THE GENERAL THEORY OF MODERATED CHARGES

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1. Introduction

A 'moderated' charge consists of nominally identical grains, each of which consists of a number of layers. The first layer is a cool slow-burning type and the succeeding layers are made of progressively faster burning propellants. The advantage of such an arrangement is that, with this, even a digressive shape like a cord, in which the surface area producing the gases decreases, may be made to behave as a constant burning, or even as a progressive shape; and at the same time, a higher density of loading can be attained.

The aim of the present paper is to develop a general theory of moderated charges, parallel to the general theory of composite charges, first developed by Clemmow (1920, 1951) and Corner (1950) and recently improved by the present author (1956*a*, 1957). The parallelism will be examined more closely in the final section.

2. The equivalent charge

Let the moderated charge consist of n layers. The layer to burn first will be called the first layer; that due to burn next will be called the second layer, and so on, so that the layer to burn out last will be called the nth layer.

For the ith layer, let

- F_i denote the force constant
- C_i denote the charge mass
- D_i denote the ballistic size
- β_i denote the rate of burning constant
- δ_i denote the propellant density
- z_i denote the fraction of the charge mass burnt
- f_i denote the fraction of the web size remaining
- θ_i denote the form-factor
- γ_i denote the ratio of the specific heats of the product gases

We shall assume that all the component charges have the same pressure index α. In particular, this condition will be satisfied, if for each component charge, the rate of burning is proportional to pressure.

Now we define the equivalent charge as that charge which gives the same ballistic equations as the moderated charge, both during and after burning. Let F, C, D, β, δ, z, f, γ, denote the corresponding quantities for the equivalent charge.

From the rate-of-burning equations, during the first stage,

$$\frac{D}{\beta}\frac{df}{dt} = \frac{D_1}{\beta_1}\frac{df_1}{dt}, \quad \ldots \quad (1)$$

we get on integration

$$\frac{D}{\beta}(1-f) = \frac{D_1}{\beta_1}(1-f_1), \quad \ldots \quad (2)$$

since at ignition, $f_1 = 1, f = 1$.

Now we put

$$\frac{D_i}{\beta_i} = D_i'[i = 1, 2, \ldots, n] \quad \ldots \quad (3a)$$

$$\frac{D}{\beta} = D', \quad \ldots \quad (3b)$$

so that (2) becomes

$$D'(1-f) = D_1'(1-f_1) \quad \ldots \quad (4)$$

We shall call $\frac{D_i}{\beta_i} = D_i'$ as the 'effective' ballistic size of the ith component charge, so that D' is the effective ballistic size of the equivalent charge.

Now during the second stage, the rate of burning equations give

$$\frac{D}{\beta}\frac{df}{dt} = \frac{D_2}{\beta_2}\frac{df_2}{dt}$$

Integrating

$$D'f = D_2'f_2 + D' - D_1' - D_2',$$

since from (4), when $f_1 = 0, f_2 = 1, f = 1 - \frac{D_1'}{D'}$.

Similarly during the rth stage,

$$D'f = D_r'f_r + D' - D_1' - D_2' - \ldots - D_r' \quad \ldots \quad (5)$$

At the end of the rth stage, (5) gives

$$f = 1 - \frac{D_1' + D_2' + \ldots + D_r'}{D'} \quad \ldots \quad (6)$$

At the end of nth stage, since the equivalent charge is all consumed, $f = 0$, so that

$$D' = D_1' + D_2' + \ldots + D_n' \quad \ldots \quad (7)$$

Thus the effective ballistic size of the equivalent charge is equal to the sum of the effective ballistic sizes of the component charges.

In particular, if $\beta_1 = \beta_2 = \ldots = \beta_n$, (7) gives

$$D = D_1 + D_2 + \ldots + D_n.$$

Also, since at all-burnt the total masses of the gas produced by the equivalent and the moderated charges are equal, we have

$$C = C_1 + C_2 + \ldots + C_n. \quad \ldots \quad (8)$$

Further, the rate of production of energy for the equivalent and the moderated charge throughout burning should be the same. Therefore, in the first stage,

$$\frac{CF}{\gamma-1}\frac{dz}{dt} = \frac{C_1F_1}{\gamma_1-1}\frac{dz}{dt}$$

Integrating

$$\frac{CF}{\gamma-1}z = \frac{C_1F_1}{\gamma_1-1}z_1, \quad \ldots \quad (9)$$

since at ignition $z_1 = 0$, $z = 0$.

Let

$$\lambda_i = \frac{\dfrac{C_iF_i}{\gamma_i-1}}{\dfrac{CF}{\gamma-1}} \, [i = 1, 2, \ldots ; n], \qquad \ldots \qquad (10)$$

then from (9) during the first stage,

$$z = \lambda_1 z_1$$

During the second stage,

$$\frac{CF}{\gamma-1}\frac{dz}{dt} = \frac{C_2F_2}{\gamma_2-1}\frac{dz_2}{dt}$$

Integrating and using (10),

$$z = \lambda_2 z_2 + \lambda_1$$

since when $z_2 = 0$, $z_1 = 1$, $z = \lambda_1$.

Similarly during rth stage

$$z = \lambda_1 + \lambda_2 + \ldots + \lambda_{r-1} + \lambda_r z_r, \qquad \ldots \qquad (11)$$

and at the end of the rth stage,

$$z = \lambda_1 + \lambda_2 + \ldots + \lambda_r \qquad \ldots \qquad (12)$$

The equivalent charge is all consumed at all-burnt (end of the nth stage), so that at this instant $z = 1$ and (12) gives

$$1 = \lambda_1 + \lambda_2 + \ldots + \lambda_n.$$

and hence from (10)

$$\frac{CF}{\gamma-1} = \sum_{i=1}^{n} \frac{C_iF_i}{\gamma_i-1}. \qquad \ldots \qquad (13)$$

(13) expresses the fact that the total energy of the equivalent charge is equal to the total energy of the component charges.

From (7), (9) and (13), we determine $\frac{D}{\beta}$, C, $\frac{F}{\gamma-1}$ for the equivalent charge. To make the reduction complete, we investigate in the next section, the form-function for the equivalent charge.

