lambda, k]$ as in case of summability.

3. In what follows we require the following lemmas.

LEMMA 1. *If $f(t)$ be integrable (L), the conditions*

$$\int_h^x t^{k-1} |f(t)|\, dt = o(x^k) \{\text{or } O(x^k)\},$$

and

$$\int_h^x |f(t)|\, dt = o(x) \{\text{or } O(x)\},$$

where $x \to \infty$, are equivalent for $h \neq 0$ and $k > 0$.

The above can be easily obtained by partial integration.

LEMMA 2. *If $f(t)$ belongs to L^p, where $p \geqslant 1$,*

$$\left|\int_h^x f(t)dt\right|^p \leqslant x^{p-1} \int_h^x |f(t)|^p dt.$$

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* Henceforth k is to be considered as > 0.

LEMMA 3. *If $f(x)$ and $g(x)$ be two functions belonging to L^p, where $p \geqslant 1$,*

$$\left\{ \int |f(x)|^p dx \right\}^{\frac{1}{p}} - \left\{ \int |g(x)|^p dx \right\}^{\frac{1}{p}}$$

$$\leqslant \left\{ \int |f(x)-g(x)|^p dx \right\}^{\frac{1}{p}}$$

$$\leqslant \left\{ \int |f(x)|^p dx \right\}^{\frac{1}{p}} + \left\{ \int |g(x)|^p dx \right\}^{\frac{1}{p}}.$$

Lemmas 2 and 3 can be obtained by Hölders and Minkowski's inequalities for integrals respectively.

LEMMA 4. (i) (Hardy and Riesz, 1915)

$$A_\lambda^{k+l}(x) = \frac{\Gamma(k+l+1)}{\Gamma(k+1)\Gamma(l)} \int_0^x (x-t)^{l-1} A_\lambda^k(t)dt,$$

where $k > 0$, $l > 0$; and

(ii)* $$A_\lambda^k(x) = k \int_0^x A_\lambda^{k-1}(t)dt,$$

where $k > 0$.

4. We now proceed to obtain results which are analogous to and generalize as well† theorems for strong Cesàro summability given from time to time by different authors.

We remark at this stage that, by Lemma 1, conditions (2.3.1), (2.3.2) and (2.3.3) are equivalent respectively to

$$\int_{\lambda_0}^x |A_\lambda^{k-1}(t) - st^{k-1}| dt = o(x^k),$$

$$\int_{\lambda_0}^x |A_\lambda^{k-1}(t) - st^{k-1}|^q dt = o\left(x^{\overline{k-1}\,q+1}\right),$$

where $kq' > 1$‡, and

$$\int_{\lambda_0}^x |A_\lambda^{k-1}(t)|^q dt = O\left(x^{\overline{k-1}\,q+1}\right),$$

where $kq' > 1$.

Henceforth these conditions instead of (2.3.1), (2.3.2) and (2.3.3) will be used wherever convenient.

THEOREM 1.§ *If the series Σa_n is summable $[R, \lambda, k]$ to sum s, then it is summable (R, λ, k) to the same sum.*

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* This is known for $k \geqslant 1$. For $0 < k < 1$ it can be easily proved.

† Since by Theorem A of the present paper strong Cesàro summability is a particular case of strong Rieszian summability.

‡ $\frac{1}{q}+\frac{1}{q'} = 1$. Since $\overline{k-1}\,q+1 > 0$ we have $kq' > 1$.

§ Corresponding results for strong Cesàro summability were given by Winn (1933), I, II.

We have

$$|C_\lambda^k(x)-s| = \left|\frac{A_\lambda^k(x)}{x^k}-s\right|$$

$$= kx^{-k}\left|\int_{\lambda_0}^{x}\left\{A_\lambda^{k-1}(t)-st^{k-1}\right\}dt\right|+x^{-k}O(1)$$

$$\leqslant kx^{-k}\int_{\lambda_0}^{x}|A_\lambda^{k-1}(t)-st^{k-1}|\,dt+x^{-k}O(1).$$

But we are given that

$$\int_{\lambda_0}^{x}|A_\lambda^{k-1}(t)-st^{k-1}|\,dt = o(x^k),$$

therefore

$$|C_\lambda^k(x)-s| = o(1),$$

as $x\to\infty$.

