ON THE CESÀRO MEANS OF ORDER r OF THE rTH DERIVED FOURIER SERIES AND ITS ALLIED SERIES

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1. Let $f(t)$ be integrable L in $(-\pi, \pi)$ and periodic with period 2π and let its Fourier series be

$$\frac{1}{2}\alpha_0+\sum_{n=1}^{\infty}(\alpha_n \cos nt+\beta_n \sin nt)=\sum_{n=0}^{\infty}A_n(t). \quad \ldots \quad \ldots \quad (1.1)$$

Then the allied series of (1.1) is

$$\sum_{n=1}^{\infty}(\beta_n \cos nt-\alpha_n \sin nt)=\sum_{1}^{\infty}B_n(t). \quad \ldots \quad \ldots \quad (1.2)$$

The series obtained by differentiating the Fourier series (1.1) r-times at $t=x$ is

$$\sum_{n=0}^{\infty}\left(\frac{d}{dx}\right)^r A_n(x) \begin{cases} =\sum(-1)^{\frac{1}{2}r}n^rA_n, \ (r \text{ even}), \text{ where } A_n=A_n(x) \\ =\sum(-1)^{\frac{1}{2}(r-1)}n^rB_n, \ (r \text{ odd}), \text{ where } B_n=B_n(x) \end{cases} \quad (1.3)$$

and the series obtained by differentiating the allied series (1.2) r-times at $t=x$ is

$$\sum_{n=1}^{\infty}\left(\frac{d}{dx}\right)^r B_n(x) \begin{cases} =\sum(-1)^{\frac{1}{2}r}n^rB_n, \ (r \text{ even}) \\ =\sum(-1)^{\frac{1}{2}(r+1)}n^rA_n, \ (r \text{ odd}) \end{cases} \quad \ldots \quad \ldots \quad (1.4)$$

Let $t_n^{(r)}$ and $\bar{t}_n^{(r)}$ denote the nth Cesàro means of order r of the rth derived Fourier series and its allied series respectively and let a polynomial

$$P(t)=\sum_{i=0}^{r-1}\frac{\theta^i}{\underline{|i}}t^i \dagger \quad \text{be found such that, for } -\pi \leqslant t \leqslant \pi$$

$$g(t)=\frac{\underline{|r}}{t^r}\left[\{f(x+t)-P(t)\}+(-1)^r\{f(x-t)-P(-t)\}\right] \ldots \quad \ldots \quad (1.5)$$

$$h(t)=\frac{\underline{|r}}{t^r}\left[\{f(x+t)-P(t)\}-(-1)^r\{f(x-t)-P(-t)\}\right] \quad \ldots \quad (1.6)$$

are integrable in $(-\pi, \pi)$. The functions $g(t)$ and $h(t)$ are defined by periodicity outside the range $(-\pi, \pi)$.

† Here θ's are suitable constants.

The object of the present note is to prove the following theorems:

THEOREM 1. If

$$\int_t^\pi \frac{|g(u)|}{u}\,du = 0\left(\log\frac{1}{t}\right)\dagger\text{, as } t\to 0 \qquad (1.7)$$

then $t_n^{(r)} = 0\,(\log n)$.

THEOREM 2. If

$$\int_t^\pi \frac{|h^*(u)|}{u}\,du = 0\left(\log\frac{1}{t}\right)\text{, as } t\to 0 \qquad (1.8)$$

where $h^*(t) = h(t)-d$, $d \equiv d(x)$

then $$\bar{t}_n^{(r)} \sim \frac{d}{\pi}\log n.$$

We require the following lemmas:

Lemma 1. The series $\Sigma(-1)^n n^k$ is summable $(C,\ k+1)$, k integral $\geqslant 0$ (Hobson, 1950; Knopp, 1949).

Lemma 2 (Obreschkoff, 1934). If $K_r\,(n, t)$ and $\bar{K}_r\,(n, t)$ denote the nth Cesàro means of order r for the sequences $\pi^{-1}\cos nt$ and $\pi^{-1}\sin nt$, then

$$\left|\left(\frac{d}{dt}\right)^r\left\{K_{r+1}(n,t)+i\bar{K}_{r+1}(n,t)\right\}\right|\begin{cases}\leqslant An^r & (1.9)\\ \leqslant An^{-1}t^{-r-1}\ddagger & (1.10)\end{cases}$$