3. Form-function for the equivalent charge

From (11), we have during the rth stage

$$z = \lambda_1 + \lambda_2 + \ldots + \lambda_{r-1} + \lambda_r (1-f_r)(1+\theta_r f_r) \qquad \ldots \qquad (14)$$

Now if we put

$$b_r = \frac{D_1' + D_2' + \ldots + D_r'}{D_r'} \, (r = 1, 2, \ldots, n), \qquad \ldots \qquad (15)$$

we get from (5)

$$f_r = b_r - \frac{D'}{D_r'}(1-f)$$

Substituting in (14), we get the form-function for the equivalent charge, during the rth stage,

$$z = \lambda_1 + \lambda_2 + \ldots + \lambda_{r-1} + \lambda_r \left[1 - b_r + \frac{D'}{D'_r}(1-f)\right]\left[1 + \theta_r b_r - \theta_r \frac{D'}{D'_r}(1-f)\right]$$

$$= A_r + B_r(1-f) - E_r(1-f)^2, \quad \ldots \quad (16)$$

where

$$A_r = \lambda_1 + \lambda_2 + \ldots + \lambda_{r-1} + \lambda_r(1-b_r)(1+\theta_r b_r) \quad \ldots \quad (17a)$$

$$B_r = \lambda_r \frac{D'}{D'_r}[1 - \theta_r + 2\theta_r b_r] \quad \ldots \quad (17b)$$

$$E_r = \lambda_r \left(\frac{D'}{D'_r}\right)^2 \theta_r \quad \ldots \quad (17c)$$

From (6) and (15), the rth stage starts when

$$f = 1 - \frac{b_{r-1} D'_{r-1}}{D'}, \quad \ldots \quad (18a)$$

and ends when

$$f = 1 - \frac{b_r D'_r}{D'}. \quad \ldots \quad (18b)$$

By substituting the value of f from (18b) in (16), and simplifying with the help of (17), we easily find that, at the end of the rth stage,

$$z = \lambda_1 + \lambda_2 + \ldots + \lambda_{r-1} + \lambda_r. \quad \ldots \quad (19)$$

Again on substituting the value of f at the beginning of the $(r+1)$th stage. from (18a) in the form-function valid for this stage, viz.

$$z = A_{r+1} + B_{r+1}(1-f) - E_{r+1}(1-f)^2,$$

we obtain the same value of z as given by (19). Thus z is a continuous function of f throughout burning.

Now from (16)

$$\frac{dz}{df} = -B_r + 2E_r(1-f)$$

$\therefore$ at the end of the rth stage, from (17b), (17c) and (18b)

$$\frac{dz}{df} = -\lambda_r \frac{D'}{D'_r}[1 - \theta_r] \quad \ldots \quad (20)$$

Similarly the value of $\frac{dz}{df}$ at the beginning of the $(r+1)$th stage is given by

$$\frac{dz}{df} = -\lambda_{r+1} \frac{D'}{D'_{r+1}}[1 + \theta_{r+1}] \quad \ldots \quad (21)$$

From (20) and (21), we find that $\frac{dz}{df}$ is, in general, discontinuous in passing from one stage to another.

From (21), the value of $\frac{dz}{df}$ at the beginning of the rth stage is

$$-\lambda_r \frac{D'}{D_r'}[1+\theta_r], \qquad (22)$$

so that the increment in $\frac{dz}{df}$ during the rth stage is

$$2\lambda_r \frac{D'}{D_r'}\theta_r \qquad (23)$$

4. Reduction to a single form-function

We have found above that there is a different form-function for each stage of burning, and consequently the solution of the equations of Internal Ballistics requires a separate integration for each stage. In order to simplify the integration, it is desirable, if possible, to find a single form-function

$$z = (1-f)(1+\theta f) \qquad (24)$$

which should be applicable equally to all the stages of burning. Two theoretically sound methods for the 'estimation' of θ have been proposed in a similar problem arising in the general theory of composite charges [Corner (1950), Kapur (1956*a*)], and we shall apply these methods in the present case also, assuming for the sake of mathematical simplicity that the moderated charge consists of only two layers.

When $n = 2$, the form-function for the first stage $\left(f \text{ decreasing from 1 to } \frac{D_2'}{D'}\right)$ is

$$z = \lambda_1(1+\theta_1)\frac{D'}{D_1'}(1-f) - \lambda_1\theta_1\frac{D'^2}{D_1'^2}(1-f)^2 \qquad (25a)$$

and that for the second stage (f decreasing from $\frac{D_2'}{D'}$ to zero) is

$$z = 1+\lambda_2\frac{D'}{D_2'}(\theta_2-1)f - \lambda_2\frac{D'^2}{D_2'^2}\theta_2 f^2 \qquad (26)$$

First method—Following Corner (1950), we choose θ so that the area under (24) between $f = 0$ and $f = 1$ is the same as the sum of the areas under the two curves (25) and (26) between the same two limits, so that

$$\int_1^0 (1-f)(1+\theta f)\,df = \int_1^{\frac{D_2'}{D'}} \left[\lambda_1(1+\theta_1)\frac{D'}{D_1'}(1-f) - \lambda_1\theta_1\frac{D'^2}{D_1'^2}(1-f)^2\right] df +$$

$$+\int_{\frac{D_2'}{D'}}^0 \left[1+\lambda_2\frac{D'}{D_2'}(\theta_2-1)f - \lambda_2\frac{D'^2}{D_2'^2}\theta_2 f_2\right] df.$$

Integrating and simplifying, we get

$$\theta = \lambda_1\theta_1\frac{D_1'}{D'} + \lambda_2\theta_2\frac{D_2'}{D'} + 3\frac{D_2'}{D'} - 3\lambda_2. \qquad (28)$$

Second method—Here we use a principle similar to the principle of least squares used in Statistics. We choose θ so that

$$\int_{1}^{\frac{D_2'}{D'}} \left\{ (1-f)(1+\theta f) - \lambda_1(1+\theta_1)\frac{D'}{D_1'}(1-f) + \lambda_1\theta_1\left(\frac{D'}{D_1'}\right)^2 (1-f)^2 \right\}^2 df$$

$$+\int_{\frac{D_2'}{D'}}^{1} \left\{ (1-f)(1+\theta f) - \left(1+\lambda_2\frac{D'}{D_2'}(\theta_2-1)f - \lambda_2\left(\frac{D'}{D_2'}\right)^2 \theta_2 f^2\right) \right\}^2 df$$

is minimum.

Differentiating under the sign of integration, we get on completing the integration and simplifying,

$$\begin{aligned}\theta\{20(m^3+n^3)-30(m^4+n^4)+12(m^5+n^5)\} &= 20[n^3-m^3+\lambda_1 m^2(1+\theta_1)+\lambda_2 n^2(\theta_2-1)] \\ &\quad -15[\lambda_1\theta_1 m^2+\lambda_1(1+\theta_1)m^3-m^4-\lambda_2 \\ &\qquad +\lambda_2\theta_2 n^2+\lambda_2\theta_2 n^3+n^4] \\ &\qquad +12[\lambda_1\theta_1 m^3+\lambda_2\theta_2 n^3], \quad \ldots \quad (29)\end{aligned}$$

where

$$m = \frac{D_1'}{D'} \quad \ldots \quad (30a)$$

$$n = \frac{D_2'}{D'} \quad \ldots \quad (30b)$$

Expression (28) for θ is simpler than (29); but we have to remember that the principle of least squares has a stronger theoretical basis than the principle used in the first method, and (29) will, in general, give better results than (28).