Hence the theorem is proved.

THEOREM 2.* *If the series Σa_n is summable (R, λ, k) to sum s, then it is summable $[R, \lambda, k+1, q]$ to the same sum.*

We are given that

$$C_\lambda^{k-1}(t)-s = o(1),$$

as $t\to\infty$. Therefore

$$|C_\lambda^{k-1}(t)-s|^q = o(1),$$

and hence

$$\int_{\lambda_0}^{x}|C_\lambda^{k-1}(t)-s|^q dt = o(x).$$

The theorem is proved.

In the following theorems we take $s=0$.

THEOREM 3.† *If Σa_n is summable $[R, \lambda, k]$ to sum s, then it is summable $[R, \lambda, k+l]$ to the same sum $(l>0)$.*

Case (i) $k\geqslant 1$. By Lemma 4

$$\int_{\lambda_0}^{x}|A_\lambda^{k+l-1}(t)|\,dt = \int_{\lambda_0}^{x}\frac{\Gamma(k+l)}{\Gamma(k)\Gamma(l)}\left|\int_{\lambda_0}^{t}(t-u)^{l-1}A_\lambda^{k-1}(u)du\right|dt.$$

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* Corresponding results for strong Cesàro summability were given by Winn (1933), I, II.

† This is the first theorem of consistency for strong Rieszian summability. For the corresponding result on strong Cesàro summability see Winn (1933), III.

But

$$\int_{\lambda_0}^{x} \left| \int_{\lambda_0}^{t} (t-u)^{l-1} A_\lambda^{k-1}(u) du \right| dt$$

$$\leqslant \int_{\lambda_0}^{x} dt \int_{\lambda_0}^{t} (t-u)^{l-1} | A_\lambda^{k-1}(u) | du$$

$$= \int_{\lambda_0}^{x} | A_\lambda^{k-1}(u) | du \int_{u}^{x} (t-u)^{l-1} dt$$

$$= \frac{1}{l} \int_{\lambda_0}^{x} | A_\lambda^{k-1}(u) | (x-u)^l du$$

$$\leqslant \frac{x^l}{l} \int_{\lambda_0}^{x} | A_\lambda^{k-1}(u) | du.$$

Hence, by hypothesis,

$$\int_{\lambda_0}^{x} | A_\lambda^{k+l-1}(t) | dt = o(x^{k+l}).$$

This gives the required result

Case (ii) $0 < k < 1$. Let

$$I(x) = \frac{\Gamma(k+l)}{\Gamma(k)\Gamma(l)} \int_{\lambda_0}^{x} A_\lambda^{k-1}(u)(x+\alpha-u)^{l-1} du,$$

where $\alpha > 0$.

We, then, have

$$\int_{\lambda_0}^{x} | I(t) | dt = \frac{\Gamma(k+1)}{\Gamma(k)\Gamma(l)} \int_{\lambda_0}^{x} \left| \int_{\lambda_0}^{t} A_\lambda^{k-1}(u)(t+\alpha-u)^{l-1} du \right| dt$$

$$\leqslant \frac{\Gamma(k+l)}{\Gamma(k)\Gamma(l)} \int_{\lambda_0}^{x} dt \int_{\lambda_0}^{t} | A_\lambda^{k-1}(u) | (t+\alpha-u)^{l-1} du$$

$$= \frac{\Gamma(k+l)}{\Gamma(k)\Gamma(l)} \int_{\lambda_0}^{x} | A_\lambda^{k-1}(u) | du \int_{u}^{x} (t+\alpha-u)^{l-1} dt$$

$$\leqslant \frac{\Gamma(k+l)}{\Gamma(k)\Gamma(l+1)} \int_{\lambda_0}^{x} | A_\lambda^{k-1}(u) | (x+\alpha-u)^l du.$$

Now, if $\alpha \to 0$, the left hand side tends to (Obreschkoff, 1929)

$$\int_{\lambda_0}^{x} | A_\lambda^{k+l-1}(t) | dt,$$

and the right hand side tends to

$$\frac{\Gamma(k+l)}{\Gamma(k)\Gamma(l+1)}\int_{\lambda_0}^{x} |A_\lambda^{k-1}(u)|\,(x-u)^l\,du.$$

The result now follows as in case (i).