2. PROOF OF THEOREM 1.

Let $a_{n,0}$ denote the nth term of the rth derived Fourier series at $t = x$.
If r be even, equal say to $2m$, we have

$$a_{n,\,0} = (-1)^m n^{2m}(\alpha_n\cos nx+\beta_n\sin nx)$$

$$= (-1)^m n^{2m}\pi^{-1}\int_0^\pi \{f(x+t)+f(x-t)\}\cos nt\,dt$$

$$= (-1)^m n^{2m}\pi^{-1}\int_0^\pi\left[\frac{1}{\underline{|2m}}t^{2m}g(t)+\{P(t)+P(-t)\}\right]\cos nt\,dt$$

$= a_n+a_{n,\,1}$ where

$$a_n = \frac{(-1)^m}{\underline{|2m}}n^{2m}\pi^{-1}\int_0^\pi t^{2m}g(t)\cos nt\,dt.$$

$$a_{n,\,1} = (-1)^m n^{2m}2\pi^{-1}\sum_{\mu=1}^{m-1}\frac{\theta_{2\mu}}{\underline{|2\mu}}\int_0^\pi t^{2\mu}\cos nt\,dt.$$

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† The relation between the Condition (1.7) and the Condition $\int_0^t |g(u)|du = 0\,(t)$ is that the latter implies the former, but is not implied by it. For this *see* Misra (1947).

‡ A is used to denote a number independent of n and t, but its value may differ from occurrence to occurrence.

Now $$\int_0^{\pi} t^{2\mu} \cos nt\, dt = (-1)^n \sum_{l=1}^{\mu} (-1)^{l-1} \frac{|2\mu}{|2\mu-2l+1} \pi^{2\mu-2l+1} n^{-2l},$$

so that

$$a_{n,1} = (-1)^n 2\pi^{-1} \sum_{l=1}^{m-1} (-1)^{m+l-1} n^{2m-2l} \sum_{\mu=l}^{m-1} \frac{\theta_{2\mu}}{|2\mu-2l+1} \pi^{2\mu-2l+1} \dagger$$

We write $t_n^{(r)} = t_{n,1}^{(r)} + t_{n,2}^{(r)}$, where .. .. (2.1)

$t_{n,1}^{(r)}$ and $t_{n,2}^{(r)}$ are the nth Cesàro means of order r of the series Σa_n and $\Sigma a_{n,1}$.

By Lemma 1, it is clear that $\Sigma a_{n,1}$ is summable $(C, r-1)$, and, a fortiori, summable (C, r).

Hence we have

$$t_{n,2}^{(r)} \to \text{a finite limit} \quad .. \quad .. \quad .. \quad .. \quad (2.2)$$

Suppose now that r is odd, equal say to $2m+1$. Then

$$a_{n,0} = (-1)^m n^{2m+1} (\beta_n \cos nx - \alpha_n \sin nx)$$

$$= (-1)^m n^{2m+1} \pi^{-1} \int_0^{\pi} \{f(x+t) - f(x-t)\} \sin nt\, dt$$

$$= (-1)^m n^{2m+1} \pi^{-1} \int_0^{\pi} \left[\frac{1}{|2m+1} t^{2m+1} g(t) + \{P(t) - P(-t)\}\right] \sin nt\, dt$$

$= a_n + a_{n,2}$, where

$$a_n = \frac{(-1)^m}{|2m+1} n^{2m+1} \pi^{-1} \int_0^{\pi} t^{2m+1} g(t) \sin nt\, dt,$$

$$a_{n,2} = (-1)^m n^{2m+1} 2\pi^{-1} \sum_{\mu=1}^{m} \frac{\theta_{2\mu-1}}{|2\mu-1} \int_0^{\pi} t^{2\mu-1} \sin nt\, dt.$$

By evaluating the integral in $a_{n,2}$ it is easy to see by Lemma 1 that $\Sigma a_{n,2}$ is summable (C, r).

We write as in (2.1) that $t_n^{(r)} = t_{n,1}^{(r)} + t_{n,3}^{(r)}$, where

$t_{n,3}^{(r)}$ is the nth Cesàro mean of order r of the series $\Sigma a_{n,2}$.

Hence we have

$$t_{n,3}^{(r)} \longrightarrow \text{a finite limit} \quad .. \quad .. \quad .. \quad .. \quad (2.3)$$

It therefore follows that whether r is even or odd, we have to consider $t_{n,1}^{(r)}$, the nth Cesàro mean of order r of the series Σa_n, where

$$a_n = \frac{\pi^{-1}}{|r} \int_0^{\pi} t^r g(t) \left(\frac{d}{dt}\right)^r \cos nt\, dt.$$

† For these reductions *see* Hyslop (1939). These are written out in detail for the sake of completeness.