5. Cylindrical moderated charge

Let the moderated charge consist of n component charges in cylindrical shells (Fig. 1). We assume that

(i) the burning of the charge obeys Piobert's law of burning by parallel layers.

(ii) Either the length l of the charge is so long that the end-burning effects can be neglected or that the ends have been inhibited for burning.

From the figure

$$D_1 = a_1-a_2,\ D_2 = a_2-a_3,\ \ldots,\ D_{n-1} = a_{n-1}-a_n,\ D_n = a_n \quad \ldots \quad (31)$$

When the outermost layer is burning

$$z_1 = \frac{\pi\left\{a_1^2-\left(a_2+\overline{a_1-a_2}f_1\right)^2\right\} l\delta_1}{\pi\left(a_1^2-a_2^2\right) l\delta_1}$$

$$= (1-f_1)\left(1+\frac{a_1-a_2}{a_1+a_2}f_1\right),$$

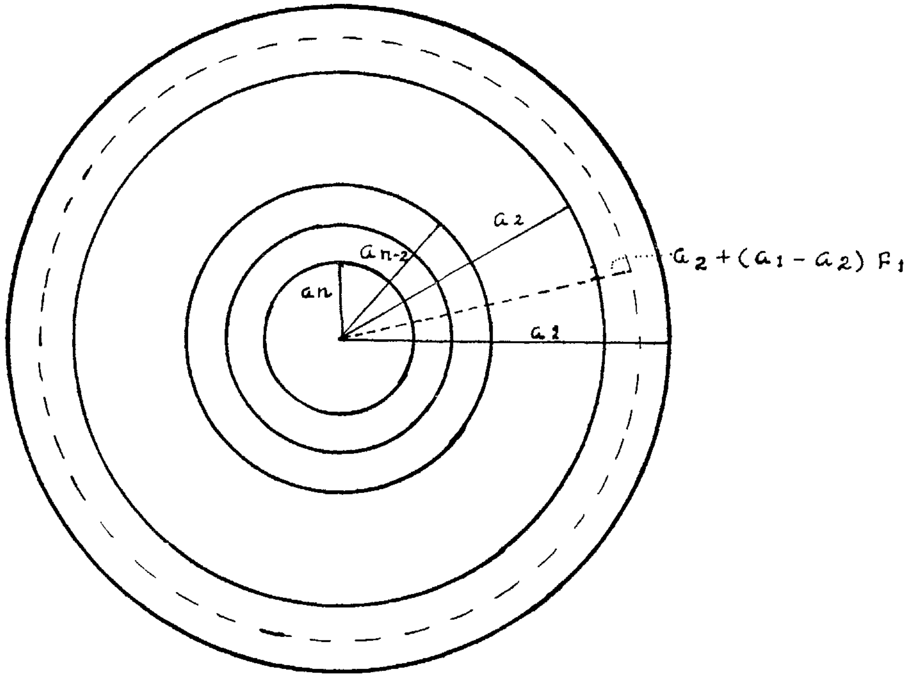

FIG. 1

so that

$$\theta_1 = \frac{a_1 - a_2}{a_1 + a_2}$$

Similarly

$$\theta_2 = \frac{a_2 - a_3}{a_2 + a_3} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

. . . .

. . . .

$$\theta_r = \frac{a_r - a_{r+1}}{a_r + a_{r+1}}$$

$$\theta_n = 1.$$

In the particular case when the layers are of equal thickness, let

$$a_n = d, \; a_{n-1} = 2d, \; \ldots\ldots, \; a_1 = nd$$

then

$$\theta_1 = \frac{1}{2n-1}, \; \theta_2 = \frac{1}{2n-3}, \; \ldots\ldots \; \theta_{n-1} = \frac{1}{3}, \; \theta_n = \frac{1}{1}, \quad \ldots \quad \ldots \quad (33)$$

so that the form-factors form an H.P.

The case of a single cord charge can be deduced as a particular case and can be used to verify the results for a moderated charge. Thus if the composition of all the component charges is the same

$$\beta_1 = \beta_2 = \ldots . = \beta_n$$

$$\frac{F_1}{\gamma_1 - 1} = \frac{F_2}{\gamma_2 - 1} = \ldots . = \frac{F_n}{\gamma_n - 1}$$

$$\therefore \quad \frac{\lambda_1}{C_1} = \frac{\lambda_2}{C_2} = \ldots . = \frac{\lambda_n}{C_n} = \frac{1}{C}$$

$$\therefore \quad \lambda_1 = \frac{a_1^2 - a_2^2}{a_1^2}, \quad \lambda_2 = \frac{a_2^2 - a_3^2}{a_1^2}, \ldots .$$

$$\lambda_r = \frac{a_r^2 - a_{r+1}^2}{a_1^2}, \ldots . , \lambda_n = \frac{a_n^2}{a_1^2}$$

Again

$$b_r = \frac{D_1' + D_2' + \ldots . + D_r'}{D_r'} = \frac{a_1 - a_{r+1}}{a_r - a_{r+1}}.$$

Substituting in (17)

$$A_r = 0, \quad B_r = 2, \quad E_r = 1,$$

so that the form-function for the rth stage is

$$z = 0 + 2(1-f) - (1-f)^2 = (1-f)(1+f) = 1 - f^2, \quad \ldots \quad \ldots \quad (35)$$

as we expect.

Substituting in (20) and (21), we see that the value of $\frac{dz}{df}$ both at the end of the rth stage and in the beginning of the $(r+1)$th stage is

$$-2\frac{a_{r+1}}{a_1},$$

and this is what we expect from (35), since at the end of the rth stage $f = \frac{a_{r+1}}{a_1}$

Again substitution in (28) and (29) gives $\theta = 1$ in each case, verifying the results of these equations.

6. Maximum pressure in the general theory of moderated charges

It can be shown, as in the general theory of composite charges, that neglecting co-volume correction terms

$$\frac{dp}{df} = \frac{FC}{A(x+l)} \left\{ \frac{dz}{df} + \gamma M(f_0 - f) \right\} \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

where f_0 is the value of f at shot-start.