THEOREM 4.* *If the series Σa_n is summable $[R, \lambda, k, q]$, then it is summable $[R, \lambda, k, q']$ for $0 < q' < q$.*

By Hölder's inequality for integrals,

$$\int_{\lambda_0}^{x} |C_\lambda^{k-1}(t)|^{q'} dt \leqslant \left[\int_{\lambda_0}^{x}\left\{|C_\lambda^{k-1}(t)|^{q'}\right\}^{\frac{q}{q'}} dt\right]^{\frac{q'}{q}} \left\{\int_{\lambda_0}^{x} dt\right\}^{1-\frac{q'}{q}}.$$

We are given that

$$\int_{\lambda_0}^{x} |C_\lambda^{k-1}(t)|^{q} dt = o(x),$$

therefore

$$\int_{\lambda_0}^{x} |C_\lambda^{k-1}(t)|^{q'} dt = o\left\{x^{\frac{q'}{q}}\,.\,x^{1-\frac{q'}{q}}\right\} = o(x),$$

as required.

THEOREM 5.† *If $p \geqslant 1$ and the series Σa_n is summable $[R, \lambda, k, p]$, then it is summable $[R, \lambda, k+l, q]$, where $l > 0$ and $0 < q \leqslant p$.*

We use Lemma 2, and proceed as in Theorem 3, and get that summability $[R, \lambda, k, p]$ implies summability $[R, \lambda, k+l, p]$. Then the result of Theorem 4 completes the proof.

THEOREM 6. *If $q > 1$, and the series Σa_n is summable $[R, \lambda, 1, q]$, then it is summable (R, λ, k) to the same sum provided $k > \frac{1}{q}$.*

We have

$$\begin{aligned} A_\lambda^k(t) &= k\int_{\lambda_0}^{t} (t-\tau)^{k-1} A_\lambda(\tau)\,d\tau \\ &\leqslant k\left[\int_{\lambda_0}^{t} |A_\lambda^0(\tau)|^q\,d\tau\right]^{\frac{1}{q}}\left[\int_{\lambda_0}^{t}(t-\tau)^{(k-1)q'}d\tau\right]^{\frac{1}{q'}} \\ &= o\left[t^{\frac{1}{q}}.\,t^{k-1+\frac{1}{q'}}\right] = o(t^k) \end{aligned}$$

by hypothesis, provided

$$(k-1)q' > -1,$$

or

$$k-1 > -\frac{1}{q'},$$

* For the corresponding result on strong Cesàro summability see Kuttner (1946).

† For analogous result on $[C, k, p]$ summability see Hyslop (1952).

or

$$k > \frac{1}{q}.$$

THEOREM 7. *If $q > 1$ and the series Σa_n is summable $[R, \lambda, k, q]$, then it is summable $(R, \lambda, k-1, \delta)$ provided $\delta > \frac{1}{q}$ and $kq' > 1$.*

This is a generalisation of Theorem 6 and can be proved similarly.

Introducing O in the place of o in the demonstration and writing $s = 0$, corresponding results for boundedness $[R, \lambda, k, q]$ can be obtained.*

5. We now prove a converse theorem on strong Rieszian summability analogous to one given for summability (R, λ, k). (Chandrasekharan and Minakshisundaram, 1952, p. 24.)