Now

$$t_{n,1}^{(r)} = \frac{\pi^{-1}}{\lfloor r} \frac{1}{A_n^r} \int_0^{\pi} t^r g(t) \left(\frac{d}{dt}\right)^r \sum_{k=0}^{n} A_{n-k}^r \cos kt \, dt$$

$$= \frac{n+r+1}{\lfloor r+1} \int_0^{\pi} t^r g(t) \left(\frac{d}{dt}\right)^r K_{r+1}(n, t)\, dt$$

$$= \left(\frac{n}{\lfloor r+1} + \frac{1}{\lfloor r}\right) \int_0^{\frac{\pi}{n}} t^r g(t) \left(\frac{d}{dt}\right)^r K_{r+1}(n, t)\, dt$$

$$+ \left(\frac{n}{\lfloor r+1} + \frac{1}{\lfloor r}\right) \int_{\frac{\pi}{n}}^{\pi} t^r g(t) \left(\frac{d}{dt}\right)^r K_{r+1}(n, t)\, dt$$

$$= I_1 + I_2, \text{ say} \qquad \qquad (2.4)$$

Using (1.7) and (1.9), we have

$$|I_1| \leqslant \left(\frac{n}{\lfloor r+1} + \frac{1}{\lfloor r}\right) A\, n^r \int_0^{\frac{\pi}{n}} t^r |g(t)|\, dt$$

$$\leqslant A\pi^r \left(\frac{n}{\lfloor r+1} + \frac{1}{\lfloor r}\right) \int_0^{\frac{\pi}{n}} |g(t)|\, dt$$

$$= o(\log n).\dagger \qquad \qquad (2.5)$$

Again using (1.7) and (1.10), we have

$$|I_2| \leqslant \left(\frac{n}{\lfloor r+1} + \frac{1}{\lfloor r}\right) A\, n^{-1} \int_{\frac{\pi}{n}}^{\pi} \frac{|g(t)|}{t}\, dt$$

$$\leqslant A \int_{\frac{\pi}{n}}^{\pi} \frac{|g(t)|}{t}\, dt$$

$$= o(\log n). \qquad \qquad (2.6)$$

Thus by (2.4), (2.5) and (2.6), we have

$$t_{n,1}^{(r)} = o(\log n). \qquad \qquad (2.7)$$

Hence by (2.1), (2.2), (2.3) and (2.7), we have

$t_n^{(r)} = O(1) + o(\log n) = o(\log n)$, which completes the proof of Theorem 1.

3. Proof of Theorem 2. It is easy to see, by an argument similar to that used at the beginning of the proof of Theorem 1, that

$$\bar{t}_n^{(r)} = O(1) + \bar{t}_{n,1}^{(r)}, \qquad \qquad (3.1)$$

† Misra (1947).

where $\bar{t}^{(r)}_{n,1}$, is the nth Cesàro mean of order r of the series $\Sigma\,\bar{a}_n$, and

$$\bar{a}_n = \frac{\pi^{-1}}{\lfloor r}\int_0^\pi t^r h(t)\left(\frac{d}{dt}\right)^r \sin nt\, dt.$$

Now we have

$$\bar{a}_n = \frac{\pi^{-1}}{\lfloor r}\int_0^\pi t^r h^*(t)\left(\frac{d}{dt}\right)^r \sin nt\, dt + \frac{d}{\pi}\frac{1}{\lfloor r}\int_0^\pi t^r\left(\frac{d}{dt}\right)^r \sin nt\, dt$$

$$= \bar{a}_{n,1} + \bar{a}_{n,2}, \text{ say.}$$

Let $\bar{t}^{(r)}_{n,1} = \bar{t}^{(r)}_{n,2} + \bar{t}^{(r)}_{n,3}$, where $\bar{t}^{(r)}_{n,2}$ and $\bar{t}^{(r)}_{n,3}$ are the nth .. .. (3.2)
Cesàro means of order r of the series $\Sigma\,\bar{a}_{n,1}$ and $\Sigma\,\bar{a}_{n,2}$ respectively.