From (16) and (17), we have, during the rth stage,

$$\frac{dz}{df} = -B_r + 2E_r(1-f)$$

or

$$\frac{dz}{df} = -\lambda_r \frac{D'}{D'_r}[1-\theta_r+2\theta_r b_r]+2\lambda_r\left(\frac{D'}{D'_r}\right)^2 \theta_r(1-f) \quad \ldots \quad \ldots \quad (37)$$

From (36) and (37)

$$\frac{dp}{df} = \frac{FC}{A(x+l)}\left\{-\lambda_r \frac{D'}{D'_r}(1-\theta_r+2\theta_r b_r)-\gamma M(1-f_0)+\right.$$

$$\left.+\left(\gamma M+2\lambda_r \frac{D'^2}{D'^2_r}\theta_r\right)(1-f)\right\} \quad \ldots \quad (38)$$

Case (i)

$$\gamma M+2\lambda_r \frac{D'^2}{D'^2_r}\theta_r > 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

In this case as f decreases, $(1-f)$ increases, $\frac{dp}{df}$ can change sign from negative to positive and not vice versa. Consequently a pressure maximum can arise during a stage, but a pressure minimum cannot arise there. This case includes the case when all θ's are positive and in particular of a cylindrical moderated charge.

Case (ii)

$$\gamma M+2\lambda_r\left(\frac{D'}{D'_r}\right)^2 \theta_r \leqslant 0, \ldots \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

and

$$1-\theta_r+2\theta_r b_r \geqslant 0$$

$\frac{dp}{df}$ remains negative throughout the rth stage and neither a pressure maximum nor a pressure minimum can arise during this stage.

Case (iii)

$$\gamma M+2\lambda_r\left(\frac{D'}{D'_r}\right)^2 \theta_r \leqslant 0, \ldots \quad \ldots \quad \ldots \quad \ldots \quad (41)$$

$$1-\theta_r-2\theta_r b_r < 0$$

In this case $\frac{dp}{df}$ can change sign from positive to negative and a pressure minimum can arise during the stage.

Cases (ii) and (iii) will arise only if the burning surface is continuously increasing, but in such a situation, the necessity of using moderated charges would not usually arise.

Now in crossing from rth stage to $(r+1)$th, the increment in $\frac{dz}{df}$ is from (20) and (21)

$$-\lambda_{r+1}\frac{D'}{D'_{r+1}}(1+\theta_{r+1})+\lambda_r \frac{D'}{D'_r}(1-\theta_r) \quad \ldots \quad \ldots \quad \ldots \quad (42)$$

If the increment is negative, $\frac{dp}{df}$ can change sign from positive to negative and a pressure minimum can arise. If, on the other hand, the increment is positive, $\frac{dp}{df}$ can change from negative to positive and a pressure maximum can arise.

From (38), a pressure maximum can occur in the rth stage if $\frac{dp}{df}$ is negative at the beginning of this stage and positive at its end, i.e. if

$$-\lambda_r \frac{D'}{D'_r}[1-\theta_r+2\theta_r b_r]-\gamma M(1-f_0) + \left[\gamma M+2\lambda_r\left(\frac{D'}{D'_r}\right)^2\theta_r\right]\frac{b_{r-1}D'_{r-1}}{D'} < 0 \quad .. \ (43a)$$

and

$$-\lambda_r \frac{D'}{D'_r}[1-\theta_r+2\theta_r b_r]-\gamma M(1-f_0) + \left[\gamma M+2\lambda_r\left(\frac{D'}{D'_r}\right)^2\theta_r\right]\frac{b_r D'_r}{D'} > 0 \ .. \quad .. \ (43b)$$

A pressure maximum will occur at the end of the rth stage if

$$-\lambda_r \frac{D'}{D'_r}[1-\theta_r+2\theta_r b_r]-\gamma M(1-f_0) + \left[\gamma M+2\lambda_r\left(\frac{D'}{D'_r}\right)^2\theta_r\right]\frac{b_r D'_r}{D'} \not< 0 \quad .. \ (44a)$$

and

$$-\lambda_{r+1} \frac{D'}{D'_{r+1}}[1-\theta_{r+1}+2\theta_{r+1}b_{r+1}]-\gamma M(1-f_0) + \left[\gamma M+2\lambda_r\left(\frac{D'}{D'_{r+1}}\right)^2\theta_{r+1}\right]\frac{b_r D'_r}{D'} > 0 \quad .. \ (44b)$$

and a pressure minimum will arise at the end of the rth stage if the inequalities in (44a) and (44b) are reversed.

Particular case—

$$D_1 = D_2 = \ldots . = D_n = d. \quad .. \quad .. \quad .. \ (45)$$

Let $D = nd$
then (7) gives

$$\frac{n}{\beta} = \frac{1}{\beta_1}+\frac{1}{\beta_2}+ \ldots . + \frac{1}{\beta_n}, \ .. \quad .. \quad .. \quad .. \ (46)$$

so that in the case when the layers are of equal thickness, β is the harmonic mean of the rates of burning constants of the component charges.

Again in this case, from (15)

$$b_r = \frac{\frac{1}{\beta_1}+\frac{1}{\beta_2}+ \ldots . +\frac{1}{\beta_r}}{\frac{1}{\beta_r}} = 1+\frac{\beta_r}{\beta_1}+\frac{\beta_r}{\beta_2}+ \ldots . +\frac{\beta_r}{\beta_{r-1}} \ .. \quad .. \ (47)$$

so that for positively moderated charges, for which

$$\beta_{r+1} > \beta_r \ .. \quad .. \quad .. \quad .. \quad .. \ (48)$$

we have

$$b_r > r \quad \ldots \quad (49)$$

Also the increment in $\frac{dz}{df}$ at the end of the rth stage is given by

$$\frac{1}{\beta}\{\lambda_r\beta_r(1-\theta_r)-\lambda_{r+1}\beta_{r+1}(1+\theta_{r+1}\} \quad \ldots \quad (50)$$

Now, for a moderated charge for which

(i) layers are of equal thickness .. .. (51*a*)

(ii) $\theta_r > 0$ for all r .. .. (51*b*)

(iii) $\lambda_r\beta_r(1-\theta_r) < \lambda_{r+1}\beta_{r+1}(1+\theta_{r+1})$ for all r .. .. (51*c*)

we find:

(*a*) from (39), that only a pressure maximum can arise during a stage.

(*b*) from (50), that only a pressure minimum can arise at the end of a stage, since the increment in $\frac{dz}{df}$ and hence in $\frac{dp}{df}$ is negative at the end of every stage.