We put

$$f(x) = C_\lambda^{k-1}(x) - s,$$

and

$$g(x) = C_\lambda^{k}(x) - s,$$

in Lemma 3; and get

$$\left\{ \int_{\lambda_0}^{x} |C_\lambda^{k-1}(t) - s|^p dt \right\}^{\frac{1}{p}} - \left\{ \int_{\lambda_0}^{x} |C_\lambda^{k}(t) - s|^p dt \right\}^{\frac{1}{p}}$$

$$\leqslant \left\{ \int_{\lambda_0}^{x} |C_\lambda^{k-1}(t) - C_\lambda^{k}(t)|^p dt \right\}^{\frac{1}{p}}$$

$$\leqslant \left\{ \int_{\lambda_0}^{x} |C_\lambda^{k-1}(t) - s|^p dt \right\}^{\frac{1}{p}} + \left\{ \int_{\lambda_0}^{x} |C_\lambda^{k}(t) - s|^p dt \right\}^{\frac{1}{p}},$$

where $p \geqslant 1$.

Now, if

$$\int_{\lambda_0}^{x} |C_\lambda^{k}(t) - s|^p dt = o(x),$$

a necessary and sufficient condition that

$$\int_{\lambda_0}^{x} |C_\lambda^{k-1}(t) - s|^p dt = o(x),$$

is

$$\int_{\lambda_0}^{x} |C_\lambda^{k-1}(t) - C_\lambda^{k}(t)|^p dt = o(x). \quad \ldots \quad \ldots \quad (5.1)$$

Condition (5.1) can also be written as

$$\int_{\lambda_0}^{x} \left| t \frac{d}{dt} C_\lambda^{k}(t) \right| dt = o(x).$$

* Theorems on boundedness $[C, k]$ were given by Winn (1933).

or

$$\int_{\lambda_0}^{x} \left|\frac{B_\lambda^{k-1}(t)}{t^k}\right|^p dt = o(x), \quad \ldots \quad (5.2)$$

where $t^{-(k-1)}B_\lambda^{k-1}(t)$ is the Riesz mean for the series $\Sigma a_n\lambda_n$ of order $k-1$.

Again by Lemma 2 condition (5.2) is equivalent to

$$\int_{\lambda_0}^{x} |B_\lambda^{k-1}(t)|^p dt = o(x^{kp+1}). \quad \ldots \quad (5.3)$$

It can be easily proved that if conditions (5.1), (5.2), (5.3) be true for $k = l > 0$, they also hold for any $k > l$.

From the above follows:

Theorem 8. *For $q \geqslant 1$, a necessary and sufficient condition that the series Σa_n, summable or bounded $[R, \lambda, k+1, q]$, should be summable or bounded $[R, \lambda, k, q]$ is that*

$$\int_{\lambda_0}^{x} \left|u\frac{d}{du}C_\lambda^k(u)\right|^q du = o(x) \text{ or } O(x).$$

Corollary 1.* *Necessary and sufficient conditions for the series Σa_n to be summable $[R, \lambda, k, q]$ to the sum s are that Σa_n be summable (R, λ, k) to s and*

$$\int_{\lambda_0}^{x} \left|u\frac{d}{du}C_\lambda^k(u)\right|^q du = o(x).$$

It is obtained directly from Theorem 8 by application of Theorems 1 and 2.

Corollary 2. *If*

$$\int_{\lambda_0}^{x} \left|u\frac{d}{du}C_\lambda^k(u)\right| du = o(x),$$

then Σa_n is either summable $[R, \lambda, k]$ or never summable.

For, if possible, suppose that Σa_n is summable (R, λ, l), $l > k$ and therefore summable $(R, \lambda, l+1)$ by Theorem 2, but not summable $[R, \lambda, k]$. We can take l to be such as to exceed k by an integer m, say. Now, since Σa_n is summable $[R, \lambda, l+1]$ and condition (5.1) is satisfied for $k = l$ (as $l > k$), it is summable $[R, \lambda, l]$ by the above theorem. Repeating the above process m times we get the required result.

It is evident from Corollary 1 that we can define summability $[R, \lambda, k, q]$ in another way as given below.

Definition 2′

If

$$\lim_{\omega\to\infty} C_\lambda^k(\omega) = s,$$

where s is finite, and

$$\int_{\lambda_0}^{\omega} \left|u\frac{d}{du}C_\lambda^k(u)\right|^q du = o(\omega),$$

as $\omega \to \infty$, we say that Σa_n is summable $[R, \lambda, k, q]$, $(k > 0, q \geqslant 1)$.