Now

$$\bar{t}^{(r)}_{n,2} = \left(\frac{n}{\lfloor r+1} + \frac{1}{\lfloor r}\right)\int_0^\pi t^r h^*(t)\left(\frac{d}{dt}\right)^r \bar{K}_{r+1}(n,t)\, dt$$

$$= \left(\frac{n}{\lfloor r+1} + \frac{1}{\lfloor r}\right)\int_0^{\frac{\pi}{n}} t^r h^*(t)\left(\frac{d}{dt}\right)^r \bar{K}_{r+1}(n,t)\, dt$$

$$+ \left(\frac{n}{\lfloor r+1} + \frac{1}{\lfloor r}\right)\int_{\frac{\pi}{n}}^{\pi} t^r h^*(t)\left(\frac{d}{dt}\right)^r \bar{K}_{r+1}(n,t)\, dt$$

$$= J_1 + J_2, \text{ say.} \qquad \text{.. .. ..} \qquad (3.3)$$

Now the discussions of J_1 and J_2 are exactly similar to those of I_1 and I_2 above and hence

$$\bar{t}^{(r)}_{n,2} = o(\log n). \qquad \text{.. .. .. ..} \qquad (3.4)$$

Lastly it remains to consider $\bar{t}^{(r)}_{n,3}$.

$$\bar{a}_{n,2} = (-1)^{\frac{1}{2}r}\frac{\pi^{-1}}{\lfloor r} n^r d\int_0^\pi t^r \sin nt\, dt, \ (r \text{ even})$$

$$= (-1)^{\frac{1}{2}(r+1)}\frac{\pi^{-1}}{\lfloor r} n^r d\int_0^\pi t^r \cos nt\, dt, \ (r \text{ odd}).$$

By evaluating the integrals above, it is easy to see that

$$\bar{a}_{n,2} = (-1)^n\left[a_{r-2}n^{r-2} + a_{r-4}n^{r-4} + \ldots + a_1 n\right] + \frac{d}{\pi}\frac{1-\cos n\pi}{n}, \ (r \text{ odd})$$

$$= (-1)^n\left[b_{r-1}n^{r-1} + b_{r-3}n^{r-3} + \ldots + b_1 n\right] + \frac{d}{\pi}\frac{1-\cos n\pi}{n}, \dagger \ (r \text{ even})$$

$$= \gamma_n + \delta_n,$$

† Here a's and b's are constants.

where γ_n stands for any of the two expressions inside the square brackets.

Let
$$\bar{t}^{(r)}_{n,3}=\bar{t}^{(r)}_{n,4}+\bar{t}^{(r)}_{n,5} \qquad (3.5)$$

where $\bar{t}^{(r)}_{n,4}$ and $\bar{t}^{(r)}_{n,5}$ are the nth Cesàro means of order r of the series $\Sigma\gamma_n$ and $\Sigma\delta_n$ respectively.

It is easy to see that $\Sigma\gamma_n$ is summable (C, r) and hence $\bar{t}^{(r)}_{n,4}\rightarrow$ a finite limit (3.6)

Now
$$\delta_n=\frac{d}{\pi}\frac{1-\cos n\pi}{n}=\frac{d}{\pi}\,\omega_n,\ \text{say} \qquad (3.7)$$

Considering the nth Cesàro mean of order 1 of $\Sigma\delta_n$, we have

$$\bar{t}^{(1)}_{n,5}=\frac{d}{\pi}\frac{1}{n+1}\sum_{k=1}^{n}(n-k+1)\,\omega_k$$
$$=\frac{d}{\pi}\frac{1}{n+1}\sum_{k=1}^{n}\Omega_k\,\dagger$$
$$=\frac{d}{\pi}\frac{1}{n+1}\sum_{1}^{n}\{\log k+C+o\,(1)\}$$
$$=\frac{d}{\pi}\{\log n+C+o\,(1)\}.$$

Similar argument shows that the nth Cesàro mean of order 2 of $\Sigma\delta_n$ is

$$\bar{t}^{(2)}_{n,5}=\frac{1}{n+1}\frac{d}{\pi}\sum_{k=1}^{n}\bar{t}^{(1)}_{k,5}$$
$$=\frac{d}{\pi}\{\log n+C+o\,(1)\}$$

By successive application of this we have

$$\bar{t}^{(r)}_{n,5}=\frac{d}{\pi}\{\log n+C+o\,(1)\}$$
$$\sim\frac{d}{\pi}\log n,\ \text{where } r\geqslant 1. \qquad (3.8)$$

Thus by (3.5), (3.6) and (3.8), we have

$$\bar{t}^{(r)}_{n,3}\sim\frac{d}{\pi}\log n. \qquad (3.9)$$

Hence by (3.2), (3.4) and (3.9)

$$\bar{t}^{(r)}_{n,1}\sim\frac{d}{\pi}\log n. \qquad (3.10)$$

† Here $\Omega_k=\sum_{r=1}^{k}\omega_r=\log k+C+\epsilon_k$, where C is a constant and $\epsilon_k\rightarrow o$ as $k\rightarrow\infty$.

Finally by (3.1) and (3.10), we find

$$\bar{t}_n^{(r)} \sim \frac{d}{\pi} \log n, \text{ which completes the proof of Theorem 2.}$$

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