For a cylindrical moderated charge, with layers of equal thickness d, for which

$$\theta_r = \frac{1}{2n-2r+1}, \quad \lambda_r = \frac{\frac{F_r}{\gamma_r-1}\pi d^2 l\delta_r\{(n-r+1)^2-(n-r)^2\}}{\frac{CF}{\gamma-1}},$$

condition (51*b*) is always satisfied and condition (51*c*) simplifies to

$$\frac{F_r\beta_r\delta_r}{\gamma_r-1} < \frac{F_{r+1}\beta_{r+1}\delta_{r+1}}{\gamma_{r+1}-1} \quad \ldots \quad (51d)$$

7. Particular case of a cylindrical moderated charge of two layers of equal thickness

For this case

$$\theta_1 = \tfrac{1}{3}, \quad \theta_2 = 1; \quad \ldots \quad (52a)$$

$$D_1' = \frac{d}{\beta_1}, \quad D_2' = \frac{d}{\beta_2}, \quad D' = \frac{2d}{\beta} \quad \ldots \quad (52b)$$

$$\frac{2}{\beta} = \frac{1}{\beta_1}+\frac{1}{\beta_2} \quad \ldots \quad (52c)$$

and

$$\frac{D_1'}{D} = \frac{\beta_2}{\beta_1+\beta_2}, \quad \frac{D_2'}{D'} = \frac{\beta_1}{\beta_1+\beta_2} \quad \ldots \quad (52d)$$

From (25) and (38), during the first stage,

$$\frac{dp}{df} = \frac{FC}{A(x+l)}\left[-\frac{4\lambda_1}{3}\frac{\beta_1+\beta_2}{\beta_2}+\frac{2\lambda_1}{3}\frac{(\beta_1+\beta_2)^2}{\beta_2^2}(1-f)+\gamma M(f_0-f)\right] \quad \ldots \quad (53)$$

and from (26) and (38), during the second stage,

$$\frac{dp}{df} = \frac{FC}{A(x+l)}\left[-2\lambda_2\frac{(\beta_1+\beta_2)^2}{\beta_1^2}f+\gamma M(f_0-f)\right] \quad \ldots \ldots \ldots (54)$$

$\therefore$ at the beginning of the first stage ($f = f_0$, $x = 0$)

$$\frac{dp}{df} = \frac{2\lambda_1}{3}\frac{FC}{Al}\left\{\left(\frac{\beta_1+\beta_2}{\beta_2}\right)\left(-1+\frac{\beta_1}{\beta_2}\right)-\frac{(\beta_1+\beta_2)^2}{\beta_2^2}f_0\right\} \ldots \ldots \ldots (55)$$

At the end of the first stage $\left(f = \frac{\beta_1}{\beta_1+\beta_2},\ x = x_1\right)$

$$\frac{dp}{df} = \frac{FC}{A(x_1+l)}X \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (56)$$

and at the beginning of the second stage $\left(f = \frac{\beta_1}{\beta_1+\beta_2},\ x = x_1\right)$

$$\frac{dp}{df} = \frac{FC}{A(x_1+l)}Y \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (57)$$

where

$$X = \gamma Mf_0-\gamma M\frac{\beta_1}{\beta_1+\beta_2}-\frac{2\lambda_1}{3}\frac{\beta_1+\beta_2}{\beta_2} \quad \ldots \ldots \ldots (58)$$

$$Y = \gamma Mf_0-\gamma M\frac{\beta_1}{\beta_1+\beta_2}-2\lambda_2\frac{\beta_1+\beta_2}{\beta_1}, \quad \ldots \ldots \ldots (59)$$

and x_1 denotes the shot-travel up to the end of the first stage.

Also if x_2 denotes the shot-travel up to the end of the second stage, we have, at all-burnt ($f = 0$, $x = x_2$),

$$\frac{dp}{df} = \frac{FC}{A(x_2+l)}\gamma Mf_0. \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots (60)$$

For progressively moderated charges $\beta_1 < \beta_2$, and, therefore, from (55) we see that $\frac{dp}{df}$ is negative at shot-start. Also from (60), $\frac{dp}{df}$ is positive at all-burnt. From (56), (57) the sign of $\frac{dp}{df}$ at the end of the first stage and at the beginning of the second depends on those of X and Y respectively. Accordingly we have the following four cases:

Case (1) :

$$X > 0, \quad Y \geqslant 0. \quad \ldots \ldots \ldots \ldots (61)$$

The maximum pressure is unique and occurs in the first stage [cf. Fig. 1].

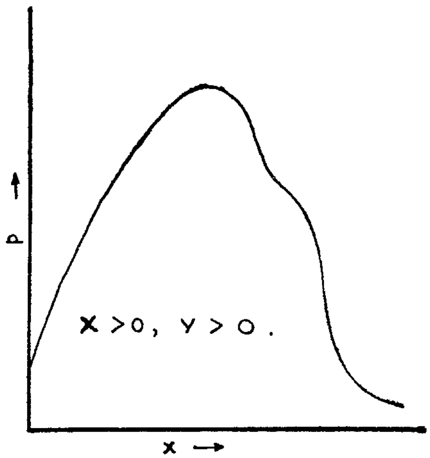

FIG. 1(a)

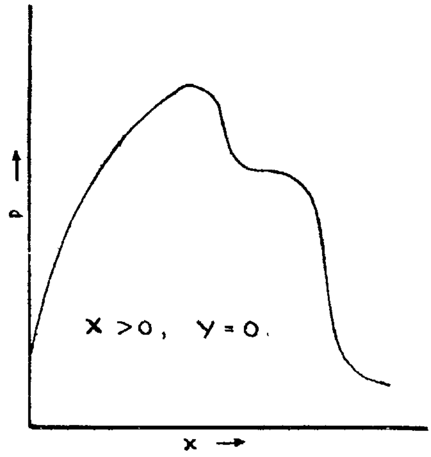

FIG. 1(b)

Case (2):

$$X > 0, \quad Y < 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (62)$$

In this case a pressure maximum occurs in the first stage; a pressure minimum occurs in crossing from first stage to second and a secondary pressure maximum occurs in the second stage [cf. Fig. 2].

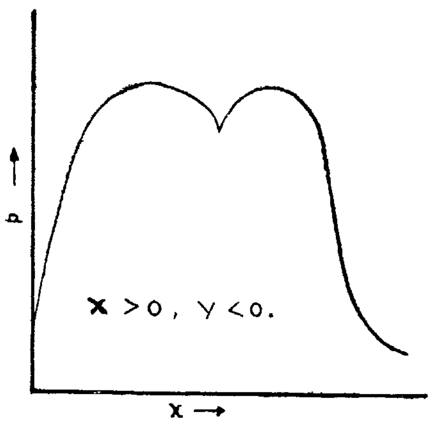

FIG. 2

Case (3):

$$X \leqslant 0, \quad Y \geqslant 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (63)$$

The maximum pressure is unique and occurs in passing from first stage to the second [cf. Fig. 3].