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* Analogous result for $[C, k, p]$ summability being given by Hyslop (1952), where summability $[C, k, p]$ stands for strong summability (C, k) with index p.

Boyd and Hyslop (1952) have introduced the above definition with $\lambda_n = n$ for strong Rieszian summability. They have proved

THEOREM A.* *If* $k > 0$, $p \geqslant 1$, $\frac{1}{p}+\frac{1}{p'} = 1$, $kp' > 1$, *then summability* $[C, k, p]$ *of the series* Σa_n *implies summability* $[R, k, p)$ † *of the series and conversely.*

Kuttner (1946) has shown by an example that a series summable $[C, 1, q]$ may not be summable $[C, \frac{1}{q}]$. Hence from the above theorem it follows that summability $[R, \lambda, 1, q]$ does not necessarily imply summability (R, λ, k) for $k = \frac{1}{q}$, though it does so for every $k > \frac{1}{q}$ by Theorem 6.

Definition 2′ when extended to the value $k = 0$ gives a meaning to summability $[R, \lambda, 0, q]$. In a recent paper Harrington and Hyslop (1953) have used a similar property to define summability $[C, 0, p]$.

To complete the first theorem of consistency for strong Rieszian summability, we prove here

THEOREM 3′. *If the series* Σa_n *is summable* $[R, \lambda, 0]$, *then it is also summable* $[R, \lambda, k]$ *for* $k > 0$.

Now, $[R, \lambda, 0]$ summability is equivalent to convergence and

$$\int_{\lambda_0}^{X} x \,|\, dC_\lambda^0(x) \,|= o(X). \qquad (5.4)$$

Condition (5.4) is the same as

$$\sum_{\lambda_n < X} |a_n \lambda_n| = o(X).$$

To prove that Σa_n is summable $[R, \lambda, k]$ we have to show that Σa_n is summable (R, λ, k), and

$$I \equiv \int_{\lambda_0}^{X} |B_\lambda^{k-1}(x)| \, dx = o(X^{k+1}).$$

Now summability (R, λ, k) follows from convergence, and by definition

$$I = \int_{\lambda_0}^{X} \left| \sum_{\lambda_n < x} a_n \lambda_n (x-\lambda_n)^{k-1} \right| dx$$

$$\leqslant \int_{\lambda_0}^{X} \left[\sum_{\lambda_n < x} |a_n \lambda_n| (x-\lambda_n)^{k-1} \right] dx.$$

* For $k = 1$, this theorem is true also with $0 < p < 1$ as can be proved easily on the lines of Theorem 10 of the present paper.

† This is summability $[R, \lambda, k, p]$ where $\lambda_n = n$ of our notation.

Interchanging the order of integration and summation, we get

$$I \leqslant \sum_{\lambda_n < X} \int_{\lambda_n}^{X} |a_n \lambda_n| (x-\lambda_n)^{k-1} dx$$

$$\leqslant \sum_{\lambda_n < X} \frac{X^k}{k} |a_n \lambda_n| = o(X^{k+1}).$$

Using the inequality for series corresponding to that for integrals given in Lemma 2, the following theorem can also be easily obtained.

THEOREM 5′. *If the series Σa_n is summable $[R, \lambda, 0, q]$, then it is also summable $[R, \lambda, k, q]$ for $q > 1$.*

We now prove

THEOREM 9.* *If the series Σa_n is summable $|R, \lambda, k|$, then it is also summable $[R, \lambda, k]$ but not vice versa.*