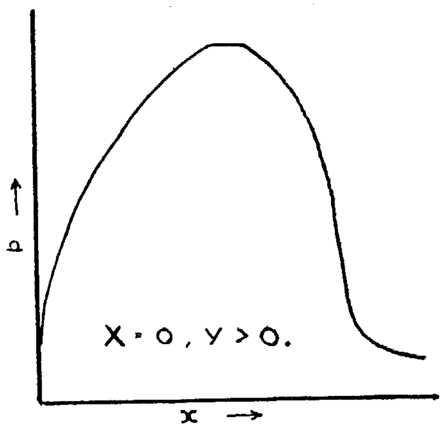

Fig. 3(a)

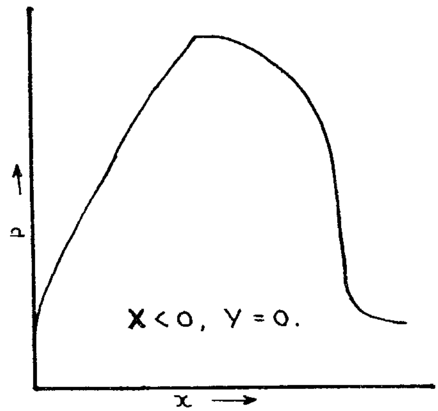

Fig. 3(b)

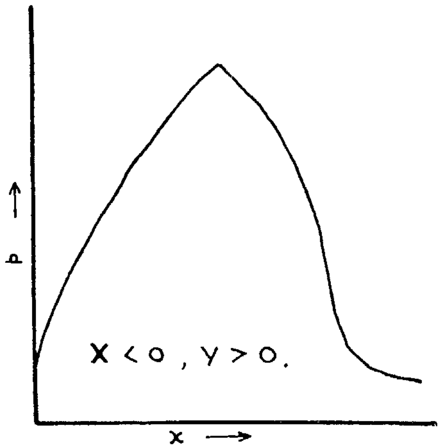

Fig. 3(c)

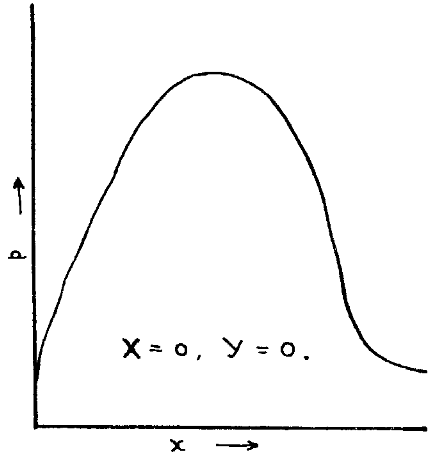

Fig. 3(d)

Case (4) :

$$X < 0, \quad Y < 0. \qquad \qquad (64)$$

The maximum pressure is unique and occurs in the second stage [cf. Fig. 4].

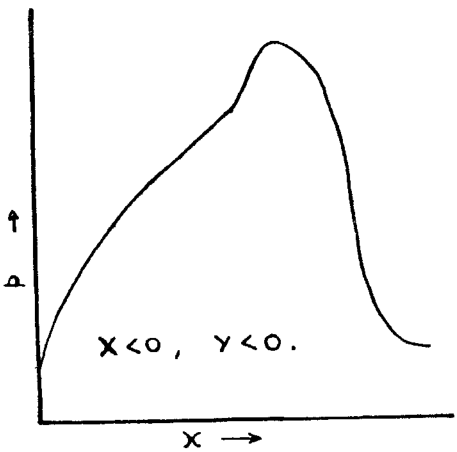

Fig. 4(a)

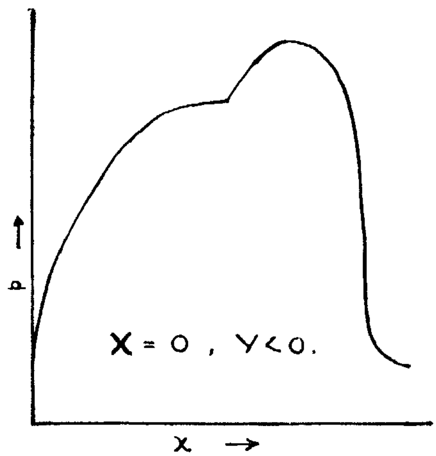

Fig. 4(b)

Thus we see that the maximum pressure is unique except in case (2) when a secondary pressure maximum can arise. This case is of special importance and we discuss it in the next section.

8. Possibility of getting a high piezometric efficiency with a moderated charge

The 'piezometric efficiency' of a given gun-charge-projectile combination refers to the 'flatness' of the pressure-space curve and may be defined as the ratio of the 'mean' pressure to the peak pressure. This may be increased by having a large chamber in the gun or a large effective ballistic size of the charge. In this section we see how a suitable moderated charge may be found to give a flat pressure-space curve.

Neglecting co-volume correction terms, the equations of Internal Ballistics for the rth stage $[r = 1, 2]$ are

$$FCz = Ap(x+l)+\tfrac{1}{2}(\gamma-1)w_1v^2 \qquad (65)$$

where

$$w_1 = 1{\cdot}05w+\tfrac{1}{3}C \qquad (66)$$

$$w_1v\frac{dv}{dx} = Ap \qquad (67)$$

$$D\frac{df}{dt} = -\beta p \qquad (68)$$

$$z = A_r+B_r(1-f)-E_r(1-f)^2 \qquad (69)$$

For a finite shot-start pressure p_0, the initial conditions are

$$x = 0,\ v = 0,\ z = z_0,\ f = f_0,\ p = p_0. \qquad (70)$$

Integrating the equations, subject to (70), we get

$$v = \frac{AD}{\beta w_1}(f_0-f) \qquad (71)$$

$$\frac{x+l}{x_{r-1}+l} = \left[\left(\frac{a_r-v_{r-1}}{a_r-v}\right)^{a_r}\left(\frac{b_r+v_{r-1}}{b_r-v}\right)^{b_r}\right]^{\frac{1}{K_r(a_r+b_r)}} \qquad (72)$$

$$p = \frac{w_1K_r}{A}\,\frac{(a_r-v)(b_r+v)}{x+l}, \qquad (73)$$

where

$$K_ra_rb_r = \frac{FC}{w_1}[A_r+B_r(1-f_0)-E_r(1-f_0)^2] \qquad (74a)$$

$$K_r(a_r-b_r) = \frac{FC\beta}{AD}[B_r-2E_r(1-f_0)] \qquad (74b)$$

$$K_r = \frac{FC\beta^2w_1}{A^2D^2}E_r+\tfrac{1}{2}(\gamma-1) = \frac{E_r}{M}+\tfrac{1}{2}(\gamma-1) \qquad (74c)$$

and x_{r-1}, v_{r-1}, p_{r-1} denote the shot-travel, velocity and pressure at the end of the $(r-1)$th stage.