It is given that

$$\int_{\lambda_0}^{\infty} |dC_\lambda^k(x)|$$

converges, to sum K, say. We then have

$$F(\omega) \equiv \int_{\lambda_0}^{\omega} \left|\frac{d}{dt} C_\lambda^k(t)\right| dt = K + o(1),$$

as $\omega \to \infty$.
Now

$$F(\omega) - \int_{\lambda_0}^{\omega} \frac{x}{\omega} |dC_\lambda^k(x)|$$

$$= \frac{1}{\omega} \int_{\lambda_0}^{\omega} (\omega - x) |dC_\lambda^k(x)|$$

$$= \frac{1}{\omega} \left| (\omega - x) F(x) \right|_{\lambda_0}^{\omega} + \frac{1}{\omega} \int_{\lambda_0}^{\omega} F(x) dx$$

$$= \frac{1}{\omega} \int_{\lambda_0}^{\omega} \{K + o(1)\} \, dx = K + o(1),$$

as $\omega \to \infty$. Hence

$$\int_{\lambda_0}^{\omega} \frac{x}{\omega} |dC_\lambda^k(x)| = o(1).$$

* Corresponding result for Cesàro's summability for the case $k = 1$ was given by Fekete (1911).

That summability $|R, \lambda, k|$ implies summability (R, λ, k) is known. Thus the first part is proved.

The second part follows from the corresponding known result for Cesàro summability (Winn, 1933).

6. Strong logarithmic summability (Hardy and Littlewood, 1935) is defined as follows.

Let

$$s_n = a_0 + a_1 + \cdots + a_n.$$

If

$$\sum_1^n \frac{|s_v - s|^q}{n} = o(\log n),$$

then the series Σa_n is said be strongly summable by logarithmic means with index q, or simply summable $[R; 1, q]$.

We prove

THEOREM 10. *If the series Σa_n is summable $[R, \lambda, 1, q]$, where $\lambda_n = \log n$, it is also summable $[R; 1, q]$ to the same sum and conversely $(q > 0)$.*

Proof

It is given that

$$\int_1^t |A_\lambda(\tau) - s|^q d\tau = o(t), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6.1)$$

and it is to be shown that

$$\sum_1^n \frac{|s_v - s|^q}{n} = o(\log n),$$

where $\lambda_n = \log n$. Since

$$A_\lambda(\log t) = A(t);$$

putting

$$\tau = \log(x),$$

and

$$t = \log \omega,$$

we have for (6.1)

$$\int_1^\omega \frac{1}{x} |A(x) - s|^q dx = o(\log \omega). \quad \ldots \quad \ldots \quad \ldots \quad (6.2)$$

Putting $\omega = n$, we get

$$\sum_1^{n-1} \int_v^{v+1} |A(x) - s|^q \frac{dx}{x} = o(\log n),$$

or

$$\sum_1^{n-1} \int_v^{v+1} \frac{dx}{x} |s_v - s|^q = o(\log n),$$

or

$$\sum_1^{n-1} |s_v - s|^q \log\left(1 + \frac{1}{v}\right) = o(\log n), \quad \ldots \quad \ldots \quad \ldots \quad (6.3)$$

But

$$\left.\begin{aligned}\log\left(1+\frac{1}{v}\right) &\simeq \frac{1}{v},\\ \log(n-1) &\simeq \log n.\end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (6.4)$$

and

Hence (6.3) implies

$$\sum_{1}^{n-1} \frac{|s_v - s|^q}{v} = o\{\log(n-1)\}.$$

Thus the first part of the theorem is proved.

Proof of the converse

Being given that

$$\sum_{1}^{n} \frac{|s_v - s|^q}{v} = o(\log n),$$

it is to be shown that

$$\int_{1}^{x} |A(t)-s|^q \frac{dt}{t} = o(\log x),$$

by (6.2). Let

$$n+1 > x \geqslant n.$$

Then

$$\int_{1}^{x} |A(t)-s|^q \frac{dt}{t} = \int_{1}^{n} + \int_{n}^{x} = I_1 + I_2,$$

say. Now it follows from (6.3) and (6.4) that

$$I_1 = o\{\log x\}.$$

And further

$$\begin{aligned} I_2 &= \int_{n}^{x} |A(t)-s|^q \frac{dt}{t} \\ &= |s_n - s|^q \log\frac{x}{n} \\ &\simeq |s_n - s|^q \frac{1}{n} \end{aligned}$$

But by hypothesis

$$|s_n - s|^q \frac{1}{n} = o(\log n).$$

Therefore

$$I_2 = o(\log x).$$

This completes the proof of the converse.

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