Our object now is to get a cylindrical moderated charge of two layers of equal thickness which would give a pressure maximum $P_{[1]}$ at shot-travel $X_{[1]}$, a pressure minimum $P_{[2]}$ at shot-travel $X_{[2]}$, and another pressure maximum $P_{[3]}$ at shot-travel

$X_{[3]}$ $[X_{[1]} < X_{[2]} < X_{[3]}]$. If we choose $P_{[1]}$ and $P_{[3]}$ nearly equal and only slightly greater than $P_{[2]}$, and choose $X_{[1]}$ and $X_{[3]}$ at some reasonable distance apart, we shall get an almost 'flat' pressure-space curve.

Now from (71) and (72),

$$\frac{X_{[2]}+l}{l} = \left[\left(\frac{a_1}{a_1 - V_{[2]}}\right)^{a_1}\left(\frac{b_1}{b_1 + V_{[2]}}\right)^{b_1}\right]^{\frac{1}{K_1(a_1+b_1)}} \quad \ldots \quad \ldots \quad (75)$$

where

$$V_{[2]} = \frac{AD}{\beta w_1}\left(f_0 - \frac{\beta_1}{\beta_1+\beta_2}\right) = \frac{AD}{\beta w_1}\left(f_0 - \frac{\mu}{1+\mu}\right) \quad \ldots \quad \ldots \quad (76)$$

and $\mu = \dfrac{\beta_1}{\beta_2}$.

From (53), the pressure maximum occurs in the first stage when f is given by

$$f_{[1]} = \frac{\frac{2\lambda_1}{3}(1+\mu)^2 - \frac{4\lambda_1}{3}(1+\mu) + \gamma M f_0}{\gamma M + \frac{2\lambda_1}{3}(1+\mu)^2}$$

From (71), $V_{[1]}$ at this instant is given by

$$V_{[1]} = \frac{AD}{\beta w_1}(f_0 - f_{[1]})$$

(72) then gives

$$\frac{X_{[1]}+l}{l} = \left[\left(\frac{a_1}{a_1 - V_{[1]}}\right)^{a_1}\left(\frac{b_1}{b_1 + V_{[1]}}\right)^{b_1}\right]^{\frac{1}{K_1(a_1+b_1)}} \quad \ldots \quad \ldots \quad (77)$$

and

$$\frac{X_{[3]}+l}{X_{[2]}+l} = \left[\left(\frac{a_2 - V_{[2]}}{a_2 - V_{[3]}}\right)^{a_2}\left(\frac{b_2 + V_{[2]}}{b_2 + V_{[3]}}\right)^{b_2}\right]^{\frac{1}{K_2(a_2+b_2)}} \quad \ldots \quad \ldots \quad (78)$$

where $V_{[3]}$ is obtained by putting $\dfrac{dp}{df} = 0$ in (54) and substituting the value of f so obtained in (71).

Again from (73), we get

$$P_{[1]}(X_{[1]}+l) = \frac{w_1 K_1}{A}(a_1 - V_{[1]})(b_1 + V_{[1]}) \quad \ldots \quad \ldots \quad (79)$$

$$P_{[2]}(X_{[2]}+l) = \frac{w_1 K_1}{A}(a_1 - V_{[2]})(b_1 + V_{[2]})$$

$$= \frac{w_1 K_2}{A}(a_2 - V_{[2]})(b_2 + V_{[2]}) \quad \ldots \quad \ldots \quad (80)$$

and

$$P_{[3]}(X_{[3]}+l) = \frac{w_1 K_2}{A}(a_2 - V_{[3]})(b_2 + V_{[3]}) \quad \ldots \quad \ldots \quad (81)$$

Again from (65), at shot-start

$$Alp_0 = FCz_0 = A_1 + B_1(1-f_0) - E_1(1-f_0)^2 \quad \ldots \quad \ldots \quad (82)$$

Now we have equations (75) and (77) to (81), from which we can solve for the six quantities F_1, F_2; C_1, C_2; $\frac{d}{\beta}$, $\frac{\beta_1}{\beta_2}$. Thus we see that, in general, we can so choose, at least theoretically, the moderated charge as to get desired values of pressure maxima and minima at desired positions.

The above discussion can be considerably simplified by making one or more of the following assumptions :

(i) $\gamma_1 = \gamma_2$
(ii) zero shot-start pressure
(iii) the Isothermal assumption.

However, in view of the limited choice of propellants at our disposal, we cannot, in practice, arbitrarily fix up $P_{[1]}$, $P_{[2]}$, $P_{[3]}$; $X_{[1]}$, $X_{[2]}$, $X_{[3]}$; but out of the propellants at our disposal, we may be able to select a combination in a cylindrical moderated charge to give us the best advantage in the direction of obtaining a flat pressure-space curve.

We get simpler results if we want the pressure maxima and pressure minima to occur at specified values of f. From (53), the pressure maximum occurs in the first stage when

$$-\frac{4\lambda_1}{3}(1+\mu)+\frac{2\lambda_1}{3}(1+\mu)^2(1-f)+\gamma M(f_0-f) = 0 \quad .. \quad .. \quad (83)$$

and it occurs in the second stage when

$$-2\lambda_2\left(1+\frac{1}{\mu}\right)^2 f+\gamma M(f_0-f) = 0 \quad .. \quad .. \quad .. \quad (84)$$

If γ, f_0, μ are specified, we can solve for M and λ_1, from (83) and (84).

Thus in particular for

$$f_0 = 1,\ \gamma = \frac{5}{4},\ \mu = \frac{2}{3},\ \frac{\mu}{1+\mu} = \cdot 4,\ f_1 = \cdot 5,\ f_2 = \cdot 3, \text{ we get } M = 1\cdot 398,\ \lambda_1 = \cdot 675$$

$$f_0 = \cdot 95,\ \gamma = 1\cdot 24,\ \mu = \frac{4}{5},\ \frac{\mu}{1+\mu} = \frac{4}{9},\ f_1 = \cdot 5,\ f_2 = \cdot 4, \text{ we get } M = 1\cdot 66,\ \lambda_1 = \cdot 711$$

$$f_0 = 1,\ \gamma = 1\cdot 25,\ \mu = \frac{2}{3},\ \frac{\mu}{1+\mu} = \cdot 4,\ f_1 = \cdot 45,\ f_2 = \cdot 35, \text{ we get } M = 1\cdot 32,\ \lambda_1 = \cdot 826.$$

The practical procedure is to tabulate for all pairs of propellants and for all values of M satisfying $X > 0$, $Y < 0$, the values of f_1 and f_2 and then to choose the most favourable case.

9. Extension to general form-function

We have, so far, considered the case when the form-function for each component charge is in the standard form

$$z = (1-f)(1+\theta f),$$

but a more general case would be when the ith layer has the form-function

$$z_i = \phi_i(f) \quad .. \quad .. \quad .. \quad .. \quad .. \quad (85)$$

then instead of (14) we have during the rth stage,

$$z = \lambda_1+\lambda_2+\dots+\lambda_{r-1}+\lambda_r\phi_r(f_r)$$

$$= \lambda_1+\lambda_2+\dots+\lambda_{r-1}+\lambda_r\phi_r\left[b_r-\frac{D'}{D'_r}(1-f)\right] \qquad (86)$$

If we consider the particular case of a moderated charge of which the layers are concentric spherical shells, the form-function for the layer with the outer and inner radii a_r and a_{r+1} is easily seen to be

$$z = (1-f)\left[1+\frac{a_r^2+a_ra_{r+1}-2a_{r+1}^2}{a_r^2+a_ra_{r+1}+a_{r+1}^2}f+\frac{a_r^2-a_{r+1}^2}{a_r^2+a_ra_{r+1}+a_{r+1}^2}f^2\right] \qquad (87)$$

In particular, if the layers are of equal thickness the form-function for the rth layer is

$$z = (1-f)\left[1+\frac{3n-3r+1}{3n^2-3n(2r-1)+3r^2-3r+1}f+\frac{f^2}{3n^2-3n(2r-1)+3r^2-3r+1}\right] \qquad (88)$$

For the case when ith component charge has the cubic form-function

$$z_i = (1-f_i)(1+\theta_if_i+\psi_if_i^2) \qquad (89)$$

(86) gives

$$z = \lambda_1+\lambda_2+\dots+\lambda_{r-1}$$

$$+\lambda_r\left\{\left[1-b_r+\frac{D'}{D'_r}(1-f)\right]\right.$$

$$\left.\times\left[1+\theta_rb_r-\frac{\theta_rD'}{D'_r}(1-f)+\psi_r\left(b_r-\frac{D'}{D'_r}(1-f)\right)^2\right]\right\}$$

$$= A'_r+B'_r(1-f)+E'_r(1-f)^2+G'_r(1-f)^3, \qquad (90)$$

where

$$A'_r = \lambda_1+\lambda_2+\dots+\lambda_{r-1}+\lambda_r(1-b_r)(1+\theta_rb_r+\psi_rb_r^2) \qquad (91a)$$

$$B'_r = \lambda_r\frac{D'}{D'_r}[1+\theta_r+2\theta_rb_r-\psi_rb_r^2+2b_r\psi_r] \qquad (91b)$$

$$C'_r = \lambda_r\frac{D'^2}{D_r'^2}[\psi_r(1+b_r)-\theta_r] \qquad (91c)$$

$$D'_r = \psi_r\left(\frac{D'}{D'_r}\right)^3 \qquad (91d)$$

10. Comparison with the general theory of composite charges

The general theory of moderated charges that we have developed is parallel to the general theory of composite charges. However, it is interesting to note certain points of distinction between the two theories:

(1) In a moderated charge, at any instant, only one of the component charges is burning, while in a composite charge, all the remaining charges burn simultaneously. In particular, if the effective ballistic sizes of the component charges of a

composite charge are equal, they burn out simultaneously. No such thing is possible for moderated charges.

(2) It has been proved in the general theory of composite charges that the maximum pressure is, under certain general conditions, unique (Kapur, 1956*b*). A similar statement is not possible for moderated charges. As a matter of fact at the end of a stage, due to the sudden rise in the rate of burning constant, the downward trend in pressure may be changed to an upward trend and secondary pressure maxima can arise.

(3) For composite charges, θ may be greater than unity, i.e. the single equivalent charge may be more digressive than a cord. This is not likely to occur for a moderated charge for the simple reason that a moderated charge will be less digressive (cf. Introduction) than an unmoderated propellant of the same general shape.

In addition, the two theories can be usefully combined in, among others, the following two cases:

(1) If in a cylindrical moderated charge, the ends are not inhibited for burning, they burn as a composite charge, while surface layers burn according to the theory of moderated charges.

(2) Consider a tubular moderated charge burning from both sides. For convenience in theory we just ensure that the rth layer counted from the outermost layer has the same effective ballistic size as the rth layer counted from the innermost layer, for all values of r. Then these two layers burn as a single charge with mass equal to sum of the masses of the two layers, energy per unit mass as the weighted average of the energies per unit mass of the two component charges, the weights being the corresponding charge masses; and the form-factor as the weighted average of the form-factors for the two component charges —the weights being the total energies in the two component charges (Kapur, 1956*a*). In this way, the moderated tubular charge with n layers can be treated as a moderated charge with $\frac{n}{2}$ layers or $\left(\frac{n+1}{2}\right)$ layers according as n is even or odd.

The form-factors for the component layers may be found as in Section 5. For a tubular charge with layers of equal thickness and number of layers even, we have

$$\theta_1 = \frac{1}{2n+1},\ \theta_2 = \frac{1}{2n-1},\ \ldots\ \theta_{\frac{n}{2}} = \frac{1}{n+1}$$

$$\theta_n = -\frac{1}{3},\ \theta_{n-1} = -\frac{1}{5},\ \ldots\ \theta_{\frac{n}{2}+1} = -\frac{1}{n-1}$$

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Summary

In this paper, the equivalent charge method in the general theory of moderated charges has been developed. The form-function for each stage has been obtained and two estimates for θ, the equivalent charge form-factor, have been given. The conditions for the occurrence of a pressure maximum in or at the end of a stage have been obtained and the results illustrated for a cylindrical moderated charge, specially that containing two layers of equal thickness. In the last

case the conditions for two pressure maxima have been obtained. We have used this case to explain, in general terms, the possibility of getting a high piezometric efficiency with a moderated charge. The results have been extended to a moderated charge consisting of component charges with general form-function and, in the final section, an interesting case where the general theories for composite and moderated charges can be usefully combined has been given.

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